COMPOSITIO MATHEMATICA

Higher Chow cycles on Abelian surfaces and a non-Archimedean analogue of the Hodge-$\mathcal{D}$-conjecture

Ramesh Sreekantan

Compositio Math. **150** (2014), 691–711

doi:10.1112/S0010437X13007690

FOUNDATION COMPOSITIO MATHEMATICA

LONDON MATHEMATICAL SOCIETY

Compositio Math. 150 (2014) 691–711
doi:10.1112/S0010437X13007690

Higher Chow cycles on Abelian surfaces and a non-Archimedean analogue of the Hodge-$\mathcal{D}$-conjecture

Ramesh Sreekantan

Abstract

We construct new indecomposable elements in the higher Chow group $CH^2(A, 1)$ of a principally polarized Abelian surface over a p-adic local field, which generalize an element constructed by Collino [*Griffiths' infinitesimal invariant and higher K-theory on hyperelliptic Jacobians*, J. Algebraic Geom. **6** (1997), 393–415]. These elements are constructed using a generalization, due to Birkenhake and Wilhelm [*Humbert surfaces and the Kummer plane*, Trans. Amer. Math. Soc. **355** (2003), 1819–1841 (electronic)], of a classical construction of Humbert. They can be used to prove a non-Archimedean analogue of the Hodge-$\mathcal{D}$-conjecture – namely, the surjectivity of the boundary map in the localization sequence – in the case where the Abelian surface has good and ordinary reduction.

1. Introduction

The aim of this paper is to prove a non-Archimedean analogue of the Hodge-$\mathcal{D}$-conjecture for Abelian surfaces. This conjecture asserts that the boundary map in the localization sequence of higher Chow groups is surjective. If an S-integral version of the Beilinson conjectures were known this would be a consequence of them; but since they seem a little out of reach at the moment it is of interest to prove this weaker statement.

The conjecture is as follows.

Conjecture 1.1. Let X be a projective scheme over a global field K. Let p be a prime in $\mathcal{O}_{\mathrm{K}}$. We think of X as a variety over K_p, the completion at p. Let $\mathcal{X}$ be a model over the ring of integers $\mathcal{O}_{\mathrm{K}_p}$ of K_p and let $\mathcal{X}_p$ be the special fibre. We assume $\mathcal{X}_p$ is smooth – that is, X has good reduction at p.

For $m, n \geqslant 1$ let

$$\Sigma_X^{m,n} := \mathrm{Ker}\{CH^m(\mathcal{X}, n-1) \longrightarrow CH^m(X, n-1)\}.$$

Then the map

$$CH^m(X, n) \otimes \mathbb{Q} \xrightarrow{\partial} CH^{m-1}(\mathcal{X}_p, n-1) \otimes \mathbb{Q}$$

is surjective and $\Sigma_X^{m,n}$ is finite.

In fact, one can formulate this conjecture more generally for primes p of semi-stable reduction [Sre08], but in this paper we will only deal with primes of good reduction. We can

Received 27 July 2012, accepted in final form 22 August 2013, published online 26 March 2014.
2010 Mathematics Subject Classification 19E15 (primary), 11G25, 14C15, 14G20 (secondary).
Keywords: higher Chow groups, rational curves, Abelian surfaces.

also consider this conjecture for X over a global field itself – looking at the boundary map for *all* primes – but that is a much harder question.

In the local case the surjectivity of ∂ alone is sufficient to show that Σ_X is finite, but in the global case it only shows that Σ_X is torsion.

A few cases of this conjecture, both the local and global version, are known.

- $m = n = 1$ and $X = \mathrm{Spec}(\mathrm{K})$ where K is a global or local field. Here $\mathcal{X} = \mathrm{Spec}(\mathcal{O}_{\mathrm{K}})$ is the ring of integers, $CH^1(X, 1) = \mathrm{K}^*$ and $CH^0(\mathcal{X}_p, 0) = CH^0(\mathcal{X}_p) \simeq \mathbb{Z}$. The conjecture is trivially true in the local case and follows from the finiteness of class number in the global case. In fact, the group $\Sigma_X^{1,1}$ is the class group.

- $m = k, n = 2k-1$, $k > 1$ and $X = \mathrm{Spec}(\mathrm{K})$ where K is a global or local field. This is an immediate consequence of the results of Quillen on the K-theory finite fields – as he showed that the higher K-groups of finite fields are finite.

- $m = 2$, $n = 1$ and $X = E \times E$ where E is an elliptic curve over $\mathbb{Q}$ and p a prime of good reduction. Mildenhall [Mil92] and Flach [Fla92] independently showed that this map is surjective in this case. Mildenhall further showed that $\Sigma_X^{2,1}$ is finite when E is a CM elliptic curve over $\mathbb{Q}$. Finiteness of $\Sigma_X^{2,1}$ in general is still not known.

- $m = 2$, $n = 1$ and $X = E_1 \times E_2$ where E_1 and E_2 are elliptic curves over a local field K_p with good reduction at p. Spiess [Spi99] showed that this map is surjective in this case.

- $m = 2$, $n = 1$ and $X = E_1 \times E_2$ where E_1 and E_2 are elliptic curves over a local field K_p with bad semi-stable reduction at p. In [Sre08] we generalized Spiess's work to the case of semi-stable reduction. Here we had to use a semi-stable model $\mathcal{X}$ of $E_1 \times E_2$.

- When $n > 1$ and p is a prime of good reduction this conjecture is a consequence of a conjecture of Parshin and Soulé which asserts that the higher Chow groups of a smooth projective variety over a finite field are torsion. In particular, in the cases when their conjecture is known, this conjecture follows.

- One can also formulate a function field variant of this conjecture – in fact this is used to formulate the Beilinson conjectures in that set-up. A special case is discussed in [Sre10].

- When $n = 1$ it is sometimes the case that all the elements of $CH^{m-1}(\mathcal{X}_p)$ are restrictions of elements of $CH^{m-1}(\mathcal{X})$ – for example, when $X = \mathbb{P}^k$. In these cases the conjecture is easily shown to the true as one can construct 'decomposable' elements in $CH^m(X, 1)$ coming from the product

$$CH^{m-1}(X) \otimes CH^1(X, 1) \to CH^m(X, 1)$$

which can be used to prove surjectivity.

The 'Archimedean' Hodge-$\mathcal{D}$-conjecture of Beilinson asserts that the regulator map to Deligne cohomology is surjective [Jan88]:

$$CH^m(X, n) \otimes \mathbb{R} \xrightarrow{r_{\mathcal{D}}} H_{\mathcal{D}}^{2m-n}(X, \mathbb{R}(m)).$$

This is false in general but was proved for $K3$ surfaces and Abelian surfaces by Chen and Lewis [CL05]. It is still expected to be true if X is defined over a number field. In fact Asakura and Saito [AS07] show that for certain generic surfaces a non-Archimedean version of this is false over a p-adic field as well. In this case too, however, one expects the conjecture to be true for varieties defined over global fields. This is why in the statement of the conjecture one has to assume X is defined over a global field.

Since the boundary map ∂ is a non-Archimedean version of the Beilinson regulator map we sometimes refer to it as the non-Archimedean regulator map. There are other reasons as well – the dimension of the target space is expected to be the same as the order of the pole of the local L-factor at p at a particular point [Con98] inasmuch as the target of the Beilinson regulator map, namely the real Deligne cohomology, has dimension equal to the order of the pole of the Archimedean local L-factor at a particular point. Our conjecture – which amounts to surjectivity of the boundary map – is sometimes referred to as the *non-Archimedean Hodge-$\mathcal{D}$-conjecture.*

We prove the following theorem.

THEOREM 1.2. *Let A be a simple, principally polarized, Abelian surface over a p-adic local field K_p. Let $\mathcal{A}$ be the Néron model of A over $\mathcal{O}_{\mathrm{K}_p}$ where $p \neq 2$ is an odd prime of good non-supersingular reduction. Let $\mathcal{A}_p$ be the special fibre. Let*

$$\Sigma_A = \mathrm{Ker}\{CH^2(\mathcal{A}) \longrightarrow CH^2(A)\}.$$

Then:

- *Σ_A is a finite p-group;*
- *the boundary map*

$$CH^2(A,1) \otimes \mathbb{Q} \xrightarrow{\partial} CH^1(\mathcal{A}_p) \otimes \mathbb{Q}$$

  is surjective.

Our theorem is hence a non-Archimedean version of the theorem of Chen and Lewis [CL05].

The outline of this paper is as follows. From Spiess [Spi99, § 4], it suffices to prove surjectivity of the boundary map, as that implies finiteness of Σ_A. He shows that it is a p-group as a consequence of the finiteness. In order to prove surjectivity, we first need to get an understanding of the target space $CH^1(\mathcal{A}_p) \otimes \mathbb{Q}$ or, equivalently, the Néron–Severi group of the special fibre $\mathcal{A}_p$. It is often the case that the rank of the Néron–Severi group of the special fibre $\mathcal{A}_p$ is greater than the rank of the Néron–Severi group of A. Hence there could be new cycles in the special fibre which do not come from the restriction of cycles in the total space. We first describe these cycles geometrically.

In order to do this we have to go to the associated Kummer surface and Kummer plane of the special fibre. This is where we use the assumption on odd characteristic. We then use a theorem of Birkenhake and Wilhelm which shows that a new cycle corresponds to a particular rational curve on the Kummer plane.

This rational curve pulls back to the union of two rational curves on the associated Kummer $K3$ surface. We then use a slight generalization of the work of Bogomolov *et al.* [BHT11, Theorem 18], to deform this sum of rational curves – which lie in the special fibre of the associated Kummer $K3$ surface of A – to an irreducible rational curve in the generic fibre. This is where we have to assume that the reduction is non-supersingular.

Finally, we use this deformed curve to construct an indecomposable higher Chow cycle in the $K3$ surface which can be transferred to the original Abelian surface. We then show that this higher Chow cycle bounds the new cycle.

Constructing 'interesting' higher Chow cycles on algebraic varieties is often difficult and finding them often has several implications. In the final section we state some immediate applications of these cycles to torsion in codimension 2. We also discuss some related questions.

2. Notation

- A – a principally polarized Abelian surface over a p-adic local field K_p.
- $\mathcal{A}$ – a model of A over the ring of integers $\mathcal{O}_{\mathrm{K}_p}$.

- $\mathcal{A}_p$ – the special fibre of $\mathcal{A}$ over the residue field k, usually assumed to be an ordinary Abelian surface.
- $\mathcal{L}_0$ – the line bundle representing the principal polarization.
- K_A – the Kummer surface associated to an Abelian surface A.
- $\tilde{K}_A$ – the associated $K3$ surface.
- $K\mathbb{P}_A$ – the associated Kummer plane.
- ϕ – the map from $A \longrightarrow K_A$ induced by $\mathcal{L}_0^2$.
- $\mathcal{C}_0$ – the line bundle corresponding to $\phi(D)$ for $D \in |\mathcal{L}_0|$.
- $\mathcal{F}_0$ – the hyperplane section on K_A. $\mathcal{F}_0 = \mathcal{C}_0^2$.
- π – the map from $K_A \longrightarrow K\mathbb{P}_A$.
- ν – the blow-up which gives the minimal resolution of singularities $\tilde{K}_A \longrightarrow K_A$.
- $\mathcal{L}_\Delta$ – a line bundle of invariant Δ.
- Q^Δ – the cycle on the Kummer plane corresponding to the line bundle of $\mathcal{L}_\Delta$.
- Q_1^Δ and Q_2^Δ – the two components of $\pi^{-1}(Q^\Delta)$ in K_A.
- $\tilde{Q}_i^\Delta$ – the strict transform of Q_i^Δ.
- $\tilde{\mathcal{Q}}^\Delta$ – the deformation of $\tilde{Q}_1^\Delta + \tilde{Q}_2^\Delta$.
- $\tilde{Q}_\eta^\Delta$ – the generic fibre of $\tilde{\mathcal{Q}}^\Delta$.
- D_i^Δ – the curve $\phi^{-1}(Q_i^\Delta)$ in A representing an extra cycle in the Néron–Severi group of $\mathcal{A}_p$.

3. Abelian surfaces

3.1 The Hodge-$\mathcal{D}$-conjecture for Abelian surfaces

Let A be an Abelian surface over a p-adic field K_p with finite residue field. In this paper we will always assume that A is principally polarized by a line bundle $\mathcal{L}_0$ – and write $(A, \mathcal{L}_0)$ when we wish to stress that fact. Let $\mathcal{A}$ be a model over the ring of integers $\mathcal{O}_{\mathrm{K}_p}$ with special fibre $\mathcal{A}_p$. We assume A has good ordinary reduction at p, so the special fibre $\mathcal{A}_p$ is smooth.

One has a map

$$\cdots \longrightarrow CH^2(A,1)\otimes\mathbb{Q} \xrightarrow{\partial} CH^1(\mathcal{A}_p)\otimes\mathbb{Q} \longrightarrow \cdots$$

coming from the localization sequence for higher Chow groups. The conjecture above asserts that the map ∂ is surjective.

In order to prove this conjecture one has to first understand the right-hand side – namely the Chow group of the special fibre – and then construct the higher Chow cycles that bound the cycles in the special fibre. In the next section we describe the Chow group of the special fibre.

3.2 The Néron–Severi group of an Abelian surface

We want to understand the group $CH^1(\mathcal{A}_p)\otimes\mathbb{Q}$. As $CH^1_{\mathrm{hom}}(\mathcal{A}_p)\otimes\mathbb{Q} = 0$, this is the same as the rational Néron–Severi group

$$CH^1(\mathcal{A}_p)\otimes\mathbb{Q} \simeq NS(\mathcal{A}_p)\otimes\mathbb{Q}.$$

It is well known that the Néron–Severi group can be identified as the part of the endomorphism algebra, $\mathrm{End}_{\mathbb{Q}}(\mathcal{A}_p)$, fixed by the Rosati involution $\dagger$

$$NS(\mathcal{A}_p)\otimes\mathbb{Q} \simeq \mathrm{End}_{\mathbb{Q}}(\mathcal{A}_p)^{\dagger}.$$

From Tate's theorem on the description of the endomorphism algebra [Tat66] one knows that the algebra of a simple Abelian surface contains a CM field of degree 4. In particular, the endomorphism algebra contains a real quadratic field. If the Abelian surface $\mathcal{A}_p$ is not simple, it contains the degenerate quadratic 'field' $\mathbb{Q} \oplus \mathbb{Q}$. On the real quadratic field as well as $\mathbb{Q} \oplus \mathbb{Q}$, the Rosati involution acts trivially so the rank of the Néron–Severi group is always at least 2.

We would like to get an explicit understanding of the generators of this Néron–Severi group. The principal polarization $\mathcal{L}_0$ of A is represented by a genus 2 curve C. In fact $A = J(C)$, the Jacobian of C. The closure of this curve in $\mathcal{A}$ restricted to the special fibre $\mathcal{A}_p$ gives one of the generators of $NS(\mathcal{A}_p) \otimes \mathbb{Q}$. However, it is often the case that there are cycles in the special fibre which are not the restrictions of the closure of cycles in the generic fibre. To get an understanding of these cycles one has to do a little more work. For this, we have to look at Kummer surface associated to A.

3.3 The Kummer surface and the Kummer plane

All the statements in this section are classical and can be found in [BW03], for example.

3.3.1. *The Kummer surface.* Let A be an Abelian surface. The *Kummer surface* of A is defined to be the hypersurface in $\mathbb{P}^3$

$$K_A = \phi_{\mathcal{L}_0^2}(A)$$

where $\phi = \phi_{\mathcal{L}_0^2}$ is the map

$$\phi_{\mathcal{L}_0^2} : A \longrightarrow \mathbb{P}^3$$

induced by the square of the principal polarization. Equivalently, this can be identified with $A/\{\pm 1\}$ – so the map $A \xrightarrow{\phi} K_A$ is a double cover ramified at the sixteen 2-torsion points of A. It is well known – see [BW03], for example – that the blow-up of K_A at these 16 points is a $K3$ surface $\nu : \tilde{K}_A \longrightarrow K_A$.

3.3.2. *The Kummer plane.* Let $\pi : \mathbb{P}^3 \backslash \{0\} \longrightarrow \mathbb{P}^2$ be the projection with centre 0, where $0 = \phi(0)$. The map π restricted to K_A is a double cover of $\mathbb{P}^2$ ramified at six lines $L_1, \ldots, L_6$. These six lines are tangent to a conic. The six lines meet at 15 points $\{q_{ij}\}$ where $q_{ij} = (L_i \cap L_j)$; see the Figure 1. These points are the images of the non-zero 2-torsion points under the map $\pi \circ \phi$. The collection $K\mathbb{P}_A = (\mathbb{P}^2, L_1, \ldots, L_6)$ is called the *associated Kummer plane* of the Abelian surface A.

The situation is summarized as

$$\begin{array}{ccccc} & & \tilde{K}_A & & \\ & & \downarrow{\scriptstyle \nu} & & \\ A & \xrightarrow[{[2:1]}]{\phi} & K_A & \xrightarrow[{[2:1]}]{\pi} & K\mathbb{P}_A \end{array}$$

3.3.3. *Humbert's theorem and its generalizations.* A classical theorem of Humbert [BW03] states that an Abelian surface $A = J(C)$ has real multiplication by $\mathbb{Z}((1+\sqrt{5})/2)$ if and only if there is a conic Q' on the Kummer plane $K\mathbb{P}_A$ which passes through five of the 15 points $\{q_{ij}\}$ *and* is tangent to one of the other lines. Further, since $\mathrm{End}(A) \otimes \mathbb{Q}^{\dagger} \simeq NS(A) \otimes \mathbb{Q}$, the

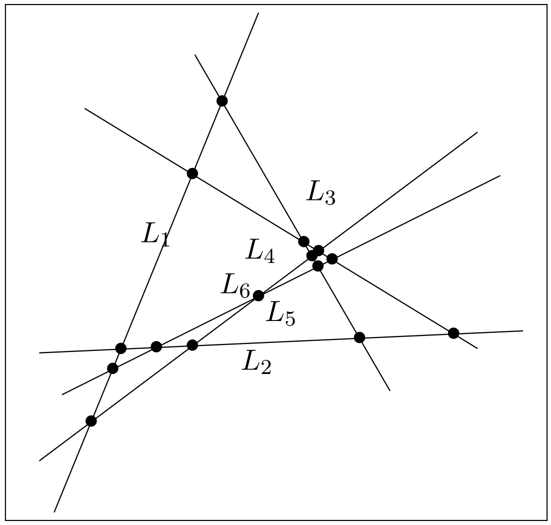

Figure 1. The six lines and 15 points on $\mathbb{P}^2$.

Néron–Severi group is of rank at least 2 and there is a curve D on A such that

$$Q' = \pi \circ \phi(D),$$

and D and C generate the part of the rational Néron–Severi group coming from $\mathbb{Z}((1+\sqrt{5})/2)$. Hence Humbert's theorem can be viewed as providing a geometric characterization of the extra cycle in the Néron–Severi group.

Birkenhake and Wilhelm [BW03] generalize this theorem and provide a geometric characterization of the cycles in the Néron–Severi group of all Abelian surfaces. In order to describe their theorem we need some definitions.

The *Humbert invariant* $\Delta(\mathcal{L})$ of a line bundle $\mathcal{L}$ is defined to be

$$\Delta(\mathcal{L}) = (\mathcal{L}.\mathcal{L}_0)^2 - 2\mathcal{L}^2$$

where (.) is the intersection pairing on $\mathrm{Pic}(A)$. This is the negative of the self-intersection number on the orthogonal complement of $\mathcal{L}_0$, hence is a positive definite quadratic form. There is a line bundle of non-zero Humbert invariant if and only if the Picard number is greater than 1. In fact, having a line bundle on A of Humbert invariant Δ is equivalent to saying that $\mathcal{O}_\Delta \subset \mathrm{End}(A)$, where $\mathcal{O}_\Delta$ is an order of discriminant Δ.

To state the theorem of Birkenhake and Wilhelm, we need to distinguish several different cases of Δ:

$$\Delta = \begin{cases} I. & 8d^2 + 9 - 2k \\ II. & 8d(d+1) + 9 - 2k \\ III. & 8d^2 + 8 - 2k \\ IV. & 8d(d+1) + 12 - 2k \\ V. & d^2 \ \ (d > 1) \end{cases} \tag{1}$$

where $d \geqslant 1$ and $k \in \{4, 6, 8, 10, 12\}$. The theorem of Birkenhake and Wilhelm is as follows.

Theorem 3.1 (Birkenhake–Wilhelm). *Let $(A, \mathcal{L}_0)$ be a principally polarized Abelian surface with a line bundle $\mathcal{L}_\Delta$ of invariant Δ. Then there exists a* **rational curve** *Q^Δ on the Kummer* *plane $K\mathbb{P}_A$ which passes through some of the points q_{ij} with no singularities at those points and meets some of the lines L_j with even multiplicity. One has the following cases.*

Case	Δ	$\deg(Q^{\Delta})$	*No. of points* q_{ij}
I.	$8d^2+9-2k$	$2d$	$k-1$
II.	$8d(d+1)+9-2k$	$2d+1$	k
III.	$8d^2+8-2k$	$2d$	k
IV.	$8d(d+1)+12-2k$	$2d+1$	$k-1$
V.	d^2	$d-1$	3

Further, $Q^{\Delta}=\pi\circ\phi(D^{\Delta})$ where D^{Δ} is a curve on A which lies in the linear system of divisors of a line bundle $\mathcal{L}$ of the form $\mathcal{L}_0^a\otimes\mathcal{L}_{\Delta}^b$ with $b\neq 0$. In particular, the class of D^{Δ} is not a multiple of the class of the principal polarization.

Remark 3.2. The curves C and D^{Δ} generate a two-dimensional subspace of the rational Néron–Severi group. While D^{Δ} need not be of Humbert invariant Δ, the divisor corresponding to the line bundle $\mathcal{L}_{\Delta}$ and D^{Δ} span the same subspace. An important aspect of their theorem is that there is a *rational curve* on $K\mathbb{P}_A$ which represents the extra cycle. Humbert's theorem above is the special case when $\Delta=5$.

We now lift this cycle to the Kummer surface. We have the following lemma, which is proved by Jakob [Jak94] in a special case.

Lemma 3.3. *$\pi^{-1}(Q^{\Delta})=Q_1^{\Delta}\cup Q_2^{\Delta}$, where Q_1^{Δ} and Q_2^{Δ} are rational curves on K_A.*

Proof. Birkenhake and Wilhelm [BW03] show that there is a curve Q_1^{Δ} such that the map $\pi:Q_1^{\Delta}\longrightarrow Q^{\Delta}$ is birational. Hence the curve Q_1^{Δ} is rational. Since the map π is a double cover, $\pi^{-1}(Q^{\Delta})=Q_1^{\Delta}\cup Q_2^{\Delta}$ where Q_2^{Δ} is another rational curve. □

3.3.4. *The involution ι.* π is a double cover so it induces an involution ι on K_A. Under this involution one has

$$\iota(Q_1^{\Delta})=Q_2^{\Delta}.$$

Since π is ramified over the lines L_i, $\pi^{-1}(L_i)$ is fixed under ι and we will abuse notation to denote the line $\pi^{-1}(L_i)$ in K_A by L_i as well. This involution also acts on the Néron–Severi group of the Abelian surface. We know that $\mathbb{Q}(\sqrt{\Delta})$ can be identified with a subgroup of $NS(A)\otimes\mathbb{Q}$, and, with respect to this identification, ι can be thought of as the non-trivial Galois conjugation.

The rational Néron–Severi group $NS(\mathbb{P}^2)\otimes\mathbb{Q}\simeq\mathbb{Q}$. So the pullback of a cycle in $K\mathbb{P}_A$ lies in subspace $\mathbb{Q}\mathcal{F}_0$. The involution ι acts trivially on the classes of such cycles and the class of a cycle in K_A lies in the $\mathbb{Q}$-span of $\mathcal{F}_0$ if and only if its class is fixed by ι.

The work of Birkenhake and Wilhelm is in the complex situation but their work is purely algebraic and carries through, *mutatis mutandis*, to the case of Abelian surfaces over finite fields as long as the characteristic is not 2.

We apply this to the case when $A=\mathcal{A}_p$, an Abelian surface over a finite field. Hence, for a line bundle $\mathcal{L}_{\Delta}$ in $\mathcal{A}_p$ with Humbert invariant $\Delta>0$ there is a rational curve Q^{Δ} in $K\mathbb{P}_{\mathcal{A}_p}$ associated to this line bundle. This cycle pulls back to $Q_1^{\Delta}\cup Q_2^{\Delta}$ on $K_{\mathcal{A}_p}$ and one of the components pulls back to a cycle D_p^{Δ} in $\mathcal{A}_p$ whose cycle class is not a multiple of the class of the principal polarization.

https://doi.org/10.1112/S0010437X13007690 Published online by Cambridge University Press

4. Rational curves on $K3$ surfaces

Our next step is to deform the rational curves $Q_1^\Delta \cup Q_2^\Delta$ above to the generic fibre. For this, we use some recent work of Bogomolov, Hassett and Tschinkel on the existence of rational curves on $K3$ surfaces. A conjecture, attributed to Mumford, states that there are infinitely many rational curves on an arbitrary $K3$ surface. The existence of even a single rational curve on a $K3$ surfaces is not always trivial.

In a recent paper, Bogomolov *et al.* [BHT11] proved a mixed characteristic generalization of a result of Mori and Mukai [MM83] which they use to construct infinitely many rational curves on certain $K3$ surfaces of Picard number 1. Their idea is to deform rational curves from a special fibre to the generic fibre. We would like to do something similar, and for our purposes we need to use a slight modification [BHT11, Theorem 18]. The proof of this was explained to us by Fakhruddin (personal communication).

Theorem 4.1. *Let (S_0, f_0) be a $K3$ surface over a finite field k of characteristic p together with a divisor class f_0 on S_0. Suppose*

$$C = C_1 + \cdots + C_r$$

is a connected union of rational curves $C_i \subset S_0$ such that $[C] = Nf_0$ for some positive integer N. Assume that the C_i are distinct and that S_0 is not uniruled. Let (S, f) be a (projective) $K3$ surface together with a divisor class defined over a finite extension W' of the ring of Witt vectors $W(k)$ reducing to the base change of (S_0, f_0) to the residue field of W'. Then there is a relative curve $R \subset S \times_{\mathrm{Spec}(W')} \mathrm{Spec}(W'')$, where W'' is a finite extension of W' such that R reduces to (the base change of) C and all irreducible components of the generic fibre of R are rational. If there are only two curves C_1 and C_2, the rational curve R is irreducible.

Proof. The proof is essentially the same as that of [BHT11, Theorem 18] – there the authors assume that f_0 is an ample class, so here we only explain, using their notation, why that assumption is unnecessary.

As in [BHT11] the dimension of the formal scheme $\overline{\mathcal{M}}_0^\circ(\mathcal{S}/B, Nf_0)$ and its image in B is at least 20; the ampleness assumption is not used anywhere for this computation. Moreover, [BHT11, Theorem 16] does not have the ampleness hypothesis, so as there we also get that the formal scheme Σ_{Nf_0} has dimension 20 and is not contained in the fibre over the closed point of $Spf(W(k))$.

By construction, the image of $\overline{\mathcal{M}}_0^\circ(\mathcal{S}/B, Nf_0)$ in B is contained in Σ_{Nf_0}, so since they have the same dimension and Σ_{Nf_0} is smooth (by [BHT11, Proposition 16]), they must be equal.

By assumption, since f reduces to f_0 it follows that the morphism $\mathrm{Spec}(W') \to B$ corresponding to the relative surface S factors through Σ_{Nf_0}, hence also through the image of $\overline{\mathcal{M}}_0^\circ(\mathcal{S}/B, Nf_0)$. Since we have assumed that S is projective over W', it follows from Grothendieck's algebraization theorem as in [BHT11] that the restriction of $\overline{\mathcal{M}}_0^\circ(\mathcal{S}/B, Nf_0)$ to $\mathrm{Spec}(W')$ is an *algebraic* stack. Moreover, this stack maps onto $\mathrm{Spec}(W')$ so the generic fibre is non-empty. It follows that there exists a morphism

$$\mathrm{Spec}(W'') \to \overline{\mathcal{M}}_0^\circ(\mathcal{S}/B, Nf_0) \times_B W',$$

with W'' a finite extension of W' such that the image of the closed point of $\mathrm{Spec}(W'')$ corresponds to the curve C. Pulling back the universal family over $\overline{\mathcal{M}}_0^\circ(\mathcal{S}/B, Nf_0)$ gives us a stable map to S (defined over W'') and the image of this gives us the desired relative curve R.

In the special case when there are only two curves C_1 and C_2 which do not deform to the generic fibre, [BHT11, Lemma 19] shows that R is irreducible. □

5. Elements of the higher Chow group

Let X be a surface over a global or local field K. The group $CH^2(X,1)$ has the following presentation [Ram89]. It is generated by formal sums of the type

$$\sum_i (C_i, f_i)$$

where C_i are curves on X and f_i are $\bar{\mathrm{K}}$-valued functions on the C_i satisfying the co-cycle condition

$$\sum_i \operatorname{div} f_i = 0.$$

Relations in this group are give by the tame symbol of pairs of functions on X.

There are some elements of this group coming from the product structure

$$\bigoplus_{\mathrm{L/K}} CH^1(X_{\mathrm{L}}) \otimes CH^1(X_{\mathrm{L}},1) \longrightarrow \bigoplus_{\mathrm{L/K}} CH^2(X_{\mathrm{L}},1) \xrightarrow{\oplus N_{\mathrm{L/K}}} CH^2(X,1)$$

where L runs through all finite extensions of K and $N_{\mathrm{L/K}}$ is the norm map. The subgroup of *decomposable elements* $CH^2_{\mathrm{dec}}(X,1)$ is the image of this map. A theorem of Bloch [Blo86, Theorem 6.1] says that $CH^1(X_{\mathrm{L}},1)$ is simply L^* where L is the field of definition of X_{L} – so such an element looks like a sum of elements of the type (C,a), where C is a curve on X_{L} and a is in L^*. The group of *indecomposable* elements of $CH^2(X,1)$ is the quotient group

$$CH^2_{\mathrm{ind}}(X,1) \simeq CH^2(X,1)/(CH^2_{\mathrm{dec}}(X,1).$$

In general it is not so easy to show that this group is non-trivial, and in some instances, for example for $X = \mathbb{P}^2$, it is trivial.

Remark 5.1. The group $CH^2(X,1)\otimes\mathbb{Q}$ is the same as the $\mathcal{K}$-cohomology group $H^1_{\mathrm{Zar}}(X,\mathcal{K}_2)\otimes\mathbb{Q}$ and the motivic cohomology group $H^3_{\mathcal{M}}(X,\mathbb{Q}(2))$ – see, for example, [Mül97].

5.1 The boundary map

Let X be as above and $\mathcal{X}$ a model of X over the ring of integers with special fibre $\mathcal{X}_p$ at a prime p. We assume $\mathcal{X}_p$ is smooth – that is, X has good reduction at p. The boundary map

$$\partial : CH^2(X,1) \longrightarrow CH^1(\mathcal{X}_p)$$

is defined by

$$\partial\Big(\sum_i (C_i, f_i)\Big) = \sum_i \operatorname{div}_{\bar{C}_i}(\bar{f}_i)$$

where $\bar{C}_i$ denotes the closure of C_i in the semi-stable model $\mathcal{X}$ of X. From the co-cycle condition, the horizontal divisor, namely, the closure $\sum_i \overline{\operatorname{div}_{C_i}(f_i)}$ of $\sum_i \operatorname{div}_{C_i}(f_i)$, is 0, so the boundary is supported on the special fibre.

For a decomposable element of the form (D,a) the boundary map is particularly simple to compute:

$$\partial((D,a)) = \operatorname{ord}_p(a)\mathcal{D}_p.$$

where $\mathcal{D}_p$ is the special fibre of the closure of D in $\mathcal{X}$.

Remark 5.2. In particular, a cycle in the special fibre which is not the restriction of a cycle in the generic fibre cannot appear in the boundary of a decomposable element.

https://doi.org/10.1112/S0010437X13007690 Published online by Cambridge University Press

6. A new element in the higher Chow group of an Abelian surface

Let $(A, \mathcal{L}_0)$ be an Abelian surface over a p-adic local field K_p as before. We assume that A has good, non-supersingular reduction at p. Then

$$\rho(\mathcal{A}_p) \geqslant \rho(A)$$

where ρ is the Picard number – the rank of the Néron–Severi group. If $\rho(\mathcal{A}_p) = \rho(A)$, then every cycle in $NS(\mathcal{A}_p)$ is the restriction of the closure of a cycle in $NS(A)$, so the surjectivity of the boundary map is trivial – as all the cycles can be obtained as boundaries of decomposable elements. Hence we assume $\rho(\mathcal{A}_p) > \rho(A)$. In other words, we assume that there are always cycles in the Néron–Severi group of the special fibre which are not the restriction to the special fibre of the closure of cycles on the generic fibre.

In this section we will construct, for such a cycle $\mathcal{D}_p$, an element $\Xi_{A,\mathcal{D}_p}$ of $CH^2(A,1)$ such that the image $\partial(\Xi_{A,\mathcal{D}_p})$ is a non-zero multiple of the cycle $\mathcal{D}_p$ in the Néron–Severi group of $\mathcal{A}_p$. In particular, from Remark 5.2, it is an *indecomposable* element in the higher Chow group.

The idea is as follows. One way to construct a higher Chow cycle on a surface S is to use a rational curve C with a node P lying on the surface. The normalization $\eta : \tilde{C} \to C$ will have two points P_1 and P_2 lying over the node. Since $\tilde{C}$ is rational one can find a function f on $\tilde{C}$ with divisor $P_1 - P_2$. Then the element $\eta_*(\tilde{C}, f)$ is an element of $CH^2(S,1)$.

Unfortunately, Abelian surfaces do not have rational curves. So we work with the associated Kummer $K3$ surface. The work of Birkenhake and Wilhelm [BW03] shows that given a cycle in the special fibre of non-zero Humbert invariant there is a rational curve in the special fibre of the $K3$ surface which represents it. We show that this curve along with a conjugate rational curve satisfies the condition of Theorem 4.1 – so we can deform them to an irreducible rational curve on the generic fibre. Further, we show that this curve has a node. Using that curve we construct a higher Chow cycle on the generic Kummer surface, transfer it to the Abelian surface and show that it has the required properties.

Since we work with related objects on the Abelian surface and its associated Kummer surface, $K3$ surface and Kummer plane, there is a lot of notation which can get a little confusing, so the reader should keep the above remarks in mind.

6.1 Lifting to the model

Let $\mathcal{A}$ be a regular model of A with special fibre $\mathcal{A}_p$. Let $K_{\mathcal{A}}$ be the Kummer surface of $\mathcal{A}$. Let $\mathcal{F}_0$ denote the hyperplane section on $K_{\mathcal{A}}$ – so $\phi^*(\mathcal{F}_0) = \mathcal{L}_0^2$. Let $(\tilde{K}_{\mathcal{A}}, \tilde{\mathcal{F}}_0)$ denote the $K3$ surface obtained by blowing up the 16 nodal points on $(K_{\mathcal{A}}, \mathcal{F}_0)$ and

$$\nu : \tilde{K}_{\mathcal{A}} \longrightarrow K_{\mathcal{A}}$$

denote the birational map from $\tilde{K}_{\mathcal{A}}$ to $K_{\mathcal{A}}$. Let $\nu^*(\mathcal{F})$ denote the total transform of a line bundle $\mathcal{F}$ on $K_{\mathcal{A}}$ and $\tilde{\mathcal{F}}_0$ denote the strict transform of $\mathcal{F}_0$. Let $\tilde{\mathcal{C}}_0$ denote the strict transform of $\mathcal{C}_0$, where $\mathcal{C}_0$ is a cycle whose class is the generator of $\mathbb{Q}\mathcal{F}_0 \cap NS(K_{\mathcal{A}})$. Its special fibre $\tilde{\mathcal{C}}_{0,p}$ is the generator of $\mathbb{Q}\mathcal{F}_{0,p} \cap NS(\tilde{K}_{\mathcal{A}_p})$ since the co-kernel of the specialization map has no torsion.

Since the cycle $\mathcal{D}_p$ in the Néron–Severi group of $\mathcal{A}_p$ is not in the span of the polarization, it has a non-zero Humbert invariant Δ. Hence from Theorem 3.1 there is a rational curve Q^{Δ} on $K\mathbb{P}_{\mathcal{A}_p}$ corresponding to this cycle. Let $Q_1^{\Delta}, Q_2^{\Delta}$ and $\tilde{Q}_1^{\Delta}, \tilde{Q}_2^{\Delta}$ be the rational curves lying above this curve in $K_{\mathcal{A}_p}$ and $\tilde{K}_{\mathcal{A}_p}$ respectively. $\tilde{Q}_i^{\Delta}$ is the strict transform of Q_i^{Δ} under the birational map ν. Let D_1^{Δ} and D_2^{Δ} denote the pullbacks of Q_1^{Δ} and Q_2^{Δ} to $\mathcal{A}_p$.

Lemma 6.1. *The cycle* $\tilde{Q}_1^{\Delta} + \tilde{Q}_2^{\Delta} \in |\tilde{\mathcal{C}}_{0,p}^N|$ *for some* $N \in \mathbb{Z}$.

Proof. Since $\mathrm{Pic}(\mathbb{P}^2) \simeq \mathbb{Z}$, the class of Q^{Δ} is a multiple of the hyperplane section. Its pullback $\pi^*(Q^{\Delta})$ is fixed by the involution ι and so its class lies in the part of $K_{\mathcal{A}_p}$ spanned by the hyperplane section, which is $\mathbb{Z}[\mathcal{C}_{0,p}]$ by the remarks in § 3.3.4. Hence the strict transform $\widetilde{\pi^*(Q^{\Delta})}$ lies in $\mathbb{Z}[\tilde{\mathcal{C}}_{0,p}]$. However, $\pi^{-1}(Q^{\Delta})) = Q_1^{\Delta} \cup Q_2^{\Delta}$. Hence its strict transform is $\tilde{Q}_1^{\Delta} \cup \tilde{Q}_2^{\Delta}$ and one has

$$\tilde{Q}_1^{\Delta} + \tilde{Q}_2^{\Delta} \in |\tilde{\mathcal{C}}_{0,p}^N|$$

for some $N \in \mathbb{Z}$. □

We would like to apply Theorem 4.1 to this situation. For that we need the curve $\tilde{Q}_1^{\Delta} + \tilde{Q}_2^{\Delta}$ to be connected. To verify that this is the case, we analyse the points of intersection of Q_1^{Δ} and Q_2^{Δ} under the blow-up. The curves Q_1^{Δ} and Q_2^{Δ} intersect at several points on $K_{\mathcal{A}_p}$ – in fact, one has the following lemma.

Lemma 6.2. *If $x \in L_i \cap Q_1^{\Delta}$ for some i then $x \in Q_1^{\Delta} \cap Q_2^{\Delta}$. Further:*

- *x lies in the image of the 2-torsion if and only if $x \in L_i \cap L_j \cap Q_1^{\Delta}(\cap Q_2^{\Delta})$ for some i and j;*
- *the other points of $L_i \cap Q_j^{\Delta}$ which are not the images of 2-torsion points appear with even multiplicity.*

Proof. If x lies on $L_i \cap Q_1^{\Delta}$, then, as ι fixes L_i, $\iota(x) = x$. Hence x lies on $\iota(Q_1^{\Delta}) = Q_2^{\Delta}$. For the second and third statements we refer to [BW03, Lemma 6.1]. □

From [BW03, Theorems 7.1–7.5] we know there are k, $k-1$ or 3 points on Q^{Δ} which are the images of the 2-torsion points of $\mathcal{A}_p$. These points lie on $L_i \cap L_j \cap Q^{\Delta}$, hence their pre-images under π lie on $Q_1^{\Delta} \cap Q_2^{\Delta}$. Under the blow-up to $\tilde{K}_{\mathcal{A}_p}$ they need no longer be points of intersection of $\tilde{Q}_1^{\Delta} \cap \tilde{Q}_2^{\Delta}$. The remaining points on $L_i \cap Q^{\Delta}$ are points of even multiplicity. Hence we need to show that either there is a point of intersection which does not lie in the image of the 2-torsion points or that a point which lies in the image of the 2-torsion points continues to lie on the intersection after the blow-up. We abuse notation and call a point of intersection of Q_1^{Δ} and Q_2^{Δ} on $K_{\mathcal{A}_p}$, or its image in $K\mathbb{P}^2_{\mathcal{A}_p}$, a *2-torsion point* if it lies in the image of the 2-torsion points of $\mathcal{A}_p$. Similarly, we call such a point a *non-2-torsion point* if it does not lie in the image of the 2-torsion points of $\mathcal{A}_p$.

Lemma 6.3. *If $d > 2$ then $\tilde{Q}_1^{\Delta} \cup \tilde{Q}_2^{\Delta}$ has at least two points of intersection and, in particular, is connected.*

Proof. From [BW03], we know that the degree of Q^{Δ} is either $2d$, in cases I and III, or $2d+1$, in cases II and IV of (1) in § 3.3.3. Further, there are at most five 2-torsion points on a line L_i – as these correspond to points of intersection with the other five lines L_j.

Case 1. The 2-torsion points on $Q^{\Delta} \cap L_i$ appear with multiplicity 1. Since the non-2-torsion points on L_i appear with even multiplicity, there are at least $2d-4$ non-2-torsion points, since if the degree is $2d$ there can be at most four 2-torsion points, and if the degree is $2d+1$ there can be five. Hence if $2d-4 > 1$, there will be at least one such point on every line L_i. Such points do not get separated in the strict transform to $\tilde{Q}_1^{\Delta} \cup \tilde{Q}_2^{\Delta}$ hence this curve has at least two points of intersection – in fact at least six.

Case 2. The 2-torsion points have multiplicity greater than 1. If at least two 2-torsion points on $(\bigcup_i L_i) \cap Q^{\Delta}$ have high multiplicity, then, under the blow-up to $\tilde{Q}_1^{\Delta} \cup \tilde{Q}_2^{\Delta}$, they will remain singular and points of intersection of $\tilde{Q}_1^{\Delta} \cup \tilde{Q}_2^{\Delta}$ – see Case 2 of Lemma 6.5 for details. If there is only one, then Case 1 above shows there is at least one non-2-torsion point on $\tilde{Q}_1^{\Delta} \cup \tilde{Q}_2^{\Delta}$.

Since there are at least two points of intersection, the curve $\tilde{Q}_1^{\Delta} \cup \tilde{Q}_2^{\Delta}$ is connected. □

https://doi.org/10.1112/S0010437X13007690 Published online by Cambridge University Press

Remark 6.4. The restriction that $d > 2$ is not a very serious one, since if $\mathcal{A}_p$ contains a line bundle of invariant Δ then $\mathcal{A}_p$ contains a line bundle of invariant $m^2\Delta$ for any $m \in \mathbb{Z}$. For $m^2\Delta$ the corresponding d will be larger. Further, even if $d = 1$ or $d = 2$, for most values of k there is no problem.

Hence $\tilde{Q}_1^\Delta \cup \tilde{Q}_2^\Delta$ along with $(\tilde{K}_\mathcal{A}, \tilde{\mathcal{C}}_0^N)$ satisfies the hypothesis of Theorem 4.1. Hence there exists an irreducible rational curve $\tilde{\mathcal{Q}}^\Delta$ with $\tilde{\mathcal{Q}}^\Delta \in |\tilde{\mathcal{C}}_0^N|$ defined on some finite extension of the Witt vectors of $W(k)$ which splits into a sum of these two rational curves mod p:

$$\tilde{\mathcal{Q}}_p^\Delta = \tilde{Q}_1^\Delta \cup \tilde{Q}_2^\Delta.$$

6.2 Deformation of singularities

A *node* is a singularity which is locally isomorphic to the singularity at the origin of the plane curve $y^2 = x^{2k}$. The number $2k$ is called the *order* of the node. A node of order 2 is called an *ordinary node*. Another way of describing it is as a point where two smooth branches of a curve meet. We will use the word 'node' to refer to a node of any order and will use the word 'ordinary node' for a node of order 2. A nodal singularity can be can be resolved by a sequence of blow-ups – each reducing the order by 2 and hence at the penultimate stage one has an ordinary node and in the normalization there will be two points lying over the node.

We would like to use the generic fibre $\tilde{Q}_\eta^\Delta$ of the curve $\tilde{\mathcal{Q}}^\Delta$ to construct a higher Chow cycle as described at the beginning of § 6. Here we will show that the curve $\tilde{Q}_\eta^\Delta$ has a node. For this we have to understand the singularities coming from the intersection of $\tilde{Q}_1^\Delta$ and $\tilde{Q}_2^\Delta$ and their deformations.

Lemma 6.5. *Let P be a singularity of the curve $\tilde{Q}_1^\Delta \cup \tilde{Q}_2^\Delta$. Assume $\nu(P)$ lies on $L_i \cap Q_1^\Delta \cap Q_2^\Delta$ for some i. Then the singularity at P is étale locally isomorphic to a singularity at the origin of a plane curve of the type $y^2 = x^{2k}$ for some positive integer k – that is, it is a higher-order node.*

Proof. From Lemma 6.2, the singularity $\nu(P)$ is either the image of a 2-torsion point or is a double intersection of a line L_i with Q_j^Δ. Let $P' = \pi(\nu(P))$, the image of P in $K\mathbb{P}^2_{\mathcal{A}_p}$ which lies on the curve Q^Δ. Since Q^Δ is a rational curve, after going to a completion we may assume that it is locally isomorphic to $y = F(x)$ near P', where $F(x)$ is a power series.

Case 1. If P' is not the image of a 2-torsion point, then from Lemma 6.2, the multiplicity of $L_i \cap Q^\Delta$ is *even*. Hence near P', Q^Δ is locally isomorphic to $y = x^r G(x)$ with $r = 2k$ even and $G(0) \neq 0$ and L_i is locally isomorphic to $y = 0$. Replacing y by $y/G(x)$, we may assume that it is locally isomorphic to

$$y = x^r.$$

The map $K_{\mathcal{A}_p} \xrightarrow{\pi} K\mathbb{P}^2_{\mathcal{A}_p}$ in a neighbourhood of $\nu(P)$ is a double cover ramified at the line L_i so it is locally isomorphic to $z^2 = y$ near $\nu(P)$, with the double cover given by $(x, y, z) \longrightarrow (x, y)$.

The pre-image of the curve Q^Δ has two components Q_1^Δ and Q_2^Δ which are birationally isomorphic to Q^Δ and are conjugate via the map $(x, y, z) \to (x, y, -z)$. So locally, near the ramified point $\nu(P)$ on $K_{\mathcal{A}_p}$, the curve is given by the equations

$$z^2 = y \quad \text{and} \quad y = x^{2k}$$

so $z^2 = x^{2k}$. Since ν is a local isomorphism near P the curve is locally isomorphic to $z^2 = x^{2k}$ near P as well. Hence it is a node of order $2k$.

https://doi.org/10.1112/S0010437X13007690 Published online by Cambridge University Press

Case 2. If P' is the image of a 2-torsion point, then, as before, on $K\mathbb{P}^2_{\mathcal{A}_p}$ the curve is locally isomorphic to

$$y = x^r$$

for some natural number r near P'. Here the two lines L_1 and L_2 are given by the axial lines $x = 0$ and $y = 0$. The double cover is ramified at both the lines $x = 0$ and $y = 0$ so is locally isomorphic to the surface $z^2 = xy$ with the map given by $(x, y, z) \longrightarrow (x, y)$.

However, unlike the earlier case, the point $\nu(P)$ is singular. The map $\nu : \tilde{K}_{\mathcal{A}_p} \longrightarrow K_{\mathcal{A}_p}$ is a blow-up in a neighbourhood of $\nu(P)$. Hence we have to see what happens to the singularity under this blow-up.

Let $[u, v, w]$ be the coordinates on the exceptional fibre $\mathbb{P}^2$ of the blow-up of $\mathbb{A}^3$ at the origin. The blow-up of $z^2 - xy$ at $(0, 0, 0)$ is given by the equations

$$\begin{aligned} xv &= yu \qquad & xw &= zu \\ yw &= zv \qquad & z^2 &= xy. \end{aligned}$$

The blow-up of $\mathbb{A}^3$ at the origin is covered by three affine open sets $\mathbb{A}^3 \times \mathbb{A}^2_u$, $\mathbb{A}^3 \times \mathbb{A}^2_v$ and $\mathbb{A}^3 \times \mathbb{A}^2_w$, where $\mathbb{A}^2_*$ is the plane given by $* = 1$ in the exceptional fibre $\mathbb{P}^2$. To understand the strict transform of the curve $y = x^r$ in the blow-up of the surface $z^2 = xy$, we first restrict to $\mathbb{A}^3 \times \mathbb{A}^2_u$. This gives us the equations

$$\begin{aligned} y &= xv \qquad & z &= xw \\ z^2 &= xy \qquad & y &= x^r \end{aligned}$$

which implies

$$z^2 = x^2w^2 = xy = x^{r+1}$$

and so

$$x^2w^2 = x^{r+1}$$

so in a neighbourhood of the point $(0, 0, 0, [1, 0, 0])$ in the blow-up of $\mathbb{A}^3$ the curve is locally isomorphic to

$$w^2 = x^{r-1}.$$

Since we know the curve $\nu^*(\pi^*(Q^\Delta)) = \tilde{Q}_1^\Delta \cup \tilde{Q}_2^\Delta$ has two components this forces $r - 1$ to be *even*. On $\mathbb{A}^3 \times \mathbb{A}^2_v$ and $\mathbb{A}^3 \times \mathbb{A}^2_w$ there are no singularities. Since we have assumed that P is a singular point we have $r > 1$ and it is a nodal singularity of order $r - 1$.

From Artin [Art69, Corollary 2.6], the original singularity and this are étale locally isomorphic. □

A local deformation of a singularity is a family $\mathcal{X}/T$ where T is the spectrum of a complete local ring such that the special fibre $\mathcal{X}_0$ is a variety with the singularity. It is said to be *miniversal* if further:

- (versal) for any other deformation $\mathcal{X}'/S$, with S the spectrum of a complete local ring, there is a morphism $\Phi : S \to T$ such that $\mathcal{X}'$ and $\mathcal{X} \times_T S$ become isomorphic after completing along the closed fibre over zero;
- (mini) although Φ may not be unique, the induced map on Zariski tangent spaces of S and T is uniquely determined.

The miniversal deformation of a singularity is étale local, so for the purposes of computing this we may assume that the singularity at P is of the form $y^2 = x^{2k}$. Using [Har10, Theorem 14.1] and [Elk74, Theoreme 8], one can see that the miniversal deformation space of the singularity

https://doi.org/10.1112/S0010437X13007690 Published online by Cambridge University Press

over $\mathrm{Spec}(W(k))$ is isomorphic to

$$\mathrm{Spec}(W(k)[x, y, a_i, \ldots, a_r]/(y^2 - F(x)))$$

where

$$F(x) = \prod_{i=1}^{r} (x - a_i)^{n_i}$$

and $\sum_{i=1}^{r} n_i a_i = 0$ and $\sum_{i=1}^{r} n_i = 2k$. So $F(x)$ is a monic polynomial of degree $2k$ with no term of degree $2k-1$. This has singularities at a_i if $n_i > 1$.

For a curve $X = \bigcup_{i=1}^{s_X} X^i$, where X^i are the irreducible components, the arithmetic genus is

$$p_a(X) = \sum_i p_g(X^i) - (s_X - 1) + \sum_P \delta_P$$

where $p_g(X^i)$ is the geometric genus (genus of the normalization) and

$$\delta_P = \sum_Q \frac{m_Q(m_Q - 1)}{2}$$

where the sum is over all the infinitely near points Q of P including P (that is, points lying over P under a series of blow-ups) with multiplicity m_Q. Note that $\delta_P \neq 0$ if and only if P is a singular point. We call δ_P the contribution from the singularity P.

Lemma 6.6. *Let $\mathcal{X}/T$ be a flat family of projective, rational curves over the spectrum of a complete discrete valuation ring. Suppose:*

- *the closed fibre $\mathcal{X}_0$ has two rational components and has at least two nodes of some order;*
- *the generic fibre X is irreducible.*

Then there is a node R in the special fibre such that

$$\sum_t \delta_{R^t} = \delta_R$$

where R^t are the singular points of $\mathcal{X}$ specializing to R. In particular, this implies that the generic fibre too has a nodal singularity.

Proof. For a singularity of multiplicity 2 at the origin of a plane curve of the type $y^2 = x^r$ one has

$$\delta_{(0,0)} = \begin{cases} r/2 & \text{if } r \text{ is even} \\ (r-1)/2 & \text{if } r \text{ is odd.} \end{cases}$$

This is because the blow-up at the origin of $y^2 = x^r$ has a singularity of type $y^2 = x^{r-2}$, so the order is lowered by 2. The contribution from this blow-up – and every subsequent blow-up – is $2(2-1)/2 = 1$. One can repeat this process till the point is non-singular – which happens when the penultimate stage is an ordinary node or a cusp. For an ordinary node $y^2 = x^2$ and for a cusp $y^2 = x^3$ one has $\delta_{0,0} = 1$ and from this the result follows.

In our case, $\mathcal{X}_0 = \mathcal{X}_0^1 \cup \mathcal{X}_0^2$, so $s_{\mathcal{X}_0} = 2$. Since the irreducible components are rational, if the singularities of $\mathcal{X}_0$ are at points P one has

$$p_a(\mathcal{X}_0) = 0 - 1 + \sum_{P \in \mathcal{X}_0} \delta_P.$$

$\mathcal{X}$ is a flat family so the arithmetic genera of the fibres are constant – that is, $p_a(\mathcal{X}_0) = p_a(X)$. From this along with $p_g(\mathcal{X}_0) = p_g(X) = 0$ and the fact that the generic fibre is irreducible, so $s_X = 1$, one has

$$-1 + \sum_{P \in \mathcal{X}_0} \delta_P = \sum_{R \in X} \delta_R. \tag{2}$$

Let $\{P^j\}_{j \in J}$ denote the points on X whose closure in $\mathcal{X}$ intersects the special fibre $\mathcal{X}_0$ at a singular point P. These points are said to *specialize* to P. Then one always has

$$\sum_j \delta_{P^j} \leqslant \delta_P.$$

Now suppose $\sum_j \delta_{P^j} < \delta_P$ for some singularity P. From (2) one has: $\sum_j \delta_{P^j} = \delta_P - 1$; and $\sum_t \delta_{R^t} = \delta_R$ for all other singularities R.

In particular, since we have assumed there are at least two nodes, there is a node R such that $\sum_t \delta_{R^t} = \delta_R$. We claim that all the singularities R^t which specialize to R are nodes.

Since R is a node, it is of the form $y^2 = x^{2k(R)}$ with $\delta_R = k(R)$. Let $(a_t, 0)$ with $n_t > 1$ be the singular points on the generic fibre which specialize to R, where locally the curve looks like

$$y^2 = \prod_t (x - a_t)^{n_t}$$

with $\sum_t n_t = 2k(R)$. If n_t is odd and greater than 1, then from the remarks above the singularity $(a_t, 0)$ will have $\delta_{(a_t,0)} = (n_t - 1)/2$. Further, as $\sum_t n_t = 2k(R)$ is even, there will have to be at least two points with odd exponent. So in that case we have

$$\sum_t \delta_{(a_t,0)} = k(R) - 1.$$

This contradicts the assumption that

$$\sum_t \delta_{R^t} = \delta_R = k(R).$$

Hence for this node, all the points R^t have to be nodes, so we have at least one node on the generic fibre. □

6.3 The higher Chow cycle

We can apply the results of the previous section to $\tilde{Q}_1^\Delta \cup \tilde{Q}_2^\Delta$ and so we may assume that there is an irreducible curve $\tilde{\mathcal{Q}} = \tilde{\mathcal{Q}}^\Delta$ defined over some extension of the Witt vectors of the residue field such that the special fibre is $\tilde{Q}_1^\Delta \cup \tilde{Q}_2^\Delta$. Further, there is a node R on the special fibre such that

$$\sum_t \delta_{R^j} = \delta_R$$

where R^j denote the points on the generic fibre specializing to R and we can assume R lies on the intersection of the two components as $\tilde{Q}_i^\Delta$ are non-singular outside the image of 0. There is another singular point $P \neq R$ such that

$$\sum_t \delta_{P^t} = \delta_P - 1.$$

Lemma 6.7. *Let R be a node in the special fibre $\tilde{Q}_1^\Delta \cup \tilde{Q}_2^\Delta$ such that*

$$\sum_j \delta_{R^j} = \delta_R. \tag{3}$$

https://doi.org/10.1112/S0010437X13007690 Published online by Cambridge University Press

Let R' be a point lying over R under a blow-up of a section $\bar{R}^j$ of $\tilde{\mathcal{Q}}$ specializing to R and R^j. Let R'^t denote the points in the generic fibre specializing to R'. Then

$$\sum_t \delta_{R'^t} = \delta_{R'}.$$

Proof. From Lemma 6.6, we know that the R^j are nodes. Consider a single blow-up of the closure $\bar{R}^1$ of a point R^1 in the model. Since R^1 is a node there is either a node of order reduced by 1, or two non-singular points lying over R^1. Suppose it is a lower-order node R'^1. Recall that

$$\delta_{R^1} = \sum_Q \frac{m_Q(m_Q-1)}{2}$$

where Q runs through all the infinitely near points of R^1 including R^1. Hence in this case,

$$\delta_{R^1} = \delta_{R'^1} + 2(2-1)/2 = 1 + \delta_{R'^1}.$$

The same holds in the special fibre – under the blow-up the point R' lying over R is a node of order reduced by 1. Hence the formula

$$\sum_j \delta_{R'^j} = \delta_{R'}$$

continues to hold as the other points R^j with $j \neq 1$ are not affected by the blow-up.

This formula also holds when there are two smooth points over R^1 as in that case R^1 is an ordinary node so $\delta_{R^1} = 1$ and δ of a smooth point is 0. □

Corollary 6.8. *If R^N is a point on the generic fibre of the normalization lying over a node R^1 which specializes to a node R in the special fibre satisfying equation (3) above, then R^N specializes to a non-singular point on the special fibre of the normalization.*

Proof. Iterating the process above, we see that as long as there is a singular point R' lying over R – which has to be a node – there are nodal points R'^j specializing to R'. In the normalization, there are no singular points in the generic fibre, hence a point on the generic fibre specializing to a point lying over R can no longer be singular. □

Let $\Psi : \mathcal{N}^\Delta \to \tilde{\mathcal{Q}}^\Delta$ denote the normalization of $\tilde{\mathcal{Q}}$ and $\Psi : N_\eta \to \tilde{Q}^\Delta_\eta$ the generic fibre of $\mathcal{N}^\Delta$. Let $R^{\tilde{Q}^\Delta}$ be a node of $\tilde{Q}^\Delta_\eta$ specializing to R. Under the normalization, there are two non-singular points R^N_1 and R^N_2 mapping to $R^{\tilde{Q}^\Delta}$. Since $\tilde{Q}^\Delta_\eta$ is rational, N_η is a smooth rational curve. Hence there is a function f_R on N_η such that

$$\operatorname{div}(f_R) = R^N_1 - R^N_2.$$

The points R^N_1 and R^N_2 are defined over some finite extension L_p of K_p. We have that $\Psi_*((N_\eta, f_R))$ is an element of $CH^2(\tilde{K}_{A_{\mathrm{L}_p}}, 1)$ as, by construction,

$$\operatorname{div}(\Psi_*(f_R)) = \Psi(R^N_1) - \Psi(R^N_2) = R^{\tilde{Q}^\Delta} - R^{\tilde{Q}^\Delta} = 0.$$

Let $\Xi^\Delta_{\tilde{K}_{A_{\mathrm{L}_p}}}$ be the element $\Psi_*((N_\eta, f_R))$. One can push forward $\Xi^\Delta_{\tilde{K}_{A_{\mathrm{L}_p}}}$ under the map ν to get an element

$$\Xi^\Delta_{K_{A_{\mathrm{L}_p}}} = \nu_*(\Xi_{\tilde{K}_{A_{\mathrm{L}_p}}})$$

of $CH^2(K_{A_{\mathrm{L}_p}}, 1)$, pull it back under ϕ and apply the norm to get an element

$$\Xi^\Delta_A = N_{\mathrm{L}_p/\mathrm{K}_p}(\phi^*(\Xi_{K_A})) \in CH^2(A, 1).$$

https://doi.org/10.1112/S0010437X13007690 Published online by Cambridge University Press

6.4 Indecomposability and the boundary

We would like to show that the boundary map in the localization sequence is surjective. In the preceding section we have shown that, given a cycle in $\mathrm{Pic}(\mathcal{A}_p)$ of Humbert invariant Δ which is not the restriction of the closure of an element of $\mathrm{Pic}(A)$, there is an an element Ξ_A^{Δ} in $CH^2(A,1)$. In this section we compute the boundary of this element.

Let D_1^{Δ} and D_2^{Δ} be the pullbacks of the cycles Q_1^{Δ} and Q_2^{Δ} to $\mathcal{A}_p$.

THEOREM 6.9. *The element Ξ_A^{Δ} is an indecomposable element of $CH^2(A,1)$. Further, the boundary under the map ∂ is the extra cycle in the special fibre $\mathcal{A}_p$,*

$$\partial(\Xi_A^{\Delta}) = mD_1^{\Delta},$$

for some $m \neq 0$, up to the boundary of an element of $CH^2_{\mathrm{dec}}(A,1)$.

Proof. Recall that, from Remark 5.2, to show indecomposability it suffices to show that the boundary of Ξ_A^{Δ} is a non-zero multiple of the cycle D_1^{Δ}, as D_1^{Δ} does not deform to a cycle on the generic fibre, hence cannot be the boundary of a decomposable element. We observe that we can work with the cycle (N_η^{Δ}, f_R) since computing the boundary commutes with proper pushforward and so it suffices to show that this boundary is a non-zero multiple of one of the components of its special fibre.

Consider the curve $\mathcal{N}^{\Delta}$. Since $\mathcal{N}^{\Delta}$ is a non-singular rational curve, the special fibre is either a non-singular curve or the union of two non-singular rational curves meeting at a point. Since $\mathcal{N}^{\Delta}$ is the normalization of $\tilde{\mathcal{Q}}^{\Delta}$ and we know the special fibre of $\tilde{\mathcal{Q}}^{\Delta}$ consists of two curves, the special fibre of $\mathcal{N}^{\Delta}$ has to consist of two non-singular rational curves N_1^{Δ} and N_2^{Δ} meeting transversally at a point,

$$\mathcal{N}_p^{\Delta} = N_1^{\Delta} \cup N_2^{\Delta}.$$

Recall that we have an involution ι on $\tilde{Q}_1^{\Delta} \cup \tilde{Q}_2^{\Delta}$ induced by the double cover. A local calculation shows that that ι extends to the normalization $N_1^{\Delta} \cup N_2^{\Delta}$ and one has

$$\iota(N_1^{\Delta}) = N_2^{\Delta}.$$

LEMMA 6.10. *If R_1 and R_2 are two points of $\mathcal{N}_p^{\Delta}$ lying over a node R of $\tilde{Q}_1^{\Delta} \cup \tilde{Q}_2^{\Delta}$, then*

$$\iota(R_1) = R_2.$$

and vice versa.

Proof. The node R is locally isomorphic to the singularity at the origin given by $y^2 = x^{2k}$. The involution ι acts by $\iota(x,y) = (x,-y)$. In one affine part the equations defining the blow-up are $y = ux$ and $y^2 = x^{2k}$ which is equivalent to $u^2 = x^{2k-2}$ and $x = 0$, with the strict transform of $y^2 = x^{2k}$ given by $u^2 = x^{2k-2}$. This is still singular if $2k-2 > 0$. The involution on the strict transform is given by $\iota((x,u)) = (x,-u)$. In the normalization, the equations are $u^2 = 1$ and $x = 0$. The involution acts the same way, hence it takes the point $(0,1)$ to $(0,-1)$ and vice versa. In our case, the points R_1 and R_2 correspond to the points $(0,1)$ and $(0,-1)$ and hence the involution interchanges them. □

LEMMA 6.11. *R_1^N and R_2^N specialize to distinct points R_1 and R_2 which lie on different components of the special fibre $N_1^{\Delta} \cup N_2^{\Delta}$.*

Proof. Suppose not. Assume both R_1 and R_2 lie on N_1^{Δ}. Then since $\Psi(R_1) = \Psi(R_2) = R$, one has $\iota(R_1) = R_2$. Since $\iota(N_1^{\Delta}) = N_2^{\Delta}$, the point R_2 lies on $N_1^{\Delta} \cap N_2^{\Delta}$. So R_2 is a singular point. This contradicts Corollary 6.8, as R is a node satisfying the conditions of Lemma 6.7. □

Hence we can assume R_1 lies on N_1^{Δ} and R_2 lies on N_2^{Δ}. We now compute $\partial((N_\eta^{\Delta}, f_R))$. From the definition of ∂,

$$\partial((N_\eta^{\Delta}, f_R)) = \operatorname{div}(\bar{f}_R) = \mathcal{H} + aN_1^{\Delta} + bN_2^{\Delta}$$

for some integers a and b, where $\mathcal{H} = \overline{\operatorname{div}(f_R)}$ is the closure of the horizontal divisor $R_1^N - R_2^N$. A decomposable element of the form (N_η^{Δ}, p^k) has boundary

$$\partial((N_\eta^{\Delta}, p^k)) = k(N_1^{\Delta} + N_2^{\Delta}).$$

Hence by adding such an element with $k = -b$ to (N_η^{Δ}, f_R) we may assume that $b = 0$ and the boundary is of the form

$$\partial((N_\eta^{\Delta}, f_R)) = \mathcal{H} + aN_1^{\Delta}.$$

We now show $a \neq 0$. From the intersection theory of arithmetic surfaces – described in Lang [Lan88, ch. III], for instance – we have that

$$\deg(\operatorname{div}(\bar{f}_R|_D)) = (\operatorname{div}(\bar{f}_R).D) = 0$$

for *all* divisors D supported in the special fibre. Applying this with $D = N_2^{\Delta}$, we have

$$(\mathcal{H}.N_2^{\Delta}) + a(N_1^{\Delta}.N_2^{\Delta}) = 0.$$

From Lemma 6.11, $\mathcal{H} \cap N_2^{\Delta} = -R_2$ and so

$$(\mathcal{H}.N_2^{\Delta}) = -1.$$

From the fact that the components of the special fibre meet at precisely one point $\tilde{P}$ with multiplicity 1, we have

$$(N_1^{\Delta}.N_2^{\Delta}) = 1.$$

Hence $-1 + a = 0$ so $a = 1$. So we have

$$\partial((N_\eta^{\Delta}, f_R)) = \mathcal{H} + N_1^{\Delta}.$$

Under the pushforward, $\Psi_*(\mathcal{H}) = 0$ and $\Psi_*(N_1^{\Delta}) = Q_1^{\Delta}$, so

$$\partial(\Xi^{\Delta}_{\tilde{K}_{A_{\mathrm{L}_p}}}) = Q_1^{\Delta}.$$

Combining this with the pullback to A and the norm map – which could introduce a non-zero scaling factor – we have that

$$\partial(\Xi_A^{\Delta}) = kD_1^{\Delta}$$

for some $k \neq 0$. All the calculations are only up to the boundary of a decomposable element. □

Returning to our original situation, from the work of Birkenhake and Wilhelm [BW03] we know that $D_1^{\Delta} \in |\mathcal{L}_0^a \otimes \mathcal{L}_{\Delta}^b|$ for some $b \neq 0$ while $\mathcal{D}_p \in |\mathcal{L}_{\Delta}|$ so by modifying Ξ_A^{Δ} by a suitable decomposable element with can construct an element $\Xi_{A,\mathcal{D}_p}$ such that

$$\partial(\Xi_{A,\mathcal{D}_p}) = m\mathcal{D}_p$$

for some $m \neq 0$.

7. The Hodge-$\mathcal{D}$-conjecture for Abelian surfaces

The Hodge-$\mathcal{D}$-conjecture asserts that the boundary map in the localization sequence from the higher Chow group of the generic fibre to the Chow group of the special fibre is surjective. Hence, given a cycle in the special fibre, we should be able to find a higher Chow cycle which bounds it.

If the cycle in the special fibre is the restriction of a cycle in the generic fibre, then it is easily seen to be the boundary of a decomposable element of the higher Chow group.

In the previous section we have shown that, given a cycle in the special fibre which is not the restriction of a generic cycle, under certain circumstances we have constructed a higher Chow cycle which bounds it. It remains to combine this result with an analysis of the possible reductions of Abelian surfaces to get our final result.

Theorem 7.1. *Let A be an Abelian surface over a p-adic local field K_p. Let $\mathcal{A}$ be a model over the ring of integers and $\mathcal{A}_p$ the special fibre. Assume A has good, non-supersingular, reduction at p. Then the map*

$$CH^2(A,1)\otimes\mathbb{Q} \xrightarrow{\partial} CH^1(\mathcal{A}_p)\otimes\mathbb{Q}$$

is surjective.

Proof. The results of the previous sections show the following. Suppose A, $\mathcal{A}$ and $\mathcal{A}_p$ are as before and suppose $\mathcal{D}_p$ is a cycle representing an element of $NS(\mathcal{A}_p)$ which is not the restriction of cycle in $NS(A)$. Then, from Theorem 6.9 there is a higher Chow cycle $\Xi_{A,\mathcal{D}_p}$ such that

$$\partial(\Xi_{A,\mathcal{D}_p}) = m\mathcal{D}_p$$

for some $m \neq 0$.

Since we are looking at the rational Chow group, it suffices to consider the different cases up to isogeny. We observe that there are three possible cases – depending on where A or $\mathcal{A}_p$ is simple. If A is simple and $\mathcal{A}_p$ is simple non-supersingular, for each element $\mathcal{D}_p$ of $NS(\mathcal{A}_p)$ which is not the restriction of a generic cycle, we have a higher Chow cycle $\Xi_{A,\mathcal{D}_p}$ from Theorem 6.9 which bounds $\mathcal{D}_p$.

If A simple and $\mathcal{A}_p \simeq \mathcal{E}_{1,p}\times\mathcal{E}_{2,p}$, case, the Humbert invariant of the extra cycle $\mathcal{D}_p$ is n^2 for some n. If $n=1$ we can use the element constructed by Collino [Col97]. If $n>1$ the argument above applies.

The case $A\simeq E_1\times E_2$ was covered by Spiess [Spi99]. □

Remark 7.2. We needed to assume non-super-singularity in order to use the Bogomolov–Hassett–Tschinkel theorem. In the case of super-singular reduction, the special fibre is unirational and that allows the possibility of the cycle being deformed within the special fibre.

Remark 7.3. This theorem can be extended to work in the case where A has semi-stable reduction. A special case was studied in [Sre08].

8. Some applications

8.1 Torsion in co-dimension two

One immediate consequence of the construction of these higher Chow cycles is the following. Let X be a smooth projective variety over a local field K and assume that it has a semi-stable model over the ring of integers $\mathcal{O}_K$. Let Σ_X be the group

$$\Sigma_X := \mathrm{Ker}(CH^2(\mathcal{X}) \longrightarrow CH^2(X)).$$

Then for X an Abelian surface A with non super-singular reduction over a p-adic field, Σ_A is torsion. This is because the long exact localization sequence gives

$$\cdots \longrightarrow CH^2(A,1) \xrightarrow{\partial} CH^1(\mathcal{A}_p) \longrightarrow CH^2(\mathcal{A}) \longrightarrow CH^2(A) \longrightarrow 0$$

https://doi.org/10.1112/S0010437X13007690 Published online by Cambridge University Press

hence the group Σ_A is the same as the co-kernel of the image of ∂. As we have shown that the $\partial \otimes \mathbb{Q}$ is surjective, this implies $\Sigma_A \otimes \mathbb{Q} = 0$, hence Σ_A is torsion.

There are a lot of consequences of the finiteness of Σ_A: they are described in the paper of Spiess [Spi99]. For example, one has the following corollary.

Corollary 8.1. *Let A be an Abelian surface over K with good, ordinary, reduction. Let $\mathcal{A}_p$ denote its reduction mod p. Then:*

- *$CH_0(A)$(prime to p) $\simeq CH_0(\mathcal{A}_p)$(prime to p) and in particular it is finite;*
- *the group Σ_A is a p-group;*
- *for every integer $n \neq 0$ which is prime to p the cycle map*

$$cl_n : CH^2(A)/n \longrightarrow H^4(A, \mathbb{Z}/n(2))$$

is injective.

Proof. [Spi99, § 4] This is a consequence of the finiteness of Σ_A and the fact that the Tate conjecture is known for Abelian surfaces over a finite field. □

8.2 Relations between CM cycles

In the geometric settings the work of Mori and Mukai [MM83] can be used in a similar manner to deform sums of rational curves on a particular $K3$ surface to rational curves on the generic fibre of the moduli of $K3$ surfaces. An argument similar to the one above can then be used to construct indecomposable higher Chow cycles in the generic Kummer surface of an Abelian surface over the Siegel modular threefold and hence can be lifted to the generic Abelian surface.

Similar to the case studied here, these cycles degenerate over Humbert surfaces. This is a generalization of the work of Collino [Col97] where he constructs a higher Chow cycle which degenerates over the moduli of products of elliptic curves – namely the Humbert surface of invariant 1. In [Sre01], Collino's elements were used to construct relations between CM cycles – certain codimension-two cycles in the CM fibres over modular and Shimura curves. These new elements can be used to get more relations. This in more detail will be the subject of another paper.

8.3 $K3$ surfaces

Since the key point of our construction was done on $K3$ surfaces and the result we used holds over fairly general $K3$ surfaces, we expect that out construction could be used to prove the non-Archimedean Hodge-$\mathcal{D}$-conjecture for such $K3$ surfaces.

Acknowledgements

I would like to thank Spencer Bloch and Chad Schoen for some initial discussions which led to this problem; Brendan Hassett for some very useful discussions on his results; Najmuddin Fakhruddin for his invaluable help, suggestions and the proof of Theorem 4.1; and Patrick Brosnan, Owen Patashnick, G.V. Ravindra and Jishnu Biswas for some useful conversations. I would also like to thank the referees for their comments and suggestions. I would like to thank the Indian Statistical Institute, Bangalore, CRM Montreal, Université Paris-Sud (Orsay) and Institut Henri Poincaré for their support and hospitality while this work was done. Finally, during the course of writing this paper two people who influenced me quite a bit – Prof. V. Radhakrishnan, whose remarkable approach to life changed mine, and Prof. D. Lal, who encouraged me to 'prove more lemmas' – have passed on. I would like to dedicate this work to their memory.

References

Art69 M. Artin, *Algebraic approximation of structures over complete local rings*, Publ. Math. Inst. Hautes Études Sci. (1969), 23–58.

AS07 M. Asakura and S. Saito, *Surfaces over a p-adic field with infinite torsion in the Chow group of 0-cycles*, Algebra Number Theory **1** (2007), 163–181.

BHT11 F. Bogomolov, B. Hassett and Y. Tschinkel, *Constructing rational curves on K3 surfaces*, Duke Math. J. **157** (2011), 535–550.

Blo86 S. Bloch, *Algebraic cycles and higher K-theory*, Adv. Math. **61** (1986), 267–304.

BW03 C. Birkenhake and H. Wilhelm, *Humbert surfaces and the Kummer plane*, Trans. Amer. Math. Soc. **355** (2003), 1819–1841 (electronic).

CL05 X. Chen and J. D. Lewis, *The Hodge-D-conjecture for K3 and abelian surfaces*, J. Algebraic Geom. **14** (2005), 213–240.

Col97 A. Collino, *Griffiths' infinitesimal invariant and higher K-theory on hyperelliptic Jacobians*, J. Algebraic Geom. **6** (1997), 393–415.

Con98 C. Consani, *Double complexes and Euler L-factors*, Compositio Math. **111** (1998), 323–358.

Elk74 R. Elkik, *Solution d'équations au-dessus d'anneaux henséliens*, in *Quelques problèmes de modules (Sém. Géom. Anal., École Norm. Sup., Paris, 1971–1972)*, Astérisque, vol. 16 (Soc. Math., France, Paris, 1974), 116–132.

Fla92 M. Flach, *A finiteness theorem for the symmetric square of an elliptic curve*, Invent. Math. **109** (1992), 307–327.

Har10 R. Hartshorne, *Deformation theory*, Graduate Texts in Mathematics, vol. 257 (Springer, New York, 2010).

Jak94 B. Jakob, *Poncelet 5-gons and abelian surfaces*, Manuscripta Math. **83** (1994), 183–198.

Jan88 U. Jannsen, *Deligne homology, Hodge-D-conjecture, and motives*, in *Beĭlinson's conjectures on special values of L-functions*, Perspectives in Mathematics, vol. 4 (Academic Press, Boston, 1988), 305–372.

Lan88 S. Lang, *Introduction to Arakelov theory* (Springer, New York, 1988).

Mil92 S. J. M. Mildenhall, *Cycles in a product of elliptic curves, and a group analogous to the class group*, Duke Math. J. **67** (1992), 387–406.

MM83 S. Mori and S. Mukai, *The uniruledness of the moduli space of curves of genus* 11, in *Algebraic geometry (Tokyo/Kyoto, 1982)*, Lecture Notes in Mathematics, vol. 1016 (Springer, Berlin, 1983), 334–353.

Mül97 S. J. Müller-Stach, *Constructing indecomposable motivic cohomology classes on algebraic surfaces*, J. Algebraic Geom. **6** (1997), 513–543.

Ram89 D. Ramakrishnan, *Regulators, algebraic cycles, and values of L-functions*, in *Algebraic K-theory and algebraic number theory (Honolulu, HI, 1987)*, Contemporary Mathematics, vol. 83 (American Mathematical Society, Providence, RI, 1989), 183–310.

Spi99 M. Spiess, *On indecomposable elements of K_1 of a product of elliptic curves*, K-Theory **17** (1999), 363–383.

Sre01 R. Sreekantan, *Relations among Heegner cycles on families of abelian surfaces*, Compositio Math. **127** (2001), 243–271.

Sre08 R. Sreekantan, *A non-Archimedean analogue of the Hodge-D-conjecture for products of elliptic curves*, J. Algebraic Geom. **17** (2008), 781–798.

Sre10 R. Sreekantan, *K_1 of products of Drinfeld modular curves and special values of L-functions*, Compositio Math. **146** (2010), 886–918.

Tat66 J. Tate, *Endomorphisms of abelian varieties over finite fields*, Invent. Math. **2** (1966), 134–144.

Ramesh Sreekantan rameshsreekantan@gmail.com
Indian Statistical Institute, Bangalore, India