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Dow's Principle and Q-Sets

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Abstract. A Q-set is a set of reals every subset of which is a relative G_δ. We investigate the combinatorics of Q-sets and discuss a question of Miller and Zhou on the size $\mathfrak{q}$ of the smallest set of reals which is not a Q-set. We show in particular that various natural lower bounds for $\mathfrak{q}$ are consistently strictly smaller than $\mathfrak{q}$.

1 Introduction and Statement of the Main Results

This work is devoted to combinatorial aspects of Q-sets. Such sets play a prominent role in general topology. In fact, the existence of an uncountable Q-set is equivalent to the existence of a separable normal non-metrizable Moore space ([He], see [Ta, Section II] for an overview).

The *Baire space* ω^ω is the set of functions from ω to ω with the product topology (where ω carries the discrete topology); similarly, the *Cantor space* 2^ω is the set of functions from ω to 2; elements of ω^ω and 2^ω are called *reals*. Thus basic open sets of 2^ω are of the form $[\sigma] := \{f \in 2^\omega ; \sigma \subseteq f\}$ where $\sigma \in 2^{<\omega}$ is a finite sequence; an analogous remark applies to ω^ω. We use $\restriction$ for restriction, and $\hat{}$ for concatenation of sequences (*e.g.* $\sigma\restriction i$, $\sigma\hat{}\langle n\rangle$). A set $B \subseteq 2^{<\omega}$ is a *branch* if it's of the form $B = \{\sigma ; \sigma \subseteq f\}$ for some $f \in 2^\omega$. We shall occasionally identify $2^{<\omega}$ and ω; the former inherits a linear order $\leq$ from the latter.

A set of reals $X \subseteq \omega^\omega$ (or $X \subseteq 2^\omega$) is called a *Q-set* iff every subset of X is a relative G_δ-set (that is, it is the intersection of a G_δ-subset of ω^ω with X). Let $\mathfrak{q}$ denote the size of the smallest set of reals which is not a Q-set. Since every countable set of reals is Q, one has $\omega_1 \leq \mathfrak{q} \leq \mathfrak{c}$ where $\mathfrak{c}$ denotes the cardinality of the continuum. A better lower bound can be gotten as follows.

As usual, let $[F]^\lambda$ denote the collection of subsets of F of size λ for $\lambda \leq |F|$; similarly, $[F]^{<\lambda}$ stands for the subsets of F of size $< \lambda$. For $A, B \subseteq \omega$, say that A is *almost contained* in B ($A \subseteq^* B$ in symbols) iff $A \setminus B$ is finite. A family $\mathcal{F} \subseteq [\omega]^\omega$ has the *finite intersection property* iff $\bigcap \mathcal{G}$ is infinite for each finite $\mathcal{G} \subseteq \mathcal{F}$. The *pseudointersection number* $\mathfrak{p}$ is the cardinality of the least $\mathcal{F} \subseteq [\omega]^\omega$ with the finite intersection property such that no $A \in [\omega]^\omega$ is almost contained in all members of $\mathcal{F}$. Let $(\mathbb{P}, \leq)$ be a notion of forcing, *i.e.*, a poset. $P \subseteq \mathbb{P}$ is called *centered* iff any finitely many members of P have a common lower bound in $\mathbb{P}$. $(\mathbb{P}, \leq)$ is *σ-centered* iff $\mathbb{P}$ can be written as a union of countably many centered subsets. By Bell's Theorem [Be], $\mathfrak{p}$ is the least cardinal for which Martin's axiom for σ-centered posets fails. Since the natural p.o. for making a subset of a given set of reals a relative G_δ is σ-centered (see [Mi 3, Section 5]), it is immediate that $\mathfrak{p} \leq \mathfrak{q}$. Miller and Zhou ([Mi 2, Problem 11.14], [Mi 3, Question 5.2]) asked whether $\mathfrak{p} = \mathfrak{q}$. This question was answered

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recently, and rather implicitly, in the negative by A. Dow [Do, Theorem 2.9]. We shall briefly outline what he proved and explain the relationship to the present problem.

Given $\mathcal{A}, \mathcal{B} \subseteq [\omega]^\omega$, we say $\mathcal{A}$ and $\mathcal{B}$ are *orthogonal* (and write $\mathcal{A} \perp \mathcal{B}$) iff $A \cap B$ is finite for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$. We say the pair $\langle \mathcal{A}, \mathcal{B} \rangle$ can be *weakly separated* iff there is $D \subseteq \omega$ such that $D \cap A$ is finite for all $A \in \mathcal{A}$, yet $D \cap B$ is infinite for all $B \in \mathcal{B}$. (Note this is not symmetric.) *Dow's principle* holds for a cardinal κ iff any orthogonal pair $\langle \mathcal{A}, \mathcal{B} \rangle$ with $|\mathcal{A} \cup \mathcal{B}| \leq \kappa$ can be weakly separated. Let dp be the least cardinal for which Dow's principle fails. Using again Bell's characterization of p, we see easily $\mathrm{p} \leq \mathrm{dp}$. Call $A, B \in [\omega]^\omega$ *almost disjoint* iff $A \cap B$ is finite; $\mathcal{A} \subseteq [\omega]^\omega$ is an *almost disjoint (a.d.) family* iff its members are pairwise almost disjoint. Say the *almost disjointness principle* holds for κ iff for any a.d. family $\mathcal{A}$ of size $\leq \kappa$ and any $\mathcal{B} \subseteq \mathcal{A}$, the pair $\langle \mathcal{B}, \mathcal{A} \setminus \mathcal{B} \rangle$ can be weakly separated. Let ap denote the smallest cardinal for which the almost disjointness principle fails. Clearly $\mathrm{dp} \leq \mathrm{ap}$. The following is also folklore.

Lemma 1.1 $\mathrm{ap} \leq \mathrm{q}$.

Proof Let $X \subseteq 2^\omega$ be a set of reals of size less than ap, and let $Y \subseteq X$. Given $f \in 2^\omega$, let $B_f = \{\sigma \in 2^{<\omega} ; \sigma \subseteq f\}$ be the branch corresponding to f. Apply the almost disjointness principle to find $D \subseteq 2^{<\omega}$ weakly separating the pair $\langle \{B_f ; f \in X \setminus Y\}, \{B_f ; f \in Y\} \rangle$. Let D_n be D with the first n elements removed. Put $U_n = \bigcup \{[\sigma] ; \sigma \in D_n\}$, and $G = \bigcap_n U_n$. It is immediate that $G \cap X = Y$. ■

Since Dow [Do, Theorem 2.9] proved the consistency of $\mathrm{p} < \mathrm{dp}$, the consistency of $\mathrm{p} < \mathrm{q}$ is now immediate. However, this raises natural questions about the relationship between dp, ap and q. The main portion of this work is devoted to answering them by showing

Theorem A *It is consistent to assume that* $\mathrm{dp} < \mathrm{ap}$.

Theorem B *It is consistent to assume that* $\mathrm{ap} < \mathrm{q}$.

Section 2 is devoted to the proofs of these results. In Section 3 we shall make a few comments about related cardinals and about upper bounds for ap and for q.

Our notation is fairly standard. We refer to [Je] and [Ku] for set theory in general and forcing in particular, to [Mi 1, Section 4], [Ta, Section II] and [vD, Section 9] for Q-sets and their relatives, and to [vD, Section 3], [Va] and [BJ] for results on cardinal invariants. *For almost all* always means *for all but* $< \kappa$ *many* where κ is a regular cardinal which is clear from the context. Further notation is introduced when needed.

2 The Main Consistency Proofs

The main technical device for our consistency proofs are rank arguments of the type common in descriptive set theory. The use of such arguments in forcing constructions can be traced back at least to [BD] where they were applied to Hechler forcing. Our approach is quite similar to the one in [BJS, Section 2] where almost disjoint families were combined with rank arguments like in the present work.

Let $\mathcal{A} = \{A_\alpha ; \alpha < \kappa\}$ be an almost disjoint family of subsets of ω. The forcing notion $\mathrm{Q}(\mathcal{A})$ consists of all triples (σ, ϕ, Γ) such that $\sigma \in \omega^{\uparrow<\omega}$, $\phi : \omega^{\uparrow<\omega} \to \omega$, $\Gamma \in [\kappa]^{<\omega}$ and $\sigma(i) = \phi(\sigma \restriction i)$ for all $i \in |\sigma|$. Here, $\omega^{\uparrow<\omega}$ denotes the set of strictly increasing finite sequences of natural numbers. The p.o. is given by: $(\tau, \psi, \Delta) \leq (\sigma, \phi, \Gamma)$ iff $\tau \supseteq \sigma$, $\psi \geq \phi$ everywhere, $\Delta \supseteq \Gamma$ and $\tau(i) \notin \bigcup_{\alpha\in\Gamma} A_\alpha$ for all $i \in |\tau| \setminus |\sigma|$. Clearly $\mathrm{Q}(\mathcal{A})$ is σ-centered. It generically adds a dominating real $d_{\mathcal{A}} \in \omega^\omega$—namely $d_{\mathcal{A}} = \bigcup\{\sigma ; (\sigma, \phi, \Gamma) \in G$ for some ϕ and $\Gamma\}$ where G is the generic filter—such that $range(d_{\mathcal{A}})$ is almost disjoint from all members of $\mathcal{A}$. (Recall that a real $d \in \omega^\omega$ is said to be *dominating* over a model V of *ZFC* iff for all $f \in \omega^\omega \cap V$, we have $d(n) \geq f(n)$ for almost all n.) Furthermore, given any $B \in [\omega]^\omega$ from the ground model which does not belong to the ideal generated by $\mathcal{A}$, we have, again by an easy genericity argument, that $range(d_{\mathcal{A}}) \cap B$ is infinite.

We now introduce a notion of rank for $\mathrm{Q}(\mathcal{A})$. Fix $\Gamma \in [\kappa]^{<\omega}$ and a $\mathrm{Q}(\mathcal{A})$-name $\dot{D}$ for a subset of ω. Given $\ell \in \omega$, we recursively define the rank $\rho_\ell = \rho_{\ell,\Gamma,\dot{D}}$ for all $\tau \in \omega^{\uparrow<\omega}$.

$$\rho_\ell(\tau) = 0 \iff q \Vdash \text{“}\ell \in \dot{D}\text{” for some } q = (\tau, \phi, \Gamma') \in \mathrm{Q}(\mathcal{A})$$

$$\text{for } \rho > 0: \quad \rho_\ell(\tau) \leq \rho \iff \rho_\ell(\tau\hat{\ }\langle n\rangle) < \rho \text{ holds for infinitely many } n \in \omega \setminus \bigcup_{\alpha\in\Gamma} A_\alpha$$

$$\rho_\ell(\tau) = \infty \iff \text{ there is no } \rho < \omega_1 \text{ such that } \rho_\ell(\tau) \leq \rho.$$

Given $\tau \in \omega^{\uparrow<\omega}$ and $i \in \omega$, we define the set

$$D_{\tau,i} = \Big\{\ell ; \rho_\ell(\tau\hat{\ }\langle n\rangle) < \omega_1 \text{ for some } n \in \omega \setminus \bigcup_{\alpha\in\Gamma} A_\alpha \text{ with } n \geq i\Big\}.$$

Sets of the form $D_{\tau,i}$ can be thought of as approximations to $\dot{D}$. They play a crucial role in the independence arguments below.

We first deal with the consistency of $\mathrm{dp} < \mathrm{ap}$. For this, we carry out a finite support iteration $\langle \mathrm{P}_\gamma, \dot{\mathrm{Q}}_\gamma ; \gamma < \lambda\rangle$ of ccc p.o.'s over a model for GCH where $\lambda \geq \omega_2$ is a regular cardinal and we have

$$\Vdash_\gamma \text{“}\dot{\mathrm{Q}}_\gamma \text{ is of the form } \mathrm{Q}(\dot{\mathcal{A}}) \text{ for some a.d. family } \dot{\mathcal{A}} \text{ of size } < \lambda\text{”}.$$

Using a standard book-keeping argument as in the consistency proof of Martin's axiom (see [Ku, Chapter VIII, Section 6]), we can guarantee that all small a.d. families are taken care of along the iteration. By the properties of the forcing $\mathrm{Q}(\mathcal{A})$, we then see that $\mathrm{ap} = \mathfrak{c} = \lambda$.

To show that $\mathrm{dp} = \omega_1$ in the resulting model, we use the following concept. Call a pair $\langle \mathcal{B}, \mathcal{C}\rangle = \langle \{B_\alpha ; \alpha < \omega_1\}, \{C_\alpha ; \alpha < \omega_1\}\rangle$ of subfamilies of $[\omega]^\omega$ *twisted* iff

- $B_\alpha \subset^* B_\beta$ for $\alpha < \beta$,
- $\mathcal{C}$ is an almost disjoint family,
- $B_\alpha \cap C_\beta$ is finite for all α, β, and
- whenever $D \in [\omega]^\omega$ has infinite intersection with uncountably many C_α's, then $D \cap B_\beta$ is infinite for some (for almost all) β's.

If there is a twisted family $\langle \mathcal{B}, \mathcal{C}\rangle$, then $\mathrm{dp} = \omega_1$ is immediate. Hence it suffices to prove the existence of a twisted family in the forcing extension. This will be done in three steps:

we show there is such a family in the ground model, we prove any twisted family remains twisted after forcing with some $\mathrm{Q}(\mathcal{A})$—this is the main technical result—, and we show that twistedness is preserved in limit steps of finite support iterations. The first and the last of these steps are standard arguments. However, we include proofs to make our work self-contained.

Lemma 2.1 *Assume CH. Then there is a twisted family* $\langle \mathcal{B}, \mathcal{C} \rangle = \langle \{B_\alpha \,;\, \alpha < \omega_1\}, \{C_\alpha : \alpha < \omega_1\} \rangle$.

Proof Let $\{D_\alpha \,;\, \alpha < \omega_1\}$ enumerate $[\omega]^\omega$. $\mathcal{B}$ and $\mathcal{C}$ will be constructed recursively to satisfy all four requirements of "twistedness". Assume $\{B_\beta \,;\, \beta < \alpha\}$ and $\{C_\beta \,;\, \beta < \alpha\}$ have been produced as required.

If D_α is almost contained in the union of finitely many C_β's, we choose B_α and C_α to satisfy the first three requirements of "twistedness". This is easy because α is countable.

If D_α is not almost contained in the union of finitely many C_β's, then we can find first $D' \subseteq D_\alpha$ infinite which is almost disjoint from all C_β's, and then B_α almost containing D' and all B_β and still almost disjoint from all C_β's. Finally choose C_α almost disjoint from all C_β's and from B_α.

This completes the recursive construction. It is immediate that the fourth requirement of "twistedness" is satisfied at the end. ■

Main Lemma 2.2 *If $\mathcal{A}$ is an almost disjoint family, then* $\mathrm{Q}(\mathcal{A})$ *preserves twisted families—i.e., whenever $\langle \mathcal{B}, \mathcal{C} \rangle$ is twisted (in the ground model), then*

$$\Vdash_{\mathrm{Q}(\mathcal{A})} \text{“}\langle \mathcal{B}, \mathcal{C} \rangle \textit{ is twisted”}.$$

Proof Let $\dot{D}$ be a $\mathrm{Q}(\mathcal{A})$-name for an infinite subset of ω, and let $p \in \mathrm{Q}(\mathcal{A})$ such that

$$p \Vdash \text{“}\dot{D} \cap B_\beta \text{ is finite for all (uncountably many) } \beta\text{'s”}.$$

We have to find $q \leq p$ such that

$$(\star) \qquad q \Vdash \text{“}\dot{D} \cap C_\alpha \text{ is finite for almost all } \alpha\text{'s”}.$$

For each $\beta < \omega_1$, we can find $q_\beta \leq p$ and $k_\beta \in \omega$ such that

$$(\star\star) \qquad q_\beta \Vdash \text{“}\dot{D} \cap B_\beta \subseteq k_\beta\text{”}.$$

Using standard pruning arguments (Δ-system lemma), we can assume without loss that, for all β, we have $k = k_\beta$ for some k, $q_\beta = (\sigma, \phi_\beta, \Gamma_\beta)$ for some σ, and $\Gamma_\beta = \Gamma \cup \Delta_\beta$ for some Γ where the Δ_β are pairwise disjoint. Since we must have $(\sigma, \chi, \Gamma) \leq p$ for some χ with $\phi_\beta \geq \chi$ for all β, we may assume $p = (\sigma, \chi, \Gamma)$.

Given $\tau \supseteq \sigma$ and $\phi' \geq \chi$ as well as $\Gamma' \supseteq \Gamma$, say that τ is *compatible with* $q' = (\sigma, \phi', \Gamma')$ iff $\tau(i) \geq \phi'(\tau \restriction i)$ and $\tau(i) \notin \bigcup_{\alpha \in \Gamma'} A_\alpha$ for all $|\sigma| \leq i < |\tau|$ (this holds of course iff $(\tau, \phi'', \Gamma') \leq (\sigma, \phi', \Gamma')$ for some ϕ''). In case $\Gamma' = \Gamma$, we also say τ is *compatible with* ϕ'.

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Note that σ is (trivially) compatible with uncountably many (in fact, with all) q_β's. Assume $\tau \supseteq \sigma$ is compatible with uncountably many q_β's. Since the Δ_β's are pairwise disjoint, we can easily find an ℓ such that $\tau\hat{}\langle n\rangle$ is compatible with uncountably many q_β's for all $n \in \omega \setminus \bigcup_{\alpha\in\Gamma} A_\alpha$ with $n \geq \ell$.

These remarks allow us to construct recursively $\phi \geq \chi$ such that whenever τ is compatible with ϕ, then it's also compatible with uncountably many q_β's. Put $q = (\sigma, \phi, \Gamma) \leq p$. (†)

For $\ell \in \omega$, let ρ_ℓ denote the rank function $\rho_{\ell,\Gamma,\dot{D}}$ as defined at the beginning of this section. The $D_{\tau,i}$ are defined accordingly. Fix τ and ℓ such that $\rho_\ell(\tau) < \omega_1$. Call $\beta \in \omega_1$ (τ,ℓ)-*bad* iff the set $\{n \in \omega \setminus \bigcup_{\alpha\in\Gamma_\beta} A_\alpha \ ; \ \rho_\ell(\tau\hat{}\langle n\rangle) < \rho_\ell(\tau)\}$ is finite. At most one β can be (τ,ℓ)-bad. Hence there are at most countably many β's which are (τ,ℓ)-bad for some τ and ℓ, and we may as well assume, without loss, that no β is (τ,ℓ)-bad for any τ,ℓ. (‡)

We now distinguish two cases the first of which yields a contradiction.

Case 1 There is τ compatible with ϕ such that for uncountably many α's, the intersection $D_{\tau,i} \cap C_\alpha$ is infinite for all i.

Fix such τ. By (†), we can assume, without loss, that τ is compatible with all q_β's. Fix i so large that there are uncountably many pairs β,β' such that $\bigcup_{\alpha\in\Delta_\beta} A_\alpha \cap \bigcup_{\alpha\in\Delta_{\beta'}} A_\alpha \subseteq i$ and $\phi_\beta(\tau), \phi_{\beta'}(\tau) \leq i$. Then use the twistedness of $\langle \mathcal{B}, \mathcal{C}\rangle$ to find β_0 such that $B_{\beta_0} \cap D_{\tau,i}$ is infinite. Next find $\beta,\beta' \geq \beta_0$ with $\bigcup_{\alpha\in\Delta_\beta} A_\alpha \cap \bigcup_{\alpha\in\Delta_{\beta'}} A_\alpha \subseteq i$ and $\phi_\beta(\tau), \phi_{\beta'}(\tau) \leq i$. Let $\bar{B}_\gamma = \{\ell \in B_{\beta_0} \ ; \ \rho_\ell(\tau\hat{}\langle n\rangle) < \omega_1$ for some $n \in \omega \setminus \bigcup_{\alpha\in\Gamma_\gamma} A_\alpha$ with $n \geq i\}$ for $\gamma \geq \beta_0$. Then $\bar{B}_\beta \cup \bar{B}_{\beta'} = B_{\beta_0} \cap D_{\tau,i}$, hence either $\bar{B}_\beta$ or $\bar{B}_{\beta'}$ is infinite. Assume without loss the former. Fix $\ell \in \bar{B}_\beta \cap B_\beta$ with $\ell \geq k$ and $n \in \omega \setminus \bigcup_{\alpha\in\Gamma_\beta} A_\alpha$, $n \geq i$, with $\rho_\ell(\tau\hat{}\langle n\rangle) < \omega_1$.

We now construct recursively natural numbers n_j, $j \in m$, with $n_0 = n < n_1 < \cdots < n_{m-1}$, $\rho_\ell(\tau\hat{}\langle n_0\rangle) > \rho_\ell(\tau\hat{}\langle n_0 n_1\rangle) > \cdots > \rho_\ell(\tau\hat{}\langle n_0\cdots n_{m-1}\rangle) = 0$ and $\tau\hat{}\langle n_0\cdots n_{m-1}\rangle$ compatible with q_β: by construction, we know that τ and $\tau\hat{}\langle n\rangle$ are compatible with q_β; since β is not $(\tau\hat{}\langle n_0\cdots n_j\rangle,\ell)$-bad by (‡), we can find n_{j+1} such that $\rho_\ell(\tau\hat{}\langle n_0\cdots n_j\rangle) > \rho_\ell(\tau\hat{}\langle n_0\cdots n_{j+1}\rangle)$ and $\tau\hat{}\langle n_0\cdots n_{j+1}\rangle$ is compatible with q_β; thus the construction can be carried out. By definition of ρ_ℓ, there is $q' = (\tau\hat{}\langle n_0\cdots n_{m-1}\rangle, \phi', \Gamma')$ such that

$$q' \Vdash \text{“}\ell \in \dot{D}\text{”}.$$

Now, q' is compatible with q_β, but, by (⋆⋆), any common extension forces contradictory statements. Hence case 1 fails.

Case 2 For all τ compatible with ϕ and all but countably many α's, there is $i = i_{\tau,\alpha}$ such that the set $D_{\tau,i}$ is almost disjoint from C_α.

For all such τ, let $\Theta_\tau = \{\alpha \ ; \ D_{\tau,i} \cap C_\alpha$ is infinite for all $i\}$. Let $\Theta = \bigcup_\tau \Theta_\tau$. Θ is countable by assumption. We claim that

$$q \Vdash \text{“}\dot{D} \cap C_\alpha \text{ is finite for } \alpha \in \omega_1 \setminus \Theta\text{”}$$

which shows (⋆).

To see this, let $r \leq q$, $r = (\tau, \psi, \Delta)$. Let $\alpha \in \omega_1 \setminus \Theta$. Put $E := D_{\tau,i_{\tau,\alpha}} \cap C_\alpha$ which is finite. We shall construct (recursively) $\psi' \geq \psi$ such that $\rho_\ell(\tau') = \infty$ for all $\tau' \supseteq \tau$ compatible with ψ' and all $\ell \in C_\alpha \setminus E$. *A fortiori*, this means that

$$r' \Vdash \text{“}\dot{D} \cap C_\alpha \subseteq E\text{”}$$

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where $r' = (\tau, \psi', \Delta) \leq r$, as required.

If $\ell \in C_\alpha \setminus E$, and thus $\ell \notin D_{\tau, i_{\tau,\alpha}}$, we have $\rho_\ell(\tau) = \infty$ by definition of ρ_ℓ and of $D_{\tau, i_{\tau,\alpha}}$. This takes care of the basic step of the recursive construction. To deal with the induction step, assume ψ' has been defined for all τ' of length $< m$ (where $m \geq |\tau|$). Fix $\tau' \supseteq \tau$ of length m compatible with ψ'; then by inductive assumption $\rho_\ell(\tau') = \infty$ for all $\ell \in C_\alpha \setminus E$. We know $\rho_\ell(\tau'^\frown\langle n\rangle) = \infty$ for all $\ell \notin D_{\tau', i_{\tau',\alpha}}$ and all $n \geq i_{\tau',\alpha}$. Also $D_{\tau', i_{\tau',\alpha}} \cap C_\alpha$ is finite. Let $\ell \in (D_{\tau', i_{\tau',\alpha}} \cap C_\alpha) \setminus E$; then $\rho_\ell(\tau') = \infty$; thus there are only finitely many n with $\rho_\ell(\tau'^\frown\langle n\rangle) < \omega_1$; hence we can find $i \geq i_{\tau',\alpha}$ such that for all $n \geq i$ and all $\ell \in C_\alpha \setminus E$, we have $\rho_\ell(\tau'^\frown\langle n\rangle) = \infty$. Let $\psi'(\tau') = i$; then ψ' is as required. This completes the construction of ψ', and the proof of the main lemma. ■

Iteration Lemma 2.3 *Twisted families are preserved in limit steps of finite support iterations of ccc p.o.'s—i.e., whenever $\langle \mathrm{P}_\gamma, \dot{\mathrm{Q}}_\gamma ; \gamma < \delta\rangle$, δ a limit ordinal, is such an iteration and $\langle \mathcal{B}, \mathcal{C}\rangle$ satisfies*

$$\Vdash_\gamma \text{“}\langle \mathcal{B}, \mathcal{C}\rangle \textit{ is twisted”}$$

for all $\gamma < \delta$, then

$$\Vdash_\delta \text{“}\langle \mathcal{B}, \mathcal{C}\rangle \textit{ is twisted”}.$$

Proof Since new countable subsets of ω can appear only in limit steps of countable cofinality in such iterations, we may assume without loss that $\delta = \omega$.

Let $\dot{D}$ be a P_ω-name for an infinite subset of ω, and let $p \in \mathrm{P}_\omega$ such that

$$p \Vdash_\omega \text{“}\dot{D} \cap B_\beta \text{ is finite for all } \beta\text{'s”}.$$

For each $\beta < \omega_1$ find $p_\beta \leq p$ and $k_\beta \in \omega$ such that

$$p_\beta \Vdash_\omega \text{“}\dot{D} \cap B_\beta \subseteq k_\beta\text{”}.$$

Without loss there is n such that $p_\beta \in \mathrm{P}_n$ for all β. Since P_n is ccc, there is a P_n-generic filter G_n such that $\{\beta \ ; \ p_\beta \in G_n\}$ is uncountable. Step into $V[G_n]$, and let $D = \{\ell \ ; \ q \Vdash \text{“}\ell \in \dot{D}\text{” for some } q \in \mathrm{P}_\omega \text{ with } q{\restriction}n \in G_n\}$. For any β with $p_\beta \in G_n$, we have $D \cap B_\beta \subseteq k_\beta$. Thus $|D \cap C_\alpha| < \omega$ for all but countably many α's. This means that

$$\Vdash_{[n,\omega)} \text{“}\dot{D} \cap C_\alpha \text{ is infinite for at most countably many } \alpha\text{'s”},$$

as required. ■

Putting together the three preceding lemmata, we can prove Theorem A.

Theorem 2.4 *Let $\lambda > \omega_1$ be regular. It is consistent that* $\mathrm{dp} = \omega_1 < \lambda = \mathrm{ap} = \mathrm{c}$. ■

We now proceed to show the consistency of $\mathrm{ap} < \mathrm{q}$. We perform again a finite support iteration of length $\lambda \geq \omega_2$ of p.o.'s of the form $\mathrm{Q}(\mathcal{A})$ over a model for GCH. However we only deal with $\mathcal{A} \subseteq [2^{<\omega}]^\omega$ of size $< \lambda$ which consist of branches, and we take care of all such $\mathcal{A}$'s by a book-keeping argument. By arguments like those at the beginning of our work (Lemma 1.1), this will guarantee that $\mathrm{q} = \mathrm{c} = \lambda$.

To see that $\mathrm{ap} = \omega_1$ after the iteration, we need the following device. Call a pair $\langle \mathcal{B}, \mathcal{C}\rangle = \langle \{B_\alpha \ ; \ \alpha < \omega_1\}, \{C_\alpha \ ; \ \alpha < \omega_1\}\rangle$ of infinite subsets of ω *intertwined* iff

- $\mathcal{B} \cup \mathcal{C}$ is an almost disjoint family, and
- whenever $D \in [\omega]^\omega$ has infinite intersection with uncountably many C_α's, then $D \cap B_\beta$ is infinite for almost all β's.

If there is an intertwined family, then obviously $\mathrm{ap} = \omega_1$. Therefore we proceed to show the existence of such a family in the forcing extension. The proof of the following lemma is tedious, but straightforward, and therefore left to the reader (*cf.* Lemma 2.1 which is similar).

Lemma 2.5 *Assume CH. Then there is an intertwined family $\langle \mathcal{B}, \mathcal{C} \rangle$.* ■

Main Lemma 2.6 *If $\mathcal{A} \subseteq [2^{<\omega}]^\omega$ is a family of branches, then $\mathrm{Q}(\mathcal{A})$ preserves intertwined families—i.e., whenever $\langle \mathcal{B}, \mathcal{C} \rangle$ is intertwined (in the ground model), then*

$$\Vdash_{\mathrm{Q}(\mathcal{A})} \text{“}\langle \mathcal{B}, \mathcal{C} \rangle \textit{ is intertwined”}.$$

Proof We approach this lemma in a fashion very similar to the proof of Lemma 2.2, and therefore try to be as brief as possible. The five first paragraphs of the former proof can be taken over almost verbatim. We refrain from giving them again, and leave the rare differences to the reader. The treatment of case 2 is also the same, hence we restrict ourselves to dealing with

Case 1 There is τ compatible with ϕ such that for uncountably many α's, $D_{\tau,i} \cap C_\alpha$ is infinite for all i.

Fix such τ compatible with all q_β's (without loss). Let $\Theta = \{\alpha \; ; \text{ all } D_{\tau,i} \cap C_\alpha \text{ are infinite}\} \in [\omega_1]^{\omega_1}$. For $u \in 2^{<\omega}$ and $i \in \omega$ define

$$D_i^u = \Big\{ \ell \; ; \rho_\ell(\tau\hat{\ }\langle t \rangle) < \omega_1 \text{ for some } t \in \omega \Big\backslash \bigcup_{\alpha \in \Gamma} A_\alpha \text{ with } u \subseteq t \text{ and } t \geq i \Big\}.$$

(Recall here that we identify $2^{<\omega}$ and ω. Thus "$u \subseteq t$" refers to the p.o. on $2^{<\omega}$, and "$t \geq i$" refers to the l.o. on ω.) Build a tree $T \subseteq 2^{<\omega}$ as follows: $u \in T$ iff for uncountably many $\alpha \in \Theta$ we have that $D_i^u \cap C_\alpha$ is infinite for all i. Note that, by assumption, $\langle\, \rangle \in T$, and if $t \in T$ then either $t\hat{\ }\langle 0 \rangle \in T$ or $t\hat{\ }\langle 1 \rangle \in T$. Hence T has an infinite branch, call it $f \in 2^\omega$. Now let, for $n \in \omega$,

$$D_{n,0} = \Big\{ \ell \; ; \rho_\ell(\tau\hat{\ }\langle t \rangle) < \omega_1 \text{ for some } t \in \omega \Big\backslash \bigcup_{\alpha \in \Gamma} A_\alpha \text{ with } f{\restriction}n \subseteq t \subseteq f \Big\} \quad \text{and}$$

$$D_{n,1} = \Big\{ \ell \; ; \rho_\ell(\tau\hat{\ }\langle t \rangle) < \omega_1 \text{ for some } t \in \omega \Big\backslash \bigcup_{\alpha \in \Gamma} A_\alpha \text{ with } f{\restriction}n \subseteq t \not\subseteq f \Big\}.$$

Again, for all n, we either have $|D_{n,0} \cap C_\alpha| = \omega$ for uncountably many α or $|D_{n,1} \cap C_\alpha| = \omega$ for uncountably many α. Note that if $\{f{\restriction}n \; ; \; n \in \omega\} = A_\alpha$ for some $\alpha \in \Gamma$, then the second case must hold, for then $D_{n,1} = D_0^{f\restriction n}$. We distinguish the two cases.

Subcase a For all n, we have $|D_{n,0} \cap C_\alpha| = \omega$ for uncountably many α.

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By the preceding remark, there is at most one β_0 such that $\{f\restriction n\,;\,n\in\omega\}=A_\alpha$ for some $\alpha\in\Gamma_{\beta_0}$. Hence we can fix n such that for uncountably many β, we have $\phi_\beta(\tau)\le f\restriction m$ and $f\restriction m\notin\bigcup_{\alpha\in\Gamma_\beta}A_\alpha$ for all $m\ge n$; without loss this is true for all β. By intertwinedness of $\langle\mathcal{B},\mathcal{C}\rangle$ we find β such that $|D_{n,0}\cap B_\beta|=\omega$. Fix $\ell\in D_{n,0}\cap B_\beta$ with $\ell>k$ and $m\ge n$ with $\rho_\ell(\tau\hat{}\langle f\restriction m\rangle)<\omega_1$. Then $\tau\hat{}\langle f\restriction m\rangle$ is compatible with q_β, and we can recursively construct a condition q' compatible with q_β such that

$$q'\Vdash \text{“}\ell\in\dot{D}\text{”},$$

a contradiction (see the corresponding argument in the proof of Lemma 2.2 for details).

Subcase b For all n, we have $|D_{n,1}\cap C_\alpha|=\omega$ for uncountably many α.

Fix n such that for uncountably many β, we have $\phi_\beta(\tau)\le u$ and $u\notin\bigcup_{\alpha\in\Gamma_\beta}A_\alpha$ for all $u\supseteq f\restriction n$ with $u\nsubseteq f$. This is possible since each A_α is a branch. Without loss this is true for all β, and we can again use the intertwinedness of $\langle\mathcal{B},\mathcal{C}\rangle$ to proceed as before in Subcase a.

This completes the proof of the Main Lemma. ■

As before a standard argument shows:

Iteration Lemma 2.7 *Intertwined families are preserved in limit steps of finite support iterations of ccc p.o.'s—i.e., whenever $\langle \mathrm{P}_\gamma,\dot{\mathrm{Q}}_\gamma\,;\,\gamma<\delta\rangle$, δ a limit ordinal, is such an iteration and $\langle\mathcal{B},\mathcal{C}\rangle$ satisfies*

$$\Vdash_\gamma \text{“}\langle\mathcal{B},\mathcal{C}\rangle \textit{ is intertwined”}$$

for all $\gamma<\delta$, then

$$\Vdash_\delta \text{“}\langle\mathcal{B},\mathcal{C}\rangle \textit{ is intertwined”}.$$

■

We conclude with Theorem B which is the consequence of the three preceding lemmata.

Theorem 2.8 *Let $\lambda>\omega_1$ be regular. It is consistent that* $\mathrm{ap}=\omega_1<\lambda=\mathrm{q}=\mathfrak{c}$. ■

Remark 2.9 Note that, by generalizing the notions of “twistedness” and “intertwinedness” appropriately, we can get the consistency of $\kappa=\mathrm{dp}<\mathrm{ap}=\lambda$ and of $\kappa=\mathrm{ap}<\mathrm{q}=\lambda$ for arbitrary regular $\kappa<\lambda$. (In fact, it suffices to use $\mathcal{B}$ of size κ (instead of ω_1). Apart from forcings of type $\mathrm{Q}(\mathcal{A})$, the iteration also involves forcings of size $<\kappa$ to guarantee $\mathrm{dp}\ge\kappa$ ($\mathrm{ap}\ge\kappa$, respectively). A standard argument shows such forcings do not destroy twistedness (intertwinedness, resp.).) In the same vein, we can even show the consistency of $\kappa=\mathrm{dp}<\lambda=\mathrm{ap}<\mu=\mathrm{q}$ for arbitrary regular $\kappa<\lambda<\mu$.

3 Comments and Questions

We shall briefly discuss a few variants of the main cardinal coefficients considered in this work, and then touch upon their relationship to some of the classical cardinal invariants of the continuum. Consider the following restricted (“countable”) versions of the cardinals. Let dp_1 be the size of the minimal $\mathcal{A}\subseteq[\omega]^\omega$ such that there is some countable $\mathcal{B}\subseteq[\omega]^\omega$ such that $\mathcal{A}\perp\mathcal{B}$ and $\langle\mathcal{A},\mathcal{B}\rangle$ cannot be weakly separated. Similarly, ap_1 is the cardinality of

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the least $\mathcal{A} \subseteq [\omega]^\omega$ such that there is some countable $\mathcal{B} \subseteq [\omega]^\omega$ such that $\mathcal{A} \cup \mathcal{B}$ is a.d. and $\langle \mathcal{A}, \mathcal{B} \rangle$ is not weakly separated. Recall that a set of reals $X \subseteq \omega^\omega$ is said to be a *σ-set* iff every F_σ-subset of X is a G_δ-set; X is called a *λ-set* iff every countable subset of X is a relative G_δ. Given a family of sets of reals $\mathcal{F} \subseteq \mathcal{P}(\omega^\omega)$, let $\mathrm{non}(\mathcal{F})$, the *uniformity of* $\mathcal{F}$, denote the size of the smallest set of reals which does not belong to $\mathcal{F}$. Notice that $\mathrm{non}(\lambda\text{-set})$ can be considered as a "countable" version of $\mathfrak{q}$.

Given $f, g \in \omega^\omega$, we say $f \leq^* g$ (*g eventually dominates f*) iff $f(n) \leq g(n)$ for all but finitely many n. Let $\mathfrak{b}$, the *unbounding number*, be the size of the smallest subfamily $\mathcal{F}$ of ω^ω such that no $g \in \omega^\omega$ eventually dominates all members of $\mathcal{F}$. The *dominating number* $\mathfrak{d}$ is the size of the smallest subfamily $\mathcal{F}$ of ω^ω such that every $g \in \omega^\omega$ is eventually dominated by some member of $\mathcal{F}$. Clearly $\mathfrak{b} \leq \mathfrak{d}$.

With these conventions one has the following well-known result.

Theorem 3.1 $\mathfrak{b} = \mathfrak{dp}_1 = \mathfrak{ap}_1 = \mathrm{non}(\sigma\text{-}set) = \mathrm{non}(\lambda\text{-}set)$.

Proof See [vD, Sections 3 and 9]. ■

This shows that the behaviour of the countable versions substantially differs from the behaviour of the unrestricted versions. The former simply coincide, while the latter are consistently different. This sheds new light on the interest of the results of Section 2.

Corollary 3.2 $\mathfrak{q} \leq \mathfrak{b}$. ■

Next consider the following restricted versions of the cardinals. $\mathfrak{dp}_2$ is the cardinality of the least $\mathcal{B} \subseteq [\omega]^\omega$ such that there is some countable $\mathcal{A} \subseteq [\omega]^\omega$ such that $\mathcal{A} \perp \mathcal{B}$ and $\langle \mathcal{A}, \mathcal{B} \rangle$ cannot be weakly separated. Similarly, $\mathfrak{ap}_2$ is the size of the smallest $\mathcal{B} \subseteq [\omega]^\omega$ such that for some countable $\mathcal{A} \subseteq [\omega]^\omega$, $\mathcal{A} \cup \mathcal{B}$ is a.d. and $\langle \mathcal{A}, \mathcal{B} \rangle$ is not weakly separated. There is no corresponding version of $\mathfrak{q}$ because every countable subset of a set of reals is a (relative) F_σ. We prove again that these cardinals give us nothing new.

Theorem 3.3 $\mathfrak{d} = \mathfrak{dp}_2 = \mathfrak{ap}_2$.

Proof $\mathfrak{dp}_2 \leq \mathfrak{ap}_2$ is trivial.

To see $\mathfrak{d} \leq \mathfrak{dp}_2$, take $\kappa < \mathfrak{d}$, $\mathcal{B} = \{B_\alpha \ ; \ \alpha < \kappa\}$ and $\mathcal{A} = \{A_n \ ; \ n \in \omega\}$ orthogonal. Given $\alpha < \kappa$, define $f_\alpha \in \omega^\omega$ by $f_\alpha(n) =$ the least $\ell > n$ such that $(\bigcup_{n \leq j < \ell} A_j \setminus \bigcup_{i<n} A_i) \cap B_\alpha$ contains an element $< \ell$. (This always exists because we may increase the A_n, if necessary, by finitely many points so that they exhaust all of ω.)

Now find $g \in \omega^\omega$ strictly increasing which is not eventually dominated by any member of the family of functions gotten from the f_α by taking finite maxima. Let $I_0 = \big[0, g(0)\big)$, and, in general, $I_n = \big[g^n(0), g^{n+1}(0)\big)$, where we put $g^{n+1}(0) = g\big(g^n(0)\big)$. Let E be the even and O the odd numbers. Then either for all $\alpha < \kappa$ there are infinitely many $n \in E$ such that there is $k \in I_n$ with $f_\alpha(k) < g(k)$, or there are infinitely many $n \in O$ with this property. (Otherwise we could find α_0 such that only finitely many $n \in E$ have this property and α_1 such that only finitely many $n \in O$ have this property. Then the maximum of f_{α_0} and f_{α_1} would eventually dominate g which contradicts the choice of the latter.) Without loss

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assume the former. Put

$$X = \bigcup_{n\in\omega}\Big[\Big(\bigcup_{g^{2n}(0)\le j<g^{2n+2}(0)} A_j \setminus \bigcup_{i<g^{2n}(0)} A_i\Big) \cap g^{2n+2}(0)\Big].$$

It's obvious that $X \cap A_n$ is finite for all $n \in \omega$.

To see $X \cap B_\alpha$ is infinite for all $\alpha < \kappa$, find $2n \in E$ and $k \in I_{2n} = [g^{2n}(0), g^{2n+1}(0))$ such that $f_\alpha(k) < g(k)$. Then $g^{2n}(0) \le k < f_\alpha(k) < g(k) < g^{2n+2}(0)$. Hence $\big(\bigcup_{g^{2n}(0)\le j<g^{2n+2}(0)} A_j \setminus \bigcup_{i<g^{2n}(0)} A_i\big) \cap B_\alpha$ contains an element less than $g^{2n+2}(0)$. This must belong to X.

Finally, we show $\mathrm{ap}_2 \le \mathfrak{d}$. Let $\kappa < \mathrm{ap}_2$, and choose $\{g_\alpha ; \alpha < \kappa\} \subseteq \omega^\omega$. We have to find $g \in \omega^\omega$ such that for all $\alpha < \kappa$, we have $g(n) \ge g_\alpha(n)$ for infinitely many n. For this let $A_n = \{n\} \times \omega$, choose $C_\alpha \subseteq \omega$ a.d. and let $B_\alpha = \{\langle i, j\rangle ; i \in C_\alpha \text{ and } j = g_\alpha(i)\}$. Clearly the A_n and the B_α are pairwise almost disjoint. Since $\kappa < \mathrm{ap}_2$, there is $D \in [\omega \times \omega]^\omega$ such that $D \cap B_\alpha$ is infinite for all α and $D \cap A_n$ is finite for all n. Let $g(n) = \max\{j ; \langle n, j\rangle \in D\}$. Clearly for all α there are infinitely many $n \in C_\alpha$ with $g_\alpha(n) \le g(n)$. ■

Note that the different characterizations of dp_1 and dp_2 (ap_1 and ap_2, resp.) shed new light on the asymmetry in the definition of Dow's principle (of the almost disjointness principle, resp.).

We saw already in Section 1 that Q-sets are closely related to families of branches in $2^{<\omega}$. This makes the following characterization of $\mathfrak{q}$ plausible.

Proposition 3.4 $\mathfrak{q} = \min\{|\mathcal{A}| ; \mathcal{A} \subseteq [2^{<\omega}]^\omega$ *is a family of branches and there is* $\mathcal{B} \subseteq \mathcal{A}$ *such that* $\langle \mathcal{B}, \mathcal{A} \setminus \mathcal{B}\rangle$ *is not weakly separated*$\}$.

Proof Let ap' denote the cardinal on the right-hand side. $\mathrm{ap}' \le \mathfrak{q}$ was proved in Lemma 1.1. To see $\mathfrak{q} \le \mathrm{ap}'$, fix $\kappa < \mathfrak{q}$ and $\mathcal{A} = \{A_\alpha ; \alpha < \kappa\}$ a family of branches in $2^{<\omega}$; *i.e.*, $A_\alpha = \{f_\alpha{\restriction}n ; n \in \omega\}$ for some $f_\alpha \in 2^\omega$. Given any $\Gamma \subseteq \kappa$, find open sets $U_n \subseteq 2^\omega$, $n \in \omega$, with $U_{n+1} \subseteq U_n$ such that $\{f_\alpha ; \alpha < \kappa\} \cap \bigcap_n U_n = \{f_\alpha ; \alpha \in \Gamma\}$. Suppose $U_n = \bigcup_i [\sigma_{n,i}]$, $\sigma_{n,i} \in 2^{<\omega}$; we can assume that $\sigma_{n,i}$ and $\sigma_{n,j}$ are incomparable for $i \ne j$ (otherwise throw out the superfluous $\sigma_{n,i}$); we can also assume that $|\sigma_{n,i}| \ge n$ for all i and all n (otherwise split shorter $\sigma_{n,i}$ into longer ones). Now let $B = \{\sigma_{n,i} ; n, i \in \omega\}$. It is easily checked that $|B \cap A_\alpha| = \omega$ for $\alpha \in \Gamma$ and $|B \cap A_\alpha| < \omega$ for $\alpha \in \kappa \setminus \Gamma$. ■

To get a better upper bound for ap we need the following two cardinals. Let $\mathcal{M}$ denote the ideal of meager subsets of either 2^ω or ω^ω. $\mathrm{cov}(\mathcal{M})$, the *covering number* of $\mathcal{M}$, stands for the cardinality of the smallest covering of the real line by meager sets, and $\mathrm{add}(\mathcal{M})$, the *additivity* of $\mathcal{M}$, denotes the size of the smallest collection of meager sets whose union is not meager. It's well-known that $\mathrm{add}(\mathcal{M}) = \min\{\mathfrak{b}, \mathrm{cov}(\mathcal{M})\}$; that $\mathrm{cov}(\mathcal{M}) \le \mathfrak{d}$; and that $\mathfrak{p} \le \mathrm{add}(\mathcal{M})$ (see [BJ, Chapter 2] for details).

Proposition 3.5 $\mathrm{ap} \le \mathrm{cov}(\mathcal{M})$.

Proof We use Bartoszyński's characterization of the cardinal $\mathrm{cov}(\mathcal{M})$; that is, $\mathrm{cov}(\mathcal{M})$ is the size of the smallest $\mathcal{F} \subseteq \omega^\omega$ such that for each $g \in \omega^\omega$ there is $f \in \mathcal{F}$ such that the set $\{n \in \omega ; f(n) = g(n)\}$ is finite. See [BJ, Theorem 2.4.1].

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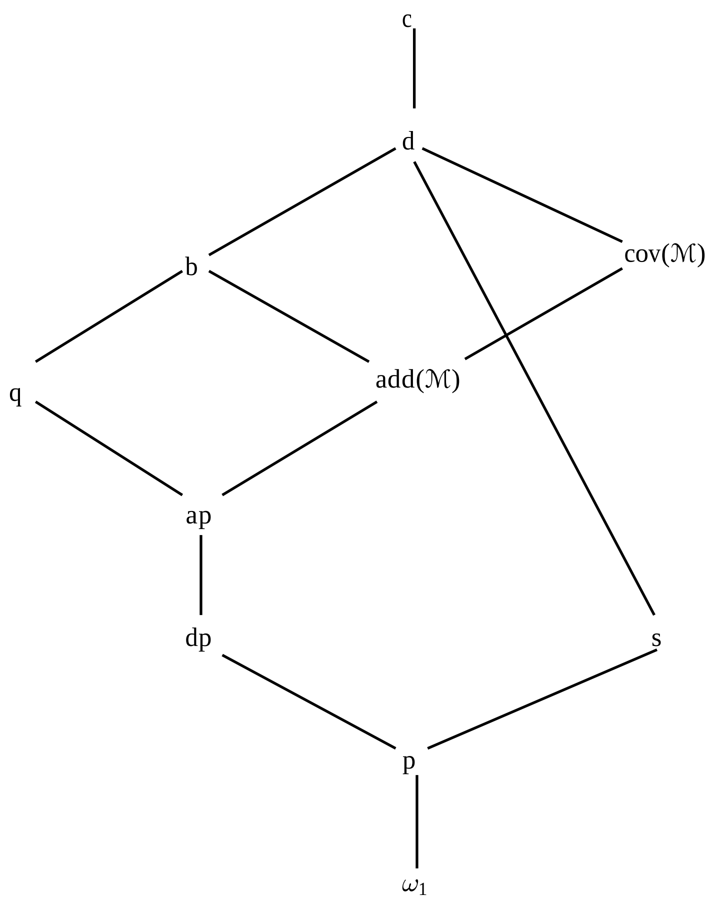

Diagram 1

Let us take $\kappa < \mathfrak{ap}$ and $\{f_\alpha \ ; \ \alpha < \kappa\} \subseteq \omega^\omega$. Choose $\{C_\alpha \ ; \ \alpha < \kappa\}$ an almost disjoint family of subsets of ω of size κ. Work in $\omega \times \omega$. For $\alpha < \kappa$, let $B_\alpha = \{\langle n, m\rangle \ ; \ n \in C_\alpha$ and $m = f_\alpha(n)\}$. Also define $A_\alpha = \{\langle n, m\rangle \ ; \ n \in C_\alpha$ and $m < f_\alpha(n)\}$ for $\alpha < \kappa$. Since $\kappa < \mathfrak{ap}$, there is $D \in [\omega \times \omega]^\omega$ which meets all A_α only finitely often but intersects all B_α infinitely often. Define $g \in \omega^\omega$ by $g(n) = \min\{m \ ; \ \langle n, m\rangle \in D\}$ if the latter set is non-empty, and arbitrary otherwise. We leave it to the reader to verify that $\{n \in C_\alpha \ ; \ f_\alpha(n) = g(n)\}$ is infinite for all $\alpha < \kappa$, as required. ■

Corollary 3.6 $\mathfrak{ap} \leq \mathrm{add}(\mathcal{M})$. ■

Whether similar results can be proved about $\mathfrak{q}$ is open. This problem was first investigated by A. Miller.

Question 3.7 (Miller) Is $\mathfrak{q} \leq \mathrm{cov}(\mathcal{M})$?

Given $A, B \in [\omega]^\omega$, we say *A splits B* iff both $A \cap B$ and $B \setminus A$ are infinite. $\mathcal{S} \subseteq [\omega]^\omega$ is a *splitting family* iff for all $B \in [\omega]^\omega$ there is $A \in \mathcal{S}$ which splits B. Let $\mathfrak{s}$ be the size of the smallest splitting family (the *splitting number*). It is well-known that $\mathfrak{p} \leq \mathfrak{s} \leq \mathfrak{d}$ [vD, Section 3]. The relationship between the cardinals discussed in this work is illustrated in Diagram 1. There, cardinals grow larger as one moves upwards along the lines. Let us notice that Dow [Do] proved (implicitly) the consistency of $\mathfrak{dp} > \mathfrak{s}$ (simply apply the techniques of [BD] to Dow's forcing). On the other hand, the consistency of $\mathfrak{q} < \min\{\mathfrak{s}, \mathrm{add}(\mathcal{M})\}$ is well-known (note that if X is an infinite Q-set, then $2^{|X|} = \mathfrak{c}$; hence one can first blow up 2^{ω_1} with countable conditions, and then iterate ccc forcing to increase $\mathfrak{s}$ and $\mathrm{add}(\mathcal{M})$; if $\mathfrak{c} < 2^{\omega_1}$, we will have $\mathfrak{q} = \omega_1$).

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