

Injective Representations of Infinite Quivers. Applications

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Abstract. In this article we study injective representations of infinite quivers. We classify the indecomposable injective representations of trees and describe Gorenstein injective and projective representations of barren trees.

1 Introduction

The major impetus for the study of the representations of quivers was given by Gabriel's study of the finite quivers of finite representation type [11] and their connection with the Dynkin diagrams associated with finite dimensional semisimple Lie algebras over the field of complex numbers.

The classical representation theory of quivers involved finite quivers and assumed that the ring was an algebraically closed field with the assumption that all vector spaces involved were finite dimensional (for example see [14]). Recently, representations by modules over more general quivers have been studied. Our concern will be these more general representations and is a continuation of the program initiated in [7] and continued in [2, 5, 6] and [9].

Many categories of graded modules over graded rings are equivalent to representations of infinite quivers. For a simple example, the category of \mathbb{Z} -graded modules over the graded ring $R[x] = R + Rx + Rx^2 + \cdots$ (here R is any ring with identity) is equivalent to the category of representations over R of the infinite line quiver. Less trivial examples can be given involving group rings $R[G]$ with the obvious grading. In [3] it was shown that the category of quasi-coherent sheaves over any scheme is equivalent to a category of representations of a quiver (with certain modifications on the representations). In many of these cases the quiver viewpoint leads to simplifications of proofs and of the descriptions of objects in the related categories. This has certainly proved true in the case of finite quivers and promises to be so when the quivers are infinite.

Our techniques are necessarily different from the usual ones concerning quivers without oriented cycles, and in general from the classic treatment of representation theory of associative algebras (see for example [1]). We consider the geometric properties of the quiver and also use infinite matrix techniques to classify projective rep-

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representations (see [2]). In [9] flat representations were studied and in [6] the so called noetherian quivers were characterized.

Our main concern in this paper is the study of injective representations over a possibly infinite quiver in terms of local properties of the representations (see properties (i) and (ii) of Proposition 2.1). As we will see in Section 6 these local properties turn out to be very useful in studying and characterizing Gorenstein injective, projective and flat representations of quivers. These are of particular interest for defining a version of relative homological algebra that is called Gorenstein homological algebra. For it has been recently proved in [5] (by using the results of [4]) that a fruitful version of Gorenstein homological algebra can be developed in the category of representations over an arbitrary quiver when the base ring is Gorenstein. Moreover by [4] we get that two model structures can be derived in the category of representations over a quiver where we use these Gorenstein injective and Gorenstein projective representations to define the cofibrations (see [12] for a deep study on model category structures).

On the other hand, infinite and barren trees (see Section 5) appear naturally in the study of finiteness conditions in the category of representations of quivers. This is especially so when we ask if this category admits a family of noetherian generators (cf. [6]) and when we characterize a Gorenstein path ring (cf. [5]). So in Section 3 we study indecomposable injective representations of trees to determine the structure of injective representations on barren trees (see the remark after Corollary 5.5).

In Sections 4 and 5 we exhibit a wide class of quivers whose injective representations admit the previous “local” characterization (the so called source injective representation quivers, see Definition 2.2). One interesting question is whether this characterization carries over to quivers with relations. For instance, in [10] it is proved that this is the case when we consider the infinite line quiver on both sides and then take the monomial relation given by the composition of N -paths. So we get a local structure theorem for injective representations in the category of N -complexes of modules ($N \geq 2$), and all the machinery of Section 6 can be applied in this situation.

2 Preliminaries

Throughout this article we will use the terminology and results of [7].

We keep the notation introduced in [1]. All rings considered in this paper will be associative with identity and, unless otherwise specified, not necessarily commutative. The letter R will usually denote a ring.

As usual we denote a quiver by Q with the understanding that a *quiver* is a four-tuple $Q = (V(Q), \Gamma(Q), s, t)$ where $V(Q)$ is a set of vertices, $\Gamma(Q)$ a set of arrows between these vertices and $s, t: \Gamma(Q) \rightarrow V(Q)$, where for each $a \in \Gamma(Q)$, $s(a)$ and $t(a)$ assign to an arrow a the source vertex and terminal vertex of a respectively. All quivers considered in this paper may be infinite, that is, one or both of the two sets $V(Q)$ or $\Gamma(Q)$ can be infinite. Note that we do not exclude loops or multiple arrows in the definition of a quiver. When Q is understood we often write V and Γ for $V(Q)$ and $\Gamma(Q)$. A *finite path* p of a quiver A is a sequence of arrows $a_n \cdots a_2 a_1$ with $t(a_i) = s(a_{i+1})$ for all $i = 0, \dots, n-1$. Therefore $s(p) = s(a_1)$ and $t(p) = t(a_n)$. Two paths p and q of a quiver can be composed, getting another path qp , whenever

$t(p) = s(q)$. A quiver can be thought of as a small category where the objects are the vertices and the morphisms are the paths. The vertices of Q can be considered as the identities of Q , that is, a vertex v of Q is a trivial path where $s(v) = t(v) = v$. A *tree* is a quiver T having a vertex v such that for another vertex w of T there exists a unique path p such that $s(p) = v$ and $t(p) = w$. Such vertex v is called the root of the tree T . A *forest* is a quiver in which every connected component is a tree. For a path p of Q we denote by $t(p)$ (resp. $s(p)$) the final (resp. the initial) vertex of p .

A representation by modules \mathcal{X} of a given quiver Q is a functor $\mathcal{X}: Q \rightarrow R\text{-Mod}$. Such a representation is determined by giving a module $\mathcal{X}(v)$ to each vertex v of Q and a homomorphism $\mathcal{X}(a): \mathcal{X}(v_1) \rightarrow \mathcal{X}(v_2)$ to each arrow $a: v_1 \rightarrow v_2$ of Q . A morphism η from \mathcal{X} to \mathcal{Y} is a natural transformation, so it will be a family of morphisms $\{\eta_v: \mathcal{X}(v) \rightarrow \mathcal{Y}(v)\}_{v \in V}$ such that $\mathcal{Y}(a) \circ \eta_{v_1} = \eta_{v_2} \circ \mathcal{X}(a)$ for any arrow $a: v_1 \rightarrow v_2$ of Q . Thus the representations of a quiver Q by modules over a ring R is a category, denoted $(Q, R\text{-Mod})$. This is a Grothendieck category with enough projectives.

The category $(Q, R\text{-Mod})$ is equivalent to the category of modules over the path algebra RQ , which is a ring with enough idempotents that in general does not have a unit (unless $|V|$ is finite).

For a given quiver one can find a family of injective cogenerators from an adjoint situation (see [7]). For every vertex $v \in V$ and the embedding morphism $\{v\} \subseteq Q$ the family $\{e_*^v(E) : v \in V\}$ is a family of injective cogenerators of $(Q, R\text{-Mod})$, whenever E is an injective cogenerator of $R\text{-Mod}$. The functor $e_*^v: R\text{-Mod} \rightarrow (Q, R\text{-Mod})$ is defined in [7, Section 4] as $e_*^v(M)(w) = \prod_{Q(w,v)} M$, where $Q(w, v)$ denotes the set of paths p in Q such that $s(p) = w$ and $t(p) = v$. If $a: w_1 \rightarrow w_2$ is an arrow, then

$$e_*^v(M)(a): \prod_{Q(w_1,v)} M \rightarrow \prod_{Q(w_2,v)} M$$

is given by the coordinate-wise function $e_*^v(M)(a) = \prod_{Q(w_2,v)a} id_{Q(w_2,v)}$. Then by [7, Theorem 4.1] e_*^v is the right adjoint functor of the evaluation functor

$$T_v: (Q, R\text{-Mod}) \rightarrow R\text{-Mod}$$

given by $T_v(\mathcal{X}) = \mathcal{X}(v)$ for any representation $\mathcal{X} \in (Q, R\text{-Mod})$.

The next result gives a property satisfied by injective representations.

Proposition 2.1 *Let Q be any quiver and let $(Q, R\text{-Mod})$ be the category of representations of Q by left R -modules. If $\mathcal{X} \in (Q, R\text{-Mod})$ is injective, then the following hold:*

- (i) $\mathcal{X}(v)$ is an injective R -module, for any vertex v of Q .
- (ii) For any vertex v the morphism

$$\mathcal{X}(v) \rightarrow \prod_{s(a)=v} \mathcal{X}(t(a))$$

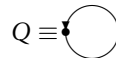
induced by $\mathcal{X}(v) \rightarrow \mathcal{X}(t(a))$ is a splitting epimorphism.

Proof We consider the injective representations $e_*^v(E)$ associated with a given vertex of Q and an injective left R -module E . By the construction given in the proof of [7, Theorem 4.1] we see that each $e_*(E)$ has the property of the proposition. As noted in [7, p. 303] the $e_*^v(E)$ (varying v and E) cogenerate $(Q, R\text{-Mod})$. So any injective representation is a retract of products of the various $e_*^v(E)$'s. The property is preserved under taking products and retracts, and so we get the desired result. ■

Since the category of representations of a quiver Q is always a Grothendieck category, it has enough injective representations. In this paper we are mainly concerned with studying when injective representations are characterized in terms of conditions (i) and (ii) of Proposition 2.1. These conditions involve local properties of the representation on the vertices and their corresponding sources, so they motivate the following definition which is pivotal for the rest of the paper.

Definition 2.2 We say that a quiver Q is a *source injective representation quiver* if for any R the injective representations of $(Q, R\text{-Mod})$ can be characterized in terms of conditions (i) and (ii) of Proposition 2.1. We will denote by \mathfrak{S} the class of all source injective representation quivers.

Example 1 The quiver



is not source injective. To see this, let us consider the category $(Q, k\text{-Mod})$ of representations of Q over k -vector spaces, (where k is a field). It is easy to see that $(Q, k\text{-Mod})$ is equivalent to the category $k[x]\text{-Mod}$. So now let us take the representation with $k[x, x^{-1}]$ in the vertex and the morphism $k[x, x^{-1}] \xrightarrow{-x} k[x, x^{-1}]$. It is obvious that it satisfies (i) and (ii) of Proposition 2.1, but $k[x, x^{-1}]$ is not divisible as a $k[x]$ -module so cannot be injective.

One of the main goals of this paper is to study the class \mathfrak{S} and more precisely to determine which trees belong to it. For this purpose, it is convenient to start Section 3 by determining the indecomposable injective representations of trees. As we will see in Section 6, there are important properties which can be derived from the fact that a quiver is in \mathfrak{S} .

Thus in Section 4 we prove that a wide class of quivers, the so called right rooted quivers (that is those which do not contain a path of the form $\bullet \rightarrow \bullet \rightarrow \dots$), are in \mathfrak{S} . But there are also important non-right rooted quivers in \mathfrak{S} . Concretely, in Section 5 we prove that A_∞^∞ , A_∞^+ and more generally any barren tree are source injective representation quivers.

Of course Proposition 2.1 can be dualized and leads to a definition dual to that of Definition 2.2.

Definition 2.3 (dual to Definition 2.2) We say that a quiver Q is a *sink projective representation quiver* if for any R the projective representation \mathcal{X} of $(Q, R\text{-Mod})$ can be characterized in terms of the dual of the conditions (i) and (ii) of Proposition 2.1, that is,

- (i') For each vertex $v \in V$, the module $\mathcal{X}(v)$ is projective.
- (ii') For a vertex $v \in V$, the morphism $\bigoplus_{t(a)=v} \mathcal{X}(s(a)) \rightarrow \mathcal{X}(v)$ (where $\mathcal{X}(s(a)) \rightarrow \mathcal{X}(v)$ is $\mathcal{X}(a)$) is a splitting monomorphism.

We will denote by \mathfrak{P} the class of all sink projective representation quivers.

Left rooted quivers are examples of quivers in \mathfrak{P} , but as is shown in [2, Theorem 4.1], A_∞ is not a sink projective representation quiver. Cyclic quivers are also examples of quivers which do not lie in \mathfrak{P} .

3 Indecomposable Injective Representations.

The main aim of this section is to characterize the indecomposable injective representations of trees. So let T be a tree. We recall that for a given v as above, e_* is the right adjoint of the restriction functor $(T, R\text{-Mod}) \rightarrow (\{v\}, R\text{-Mod})$ (where the last category is essentially $R\text{-Mod}$).

Since our quiver is a tree, these representations have an especially simple form. These are such that $e_*^v(E)(w) = E$ if there is a path (necessarily unique) from w to v , and such that $e_*^v(E)(w) = 0$ otherwise. Also $e_*^v(E)(a) = id_E$ for any arrow a such that $e_*^v(E)(t(a)) = e_*^v(E)(s(a)) = E$. This follows from [7, Theorem 4.1]. It is easy to see that if E is an indecomposable injective module, then each $e_*^v(E)$ is also an indecomposable injective object of $(T, R\text{-Mod})$. But in general these $e_*^v(E)$ do not give all such objects. However, if we introduce the notion of vertices w at infinity of T and modify the construction of $e_*^v(E)$ to include objects $e_*^w(E)$ for such vertices at infinity, we will get all the indecomposable injective objects.

We begin by noting that given the vertex v of T there is a unique path $p = a_n a_{n-1} \cdots a_1$ of Q such that $s(a_1)$ is the root of T and such that $t(a_n) = v$. Then $e_*^v(E)$ is such that $e_*^v(E)(a_i) = id_E$ for $i = 1, 2, \dots, n$ and such that $e_*^v(v') = 0$ when v' is none of $s(a_1), t(a_1), \dots, s(a_n), t(a_n)$. We consider infinite paths $p = \cdots a_3 a_2 a_1$ such that $s(a_1)$ is the root of T . For each such path we associate a vertex w (distinct from all the vertices of T) that we call a vertex at infinity. We associate distinct vertices at infinity with such distinct paths (we note the analogy with the procedure of adding points at infinity to form a projective space from an affine space).

Now let $p = \cdots a_3 a_2 a_1$ be an infinite path with $s(a_1)$ as the root of the tree and with w as its associated vertex at ∞ . For an injective left R -module E we define the object $e_*^w(E)$ of $(T, R\text{-Mod})$ to be that unique object such that $e_*^w(E)(a_i) = id_E$ for each a_1, a_2, a_3, \dots and such that $e_*^w(E)(v) = 0$ if v is any vertex distinct from all the vertices $s(a_j), t(a_j)$ for $j = 1, 2, 3, \dots$

We now shall prove the following.

Proposition 3.1 *If w is a vertex at infinity for T and if E is an injective left R -module, then $e_*^w(E)$ is an injective object of $(T, R\text{-Mod})$.*

Proof We first prove the claim when our tree is $A_\infty^+ \equiv \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \cdots$ (the infinite line to the right). Note that here we have one vertex at infinity, say w , and that then for E , $e_*^w(E)$ is just $E \xrightarrow{id} E \xrightarrow{id} E \xrightarrow{id} E \rightarrow \cdots$. Then note that if $\mathcal{M} = M_0 \rightarrow M_1 \rightarrow M_2 \rightarrow \cdots$ is any object of $(A_\infty^+, R\text{-Mod})$, then the morphisms $\mathcal{M} \rightarrow e_*^w(E)$ are in one-to-one correspondence with maps $\lim_{\rightarrow} M_n \rightarrow E$. So then if $\mathcal{S} = S_0 \rightarrow S_1 \rightarrow S_2 \rightarrow \cdots$

is a subobject of M , any morphism $S \rightarrow e_*^w(E)$ gives $\lim S_n \rightarrow E$. But E is injective and \lim is left exact, so $\lim S_n \rightarrow E$ can be extended to $\lim M_n \rightarrow E$. This in turn gives an extension $M \rightarrow e_*^w(E)$ and shows that $e_*^w(E)$ is injective.

Now we consider an arbitrary tree T . Given a vertex w at infinity of T there is an embedding $A_\infty^+ \subseteq T$ that identifies A_∞^+ with the infinite path p from the root of T and having w as its vertex at infinity. We now apply the comment at the bottom of page 302 of [7]. Letting \mathcal{X} be our $e_*^w(E)$ (as an object in $(A_\infty^+, R\text{-Mod})$) and letting $f: A_\infty^+ \rightarrow T$ be the embedding above we see that $f_*(e_*^w(E))$ is precisely what is denoted as $e_*^w(E)$ (as an object of $(T, R\text{-Mod})$). Since these right adjoint functors preserve injectivity (see [7, Section 5, p. 303]), we get that $e_*^w(E)$ is injective. ■

Note that if E is furthermore indecomposable, then $e_*^w(E)$ is an indecomposable injective object of $(T, R\text{-Mod})$.

Theorem 3.2 *If \mathcal{X} is an indecomposable injective object of $(T, R\text{-Mod})$ where T is a tree, then $\mathcal{X} \cong e_*^w(E)$ for some vertex w of T (finite or infinite) and some indecomposable injective left R -module E .*

Proof We will use Proposition 2.1. Let v_0 be the root of T . If

$$\mathcal{X}(v_0) \rightarrow \prod_{s(a)=v_0} \mathcal{X}(t(a))$$

is not an isomorphism, let E be its kernel. Then E is an injective left R -module. Also $e_*^{v_0}(E)$ is a nonzero injective subobject of \mathcal{X} . So since \mathcal{X} is indecomposable, $\mathcal{X} = e_*^{v_0}(E)$.

So now suppose $\mathcal{X}(v_0) \rightarrow \prod_{s(a)=v_0} \mathcal{X}(t(a))$ is an isomorphism. For any arrow b with $s(b) = v_0$ we then let $\mathcal{X}(v_0) = E' \oplus E''$ corresponding to the decomposition $\prod \mathcal{X}(t(a)) = \mathcal{X}(t(b)) \oplus \prod_{a \neq b} \mathcal{X}(t(a))$. Consider the subobject \mathcal{X}' of \mathcal{X} such that $\mathcal{X}'(v_0) = E'$ and then such that for $v \neq v_0$, $\mathcal{X}'(v) = 0$ if the unique path from v_0 to v does not go through b and such that $\mathcal{X}'(v) = \mathcal{X}(v)$ if it does.

Then let \mathcal{X}'' be the subobject of \mathcal{X} such that $\mathcal{X}''(v_0) = E''$ and then such that for $v \neq v_0$, $\mathcal{X}''(v) = 0$ if the path from v_0 to v does go through b and such that $\mathcal{X}''(v) = \mathcal{X}(v)$ otherwise. Checking the compatibility conditions we see that in fact \mathcal{X}' and \mathcal{X}'' are subobjects and then that $\mathcal{X} = \mathcal{X}' \oplus \mathcal{X}''$. So by the indecomposability of \mathcal{X} we get $\mathcal{X} = \mathcal{X}'$ or $\mathcal{X} = \mathcal{X}''$.

Now since $\mathcal{X}(v_0) \rightarrow \prod \mathcal{X}(t(a))$ is an isomorphism and since $\mathcal{X}(v_0) \neq 0$, there is at least one b such that $\mathcal{X}(t(b)) \neq 0$. Choosing this b we get that $\mathcal{X}' \neq 0$, and so that $\mathcal{X} = \mathcal{X}'$.

We note that our argument shows that for exactly one arrow b with $s(b) = v_0$ we get that $\mathcal{X}(b)$ is an isomorphism and that $\mathcal{X}(a) = 0$ for all $a \neq b$ with $s(a) = v_0$. Let $t(b) = v_1$ for this b . Now consider the subtree $T' \subset T$ of T with root v_1 and which contains all paths of T beginning at v_1 . If we repeat the arguments above with T' instead of T , we see that if we are in the first situation we get $\mathcal{X} = e_*^{v_1}(\mathcal{X}(v_0))$. If this does not occur, then the procedure above will give us a v_2 and a corresponding subtree $T'' \subset T$. Continuing, we see that if the procedure stops, we get that $\mathcal{X} = e_*^{v_n}(\mathcal{X}(v_0))$ for some vertex v_n . If the procedure does not stop, then the vertices v_0, v_1, v_2, \dots will determine an infinite path beginning at v_0 and so a vertex w at infinity such that $\mathcal{X} = e_*^w(\mathcal{X}(v_0))$. ■

Remark. The E and w such that $\mathcal{X} \cong e_*^w(E)$ are uniquely determined up to isomorphisms.

4 Injective Representations of Right Rooted Quivers

In this section we prove that a wide class of quivers, the class of so-called right rooted quivers, is contained in the class \mathfrak{I} of all source injective representation quivers.

The following definition is equivalent to the original one.

Definition 4.1 [9, Proposition 3.6] A quiver Q is *right rooted* (resp. *left rooted*) if and only if there exists no path of the form $\bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \dots$ in Q (resp. there exists no path of the form $\dots \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet$ in Q).

We use the same notation as that introduced in [9] to characterize right rooted quivers in terms of certain subsets of the set of vertices, that is, for a quiver $Q = (V, \Gamma)$ we shall define by transfinite induction the following subsets of V ,

$$V_0 = \{v \in V; \text{there exists no arrow } a \text{ of } Q \text{ with } s(a) = v\}.$$

For a successor ordinal α , we define

$$V_\alpha = \{v \in V^{\alpha-1}; \text{there exists no arrow } a \text{ of } Q^{\alpha-1} \text{ with } s(a) = v\},$$

where $Q^{\alpha-1} = (V^{\alpha-1}, \Gamma^{\alpha-1})$ is the subquiver of Q with $V^{\alpha-1} = V \setminus V_{\alpha-1}$ and $\Gamma^{\alpha-1} = \Gamma \setminus \{a \in \Gamma; t(a) \in V_{\alpha-1}\}$.

For a limit ordinal we define $V_\omega = \lim_{\rightarrow} V_\alpha = \cup_{\alpha < \omega} V_\alpha$ and $Q^\omega = (V^\omega, \Gamma^\omega)$ the subquiver of Q , where $V^\omega = V \setminus \cup_{\alpha < \omega} V_\alpha$ and $\Gamma^\omega = \Gamma \setminus \{a \in \Gamma; t(a) \in \cup_{\alpha < \omega} V_\alpha\}$.

Then if a quiver is right rooted there will exist an ordinal number λ such that $V = \cup_{\alpha < \lambda} V_\alpha$. The converse is also true by [9, Proposition 3.6].

Theorem 4.2 *Let Q be a right rooted quiver. Then Q is a source injective representation quiver.*

Proof We check that conditions (i) and (ii) of Proposition 2.1 are sufficient for guaranteeing that a representation is injective. So let \mathcal{E} be a representation of Q satisfying (i) and (ii) of Proposition 2.1. By transfinite induction on the set of vertices V_α , we shall construct a family of injective subrepresentations of \mathcal{E} whose direct product coincides with \mathcal{E} . Let $\mathcal{E}_0 = \prod_{v \in V_0} e_*^v(\mathcal{E}(v))$. Now, if α is a successor ordinal and $v \in V_{\alpha+1}$, it follows by hypothesis that

$$\mathcal{E}(v) = \left(\prod_{s(a)=v} \mathcal{E}(t(a)) \right) \oplus E_v^{\alpha+1}$$

(where the product $\prod_{s(a)=v} \mathcal{E}(t(a))$ is actually the product of $\mathcal{E}_\mu(v)$ with $\mu \leq \alpha$). Then we define

$$\mathcal{E}_{\alpha+1} = \prod_{v \in V_{\alpha+1}} e_*^v(E_v^{\alpha+1}).$$

Notice that $\mathcal{E}_{\alpha+1}(v) = 0$ for all $v \in V_\mu, \mu < \alpha + 1$. Now if γ is a limit ordinal, we define

$$\mathcal{E}_\gamma = \prod_{\alpha < \gamma} \mathcal{E}_\alpha.$$

Since the quiver is right rooted, it is easy to see that there will exist an ordinal number λ such that

$$\mathcal{E} = \prod_{\alpha < \lambda} \mathcal{E}_\alpha$$

and, since every \mathcal{E}_α is injective (e_* preserves injective objects), it follows that \mathcal{E} will be also injective. ■

Example 2 Consider the quiver $T \equiv \bullet \rightarrow \bullet$ and the representations $\mathcal{S} = S_1 \xrightarrow{\beta} S_0, \mathcal{X} = X_1 \xrightarrow{\alpha} X_0$. If

$$\begin{array}{ccc} S_1 & \xrightarrow{\beta} & S_0 \\ \nu_1 \downarrow & & \downarrow \nu_0 \\ X_1 & \xrightarrow{\alpha} & X_0 \end{array}$$

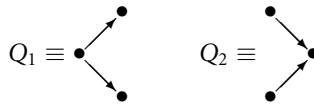
is a monomorphism of representations and

$$\begin{array}{ccc} S_1 & \xrightarrow{\beta} & S_0 \\ h_1 \downarrow & & \downarrow h_0 \\ E_1 & \xrightarrow{f} & E_0 \end{array}$$

is a morphism of representations with an injective representation $E_1 \xrightarrow{f} E_0$, we shall explicitly describe a morphism of representations (t_1, t_0) extending (h_1, h_0) via (ν_1, ν_0) . Since E_0 is injective, there exists $t_0: X_0 \rightarrow E_0$ such that $t_0 \circ \nu_0 = h_0$. On the other hand, since f splits, let g be a section. Then it follows that $E_1 = \text{Ker } f \oplus \text{Im } g$, so we may define a morphism $\gamma: X_1 \rightarrow \text{Ker } f$ such that $\gamma \circ \nu_1 = p_{\text{Ker } f} \circ h_1$ (where $p_{\text{Ker } f}$ denotes the canonical projection of E_1 over $\text{Ker } f$). Now we define $t_1: X_1 \rightarrow E_1$ as $t_1 = \gamma + g \circ t_0 \circ \alpha$. It is immediate that $f \circ t_1 = t_0 \circ \alpha$. Let us see that $t_1 \circ \nu_1 = h_1$, so let $s_1 \in S_1$ and suppose $h_1(s_1) = a + b$, with $a \in \text{Ker } f$ and $b \in \text{Im } g$ (so $b = g(x)$ for some $x \in E_0$). Then

$$\begin{aligned} (t_1 \circ \nu_1)(s_1) &= (\gamma \nu_1)(s_1) + (g t_0 \alpha \nu_1)(s_1) = p_{\text{Ker } f}(h_1(s_1)) + (g t_0 \nu_0 \beta)(s_1) \\ &= a + (g h_0 \beta)(s_1) = a + (g f h_1)(s_1) = a + (g f)(a + b) \\ &= a + (g f)(g(x)) = a + g(x) = a + b = h_1(s_1). \end{aligned}$$

Example 3 The method of constructing extensions given in the previous example can be easily generalized to many other right rooted quivers. For example,



$Q_3 \equiv \bullet \rightrightarrows \bullet$ or more generally to any dual tree or finite multiple (dual) tree. By the dual tree of a given one we mean a tree whose arrows are inverted, that is, a quiver with a vertex v_0 such that for any other vertex w there exists a unique path from w to v_0 . A multiple (dual) tree is defined as a (dual) tree that admits multiple arrows from a vertex another (so it is not properly a (dual) tree).

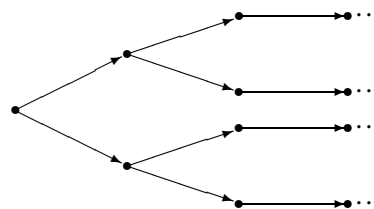
5 Injective Representations of A_∞^+

In this section we develop new techniques (necessarily very different from those of Section 4) to characterize injective representations of non rooted right quivers. We focus our attention on the quiver A_∞^+ and then generalize this argument to any quiver such that the connected components are barren trees (so a forest of barren trees). Barren trees are important because they appear naturally in the study of the existence of injective covers in $(Q, R\text{-Mod})$ (see [6, Theorem 3.6]) and also appear in the study of Gorenstein path rings, that is path rings which are left and right noetherian and whose projective objects have finite injective dimension (see [13]). We recall from [6] the definition of a barren tree: if T is a tree with root v , we can divide the set of vertices into “states” in such a way that the first state $V_1 = \{v\}$ and, for $i \in \mathbb{N}$,

$$V_{i+1} = \{w \in V : \text{there exists } a: v_i \rightarrow w, v_i \in V_i\}.$$

Then T is barren if the sequence $(n_i)_{i \in \mathbb{N}}$ stabilizes (where $n_i = |V_i|$).

Example 4 The following is an example of an infinite barren tree.



Let $(N_i)_{i \geq 0}$ be any family of modules. Let $U_n = \prod_{i=n}^\infty N_i$ and let $\mathcal{U} = U_0 \rightarrow U_1 \rightarrow \dots$ where each $U_n \rightarrow U_{n+1}$ is the projection map. Then given $\mathcal{M} = M_0 \xrightarrow{f_0} M_1 \xrightarrow{f_1} \dots$ and linear maps $\sigma_n: M_n \rightarrow N_n$, for each $n \geq 0$ there is a unique corresponding $\mathcal{M} \rightarrow \mathcal{U}$ where $M_n \rightarrow U_n = \prod_{i=n}^\infty N_i$ is such that the composition $M_n \rightarrow \prod_{i=n}^\infty N_i \rightarrow N_m$ for $m \geq n$ is σ_n if $m = n$ and is $\sigma_m \circ f_{m-1} \circ \dots \circ f_n$ if $m > n$. And in fact every morphism $\mathcal{M} \rightarrow \mathcal{U}$ is of this form. So it is easy to see that if each N_i is injective then \mathcal{U} is injective. Also by choosing the σ 's injective we can get an embedding into an injective representation. So every injective representation is a direct summand of such a \mathcal{U} .

Hence we consider direct sum decompositions of arbitrary representations \mathcal{M} , say $\mathcal{M} = \mathcal{M}' \oplus \mathcal{M}''$ where $\mathcal{M} = M_0 \rightarrow \dots$. Such a decomposition is given by decompositions $M_n = M'_n \oplus M''_n$ for each n , but which are compatible with the maps $M_n \rightarrow M_{n+1}$, i.e., which map M'_n into M'_{n+1} and M''_n into M''_{n+1} . So this just says $M_n \rightarrow M_{n+1}$ is homogeneous with respect to the decompositions. Given such a decomposition we easily see that each $\text{Ker}(M_n \rightarrow M_m)$ ($m \geq n$) is a homogeneous submodule (again with respect to the decomposition). This means that $K = (K \cap M'_n) \oplus (K \cap M''_n)$ where K is this kernel.

If now in fact \mathcal{M} is such that each $M_n \rightarrow M_{n+1}$ is surjective, then letting $\mathcal{M} = M \rightarrow M/K_0 \rightarrow M/K_1 \dots$ with $K_0 \subset K_1 \dots$ it is not hard to see that to get a decomposition of \mathcal{M} we only need give a direct sum decomposition $M = M' \oplus M''$ such that each K_i is a homogeneous submodule. Clearly this remark applies to any $M_0 \rightarrow M_1 \rightarrow \dots$ where all the maps are surjective. This means that to give a decomposition of this representation we only need give one of M_0 such that all the kernels $M_0 \rightarrow M_n$ are homogeneous for every $n \geq 1$.

We now apply these remarks.

Theorem 5.1 *Let $(E_i)_{i \geq 0}$ be a family of injective modules. Let $\bigoplus_{i=0}^{\infty} E_i \subset E$ be an injective envelope. Then the representation*

$$\bar{\mathcal{E}} = E \rightarrow E/E_0 \rightarrow E/(E_0 \oplus E_1) \rightarrow \dots$$

of the quiver A_{∞}^+ is an injective envelope of the representation

$$\mathcal{E} = \bigoplus_{i=0}^{\infty} E_i \rightarrow \bigoplus_{i=1}^{\infty} E_i \rightarrow \dots$$

of the quiver A_{∞}^+ .

Proof We use the earlier remarks to construct a known injective representation that is isomorphic to $\bar{\mathcal{E}}$. We let \mathcal{U} be the representation $\mathcal{U} = \prod_{i=0}^{\infty} E_i \rightarrow \prod_{i=1}^{\infty} E_i \rightarrow \dots$. This representation is injective. We will try to find a direct summand which is isomorphic to the representation of the theorem. So we construct a direct sum decomposition of \mathcal{U} . This representation is such that the maps $\prod_{i=n}^{\infty} E_i \rightarrow \prod_{i=n+1}^{\infty} E_i$ are all surjective. So we use this fact to construct our decomposition. Let $\bigoplus_{i=0}^{\infty} E_i \subset E' \subset \prod_{i=0}^{\infty} E_i$ where E' is an injective envelope of $\bigoplus_{i=0}^{\infty} E_i$.

Since E' is injective we have a decomposition $\prod_{i=0}^{\infty} E_i = E' \oplus E''$. We claim

$$\text{Ker}\left(\prod_{i=0}^{\infty} E_i \rightarrow \prod_{i=n}^{\infty} E_i\right) = E_0 \oplus \dots \oplus E_{n-1}$$

is homogeneous with respect to this decomposition. This is trivial since $E_0 \oplus \dots \oplus E_{n-1} \subset E'$ for each $n \geq 1$. So we now use our procedure and construct the corresponding decomposition $\mathcal{E} = \mathcal{E}' \oplus \mathcal{E}''$. By construction it is clear that \mathcal{E}' is isomorphic to the representation $\bar{\mathcal{E}} = E \rightarrow E/E_0 \rightarrow E/(E_0 \oplus E_1) \rightarrow \dots$. Hence this representation is injective.

To see that it is an envelope, note that $E/(E_0 \oplus \dots \oplus E_{n-1})$ is an injective envelope of $\bigoplus_{i=n}^{\infty} E_i$ for each n . Hence any $\bar{\mathcal{E}} \rightarrow \bar{\mathcal{E}}$ that makes the obvious diagram commutative gives automorphisms of each $E/(E_0 \oplus \dots \oplus E_{n-1})$ and so $\mathcal{E} \rightarrow \mathcal{E}$ is an automorphism. ■

We now want to give a characterization of injective representations. We will need the following lemma.

Lemma 5.2 *If E is an injective module then $\mathcal{E} = E \rightarrow E \rightarrow \dots$ is an injective representation.*

Proof This follows from the observation (made in the first part of the proof of Proposition 3.1) that morphisms $\mathcal{M} \rightarrow \mathcal{E}$ are given by maps $\varinjlim M_n \rightarrow E$. ■

So we note that any $\mathcal{E} = E_0 \rightarrow E_1 \rightarrow \dots$ with all E_n injective and each $E_n \rightarrow E_{n+1}$ an isomorphism is also an injective representation.

We note that there is a natural torsion pair on the category of representations where $\mathcal{M} = M_0 \rightarrow \dots$ is torsion if for each n and each $x \in M_n$ there is an $m > n$ such that $x \in \text{Ker}(M_n \rightarrow M_m)$. Then for an arbitrary representation \mathcal{M} we let $t(\mathcal{M})$ be the torsion subrepresentation. Note that $N_0 \rightarrow N_1 \rightarrow \dots$ torsion free just means that each $N_n \rightarrow N_{n+1}$ is an injection.

Theorem 5.3 *A representation $\mathcal{G} = G_0 \rightarrow \dots$ is injective if and only if each G_n is injective and each $G_n \rightarrow G_{n+1}$ is surjective with an injective kernel, that is, $A_\infty^+ \in \mathfrak{I}$.*

Proof By Proposition 2.1 we know the conditions are necessary, so assume these conditions. We want to argue that \mathcal{G} is injective. Let $E_0 = \text{ker}(G_0 \rightarrow G_1)$. Then E_0 is injective and is a submodule of the injective $\text{Ker}(G_0 \rightarrow G_2)$. So we can find an E_1 so that $E_0 \oplus E_1 = \text{Ker}(G_0 \rightarrow G_2)$. We proceed in this manner and find E_n for all n so that $\bigoplus_{i=0}^{n-1} E_i = \text{Ker}(G_0 \rightarrow G_n)$. Let $\bigoplus_{i=0}^\infty E_i \subset E \subset G_0$ where E is an injective envelope of $\bigoplus_{i=0}^\infty E_i$. Then up to isomorphism, $\bar{\mathcal{E}} = E \rightarrow E/E_0 \rightarrow E/(E_0 \oplus E_1) \rightarrow \dots$ is a subrepresentation of \mathcal{G} . By the above it is an injective representation and in fact is the injective envelope of $\mathcal{E} = \bigoplus_{i=0}^\infty E_i \rightarrow \bigoplus_{i=1}^\infty E_i \rightarrow \dots$. Clearly \mathcal{E} is $t(\mathcal{G})$. So $\mathcal{G} = \bar{\mathcal{E}} \oplus \mathcal{G}'$. Since $\bar{\mathcal{E}} \supset t(\mathcal{G})$, \mathcal{G}' is torsion free. But then \mathcal{G}' has all its terms injective and all its maps surjective. But this means all its maps are isomorphisms. Hence \mathcal{G}' is injective and so \mathcal{G} as the direct sum of two injectives is injective. ■

Remark 1. From the arguments it is easy to see that an injective representation is uniquely determined up to isomorphism by the family $(E_i)_{i \geq 0}$ as above and the single injective module E so that the torsion free quotient of the injective is isomorphic to $E \rightarrow E \rightarrow \dots$.

Remark 2. Since a torsion pair is stable if and only if injective envelopes of torsion objects are torsion we see that if our theory is stable, then using the notation above, $\bigoplus_{i=0}^\infty E_i$ must be its own injective envelope for any family $(E_i)_{i \geq 0}$ of injective modules. So this means the ring R must be left noetherian. If conversely R is left noetherian then by our arguments we see that for any injective \mathcal{G} , $t(\mathcal{G})$ is also injective. This quickly gives that injective envelopes of torsion objects are also torsion. So the torsion pair is stable. Hence we get that R is left noetherian if and only if our torsion pair is stable.

Theorem 5.3 allows us to prove that some quivers that are not right rooted quivers are also source injective representation quivers.

Corollary 5.4 *Let $\mathcal{E} = \cdots \rightarrow E_{-2} \rightarrow E_{-1} \rightarrow E_0 \rightarrow E_1 \rightarrow E_2 \rightarrow \cdots$ be a representation of the quiver*

$$A_\infty^\infty \equiv \cdots \rightarrow \bullet \rightarrow \bullet \rightarrow \cdots .$$

Then \mathcal{E} is injective if and only if E_i is an injective R -module and $E_i \rightarrow E_{i+1}$ is an splitting epimorphism for all $i \in \mathbb{Z}$, that is, the quiver A_∞^∞ lies in the class \mathfrak{I} .

Proof Let $0 \rightarrow \mathcal{S} \xrightarrow{g} \mathcal{X}$ and $\mathcal{S} \xrightarrow{h} \mathcal{E}$ be a morphism of representations of A_∞ (so $g = (g_i)_{i \in \mathbb{Z}}$ and $h = (h_i)_{i \in \mathbb{Z}}$). By Theorem 5.3 there exist morphisms $(t_i)_{i \geq 0}$ such that $(h_i) = (t_i) \circ (g_i)$ and (t_i) satisfying the usual compatibility conditions, for all $i \geq 0$. Now from $t_0: X_0 \rightarrow E_0$ we get, by Example 2, a morphism $t_{-1}: X_{-1} \rightarrow E_{-1}$ such that (t_{-1}, t_0) extends (h_{-1}, h_0) and verifies the compatibility condition. Now proceed inductively to get a family $t_{-i}: X_{-i} \rightarrow E_{-i}$, for all $i \geq 1$ by applying Example 2 iteratively. ■

We finish by characterizing injective representations for infinite barren trees.

Corollary 5.5 *Let Q be a forest whose connected components are barren trees. Then $Q \in \mathfrak{I}$.*

Proof We need to check that conditions (i) and (ii) of Proposition 2.1 are sufficient to get an injective representation.

It is clear that we only need to prove the result for a barren tree T . Since T is barren there exists $k \in \mathbb{N}$ such that $n_k = n_{k+i}$, for all $i \geq 0$. Now suppose that $V_k = \{v_1, \dots, v_m, v_{m+1}, \dots, v_{n_k}\}$, where we denote by v_1, \dots, v_m the vertices at infinity, so we have $v_j \equiv v_j^0 \rightarrow v_j^1 \rightarrow v_j^2 \cdots$ for $1 \leq j \leq m$ and suppose (without loss of generality) that $v_j^0 \in V_k$, for all $1 \leq j \leq m$.

Now let \mathcal{E} be a representation of T satisfying (i) and (ii) of Proposition 2.1 and let $0 \rightarrow \mathcal{S} \xrightarrow{g} \mathcal{X}, \mathcal{S} \xrightarrow{h} \mathcal{E}$ two morphisms of representations over T . Let us consider the corresponding restrictions $g(v_j)$ and $h(v_j)$ to the vertices at infinity of $V_k, 1 \leq j \leq m$. Then by Theorem 5.3, for each $1 \leq j \leq m$ there exists a family of morphisms

$$t(v_j) = \{t(v_j^i) : i \in \mathbb{N}\}$$

such that $t(v_j^i) \circ g(v_j^i) = h(v_j^i)$, for all $1 \leq j \leq m, i \in \mathbb{N}$ and such that for a fixed $1 \leq j \leq m$ the morphisms $\{t(v_j^i) : i \in \mathbb{N}\}$ induce a morphism of representations over the quiver $v_j \equiv v_j^0 \rightarrow v_j^1 \rightarrow v_j^2 \cdots$.

Now from the morphisms

$$\{t(v_j^0) : 1 \leq j \leq m\} \cup \{t(v_j) : m \leq j \leq n_k\}$$

relative to the vertices of V_k we may construct the corresponding extensions to the vertices of V_{k-1} as it is shown in Example 2 (and by the comments made in Example 3) and repeating until we reach the root $V_1 = \{v\}$. So at the end we will have a representation $t: \mathcal{X} \rightarrow \mathcal{E}$ extending $h: \mathcal{S} \rightarrow \mathcal{E}$ via $g: \mathcal{S} \rightarrow \mathcal{X}$ and satisfying the compatibility condition, that is, if $a: v \rightarrow w$ is an arrow of T , the diagram

$$\begin{array}{ccc}
 \mathcal{X}(v) & \xrightarrow{\mathcal{X}(a)} & \mathcal{X}(w) \\
 \downarrow t(v) & & \downarrow t(w) \\
 \mathcal{E}(v) & \xrightarrow{\mathcal{E}(a)} & \mathcal{E}(w)
 \end{array}$$

is commutative. ■

Remark. Let R be a fixed nonzero ring. If R is left noetherian and again our quiver is a tree T , then in [6] it was proved that $(T, R\text{-Mod})$ is a locally noetherian Grothendieck category if and only if T is barren. In this case every injective object of $(T, R\text{-Mod})$ is uniquely up to isomorphism the direct sum of the indecomposable injectives $e_*^w(E)$ of Section 3.

6 Applications: Gorenstein Representations

As an application of the results of the previous sections, we study and characterize the representations of finite injective dimension (resp. finite projective dimension) and its right orthogonal class (resp. its left orthogonal class) with respect to the Ext^1 functor, that is, the class of Gorenstein injective (resp. Gorenstein projective) representations. We also study Gorenstein flat representations and give an upper bound of the Gorenstein injective dimension of a representation having finite Gorenstein injective dimension on every module associated with each vertex.

We recall the following definition (see, for example, [8]).

Definition 6.1 An object M of an abelian category \mathcal{A} is said to be *Gorenstein injective* if there is an exact sequence

$$\cdots \rightarrow E_{-3} \rightarrow E_{-2} \rightarrow E_{-1} \rightarrow E_0 \rightarrow E_1 \rightarrow E_2 \rightarrow \cdots$$

of injective objects such that $M = \text{Ker}(E_0 \rightarrow E_1)$ and such that the sequence is $\text{Hom}(E, -)$ -exact for every injective object E (i.e., the functor $\text{Hom}(E, -)$ leaves the sequence exact).

We also recall the definition of a Gorenstein category given in [4].

Definition 6.2 We will say that a Grothendieck category \mathcal{A} is a *Gorenstein category* if the following hold:

1. For any object L of \mathcal{A} , $\text{projdim } L < \infty$ if and only if $\text{injdim } L < \infty$.
2. Finitistic projective dimension and finitistic injective dimension of \mathcal{A} are both finite, that is,

$$\text{FPD}(\mathcal{A}) = \sup\{\text{projdim}(M) : \text{projdim}(M) < \infty\}$$

and

$$\text{FID}(\mathcal{A}) = \sup\{\text{injdim}(M) : \text{injdim}(M) < \infty\}.$$

3. \mathcal{A} has a generator L such that $\text{projdim } L < \infty$.

Then it was proved in [5] that, for any arbitrary quiver Q , if R is such that $R\text{-Mod}$ is Gorenstein (for example if R is an n -Gorenstein ring), then $(Q, R\text{-Mod})$ is a Gorenstein category and so by [4, Theorem 2.25] the pair of classes $(\mathfrak{Q}, \mathfrak{Q}^\perp)$ where \mathfrak{Q} is the class of representations of finite injective dimension and \mathfrak{Q}^\perp is the class of Gorenstein injective representations, is a complete cotorsion pair. Analogously, the pair $({}^\perp\mathfrak{D}, \mathfrak{D})$ where \mathfrak{D} is the class of objects of finite projective dimension and ${}^\perp\mathfrak{D}$ is the class of Gorenstein projective representations, is also a complete cotorsion pair.

We now characterize the representations of the classes \mathfrak{Q} and \mathfrak{Q}^\perp (resp. ${}^\perp\mathfrak{D}$ and \mathfrak{D}) for source injective representation quivers.

Corollary 6.3 *Let Q be a source injective representation quiver and R be a Gorenstein ring (see [13]). A representation \mathcal{M} of the quiver Q is Gorenstein injective if and only if for each vertex v , $\mathcal{M}(v)$ is a Gorenstein injective R -module and the sequence*

$$0 \rightarrow \text{Ker}(f_v) \rightarrow \mathcal{M}(v) \xrightarrow{f_v} \prod_{s(a)=v} \mathcal{M}(t(a)) \rightarrow 0$$

is exact with $\text{Ker}(f_v)$ Gorenstein injective (where $f_v: \mathcal{M}(v) \rightarrow \prod_{s(a)=v} \mathcal{M}(t(a))$ is induced by the $\mathcal{M}(a): \mathcal{M}(v) \rightarrow \mathcal{M}(t(a))$).

Proof Suppose \mathcal{M} is Gorenstein injective, then there will exist an exact sequence of injectives

$$(6.1) \quad \cdots \rightarrow \mathcal{E}_{-2} \rightarrow \mathcal{E}_{-1} \rightarrow \mathcal{E}_0 \rightarrow \mathcal{E}_1 \rightarrow \mathcal{E}_2 \rightarrow \cdots$$

with $\mathcal{M} = \text{Ker}(\mathcal{E}_0 \rightarrow \mathcal{E}_1)$, which is $\text{Hom}(\mathcal{E}, -)$ exact, for all injective representations. Then, for a fixed vertex v we have the corresponding exact sequence of injective modules

$$\cdots \rightarrow \mathcal{E}_{-2}(v) \rightarrow \mathcal{E}_{-1}(v) \rightarrow \mathcal{E}_0(v) \rightarrow \mathcal{E}_1(v) \rightarrow \mathcal{E}_2(v) \rightarrow \cdots$$

with $\mathcal{M}(v) = \text{Ker}(\mathcal{E}_0(v) \rightarrow \mathcal{E}_1(v))$. Then, taking an integer sufficiently large in absolute value and applying [8, Theorem 9.1.11(7)], we will have that $\mathcal{M}(v)$ is Gorenstein injective for all vertex v . Now, since \mathcal{M} is Gorenstein injective, there will exist a short exact sequence of representations $0 \rightarrow \mathcal{U} \rightarrow \mathcal{E} \rightarrow \mathcal{M} \rightarrow 0$, with \mathcal{E} injective and \mathcal{U} Gorenstein injective. By Proposition 2.1, $\mathcal{E}(v) \rightarrow \prod_{s(a)=v} \mathcal{E}(t(a))$ is a splitting epimorphism, so it easily follows that $\mathcal{M}(v) \rightarrow \prod_{s(a)=v} \mathcal{M}(t(a))$ is an epimorphism. Finally let us show that $\text{Ker}(f_v)$ is Gorenstein injective. By Proposition 2.1 we have, from (6.1) the exact sequence of injective kernels:

$$\cdots \rightarrow \text{Ker}(t_v)^{-2} \rightarrow \text{Ker}(t_v)^{-1} \rightarrow \text{Ker}(t_v)^0 \rightarrow \text{Ker}(t_v)^1 \rightarrow \text{Ker}(t_v)^2 \rightarrow \cdots$$

where

$$0 \rightarrow \text{Ker}(t_v)^j \rightarrow E_j(v) \xrightarrow{t_v} \prod_{s(a)=v} E_j(t(a)) \rightarrow 0,$$

for all $j \in \mathbb{Z}$, such that $\text{Ker}(f_v) = \text{Ker}(\text{Ker}(t_v)^0 \rightarrow \text{Ker}(t_v)^1)$ which is $\text{Hom}(E, -)$ exact for all injective R -modules E , by a reasoning similar to the preceding.

Now we check that the conditions are sufficient. Since $(Q, R\text{-Mod})$ is a Gorenstein category we may find a short exact sequence $0 \rightarrow \mathcal{G} \rightarrow \mathcal{L} \rightarrow \mathcal{M} \rightarrow 0$, with \mathcal{G} a Gorenstein injective representation and \mathcal{L} a representation of finite injective dimension. By the previous proof $\mathcal{G}(v)$ is a Gorenstein injective R -module and $\text{Ker}(h_v)$ in the exact sequence

$$0 \rightarrow \text{Ker}(h_v) \rightarrow \mathcal{G}(v) \xrightarrow{h_v} \prod_{s(a)=v} \mathcal{G}(t(a)) \rightarrow 0$$

is also Gorenstein injective. Moreover, since \mathcal{L} is of finite injective dimension, it is obvious that $\mathcal{L}(v)$ and the kernel of $\mathcal{L}(v) \rightarrow \prod_{s(a)=v} \mathcal{L}(t(a))$ are also of finite injective dimension (and this map is a surjection). So by the definition of a source injective representation quiver it follows that \mathcal{L} is an injective representation, so \mathcal{M} will also be Gorenstein injective. ■

In a dual manner and using the results of [2] we can characterize Gorenstein projective representations of quivers.

Corollary 6.4 *Let Q be a sink projective representation quiver and let R be a Gorenstein ring. A representation \mathcal{M} of the quiver Q is Gorenstein projective if and only if $\mathcal{M}(v)$ is a Gorenstein projective R -module, the sequence*

$$0 \rightarrow \bigoplus_{t(a)=v} \mathcal{M}(s(a)) \xrightarrow{f_v} \mathcal{M}(v) \rightarrow \mathcal{C}(v) \rightarrow 0$$

is exact, and $\mathcal{C}(v)$ is Gorenstein projective (where $f_v: \bigoplus_{t(a)=v} \mathcal{M}(s(a)) \rightarrow \mathcal{M}(v)$ is induced by $\mathcal{M}(s(a)) \rightarrow \mathcal{M}(v)$).

We can also characterize the class of all representations of finite injective dimension for quivers in \mathfrak{S} .

Proposition 6.5 *A representation \mathcal{M} of a source injective representation quiver is of finite injective dimension if and only if $\sup\{\text{injdim}_R \mathcal{M}(v) < \infty : v \in V\}$ is finite.*

Proof It is obvious that the condition is necessary, for if $\text{injdim}_Q \mathcal{M} \leq n$, then, for all vertex $v \in V$, $\text{injdim}_R \mathcal{M}(v) \leq n$.

Conversely, suppose that $n = \sup\{\text{injdim}_R \mathcal{M}(v) < \infty : v \in V\}$, and let $v \in V$ be a vertex, then there exists an exact sequence of injective R -modules,

$$0 \rightarrow \mathcal{M}(v) \rightarrow E_0^v \rightarrow E_1^v \rightarrow \dots \rightarrow E_n^v \rightarrow 0$$

But then we have the short exact sequence of representations

$$0 \rightarrow \mathcal{M} \rightarrow \prod_{v \in V} e_*^v(E_0^v) \rightarrow \mathcal{M}_1 \rightarrow 0$$

such that $\text{injdim}_R \mathcal{M}_1(v) \leq n - 1$, for every $v \in V$. Now we repeat the same procedure to get \mathcal{M}_2 from \mathcal{M}_1 satisfying $\text{injdim}_R \mathcal{M}_2(v) \leq n - 2$, for all $v \in V$. So finally we get the exact

$$0 \rightarrow \mathcal{M} \rightarrow \mathcal{E}_0 \rightarrow \mathcal{E}_1 \rightarrow \dots \rightarrow \mathcal{E}_{n-1} \rightarrow \mathcal{M}_n \rightarrow 0$$

such that $\mathcal{M}_n(v)$ is injective for all vertex $v \in V$. We now show that $\text{injdim}_Q \mathcal{M}_n \leq 1$. If $0 \rightarrow \mathcal{M}_n \rightarrow \mathcal{E}_n \rightarrow \mathcal{C} \rightarrow 0$ is an exact sequence with \mathcal{E}_n injective, by Proposition 2.1, we have that $\mathcal{C}(v)$ is an injective R -module and $\mathcal{C}(v) \rightarrow \prod_{s(a)=v} \mathcal{C}(t(a))$ is an epimorphism with injective kernel, for all $v \in V$, so since the quiver is a source injective representation quiver, \mathcal{C} will be an injective representation. ■

Remark. Notice that the previous proposition shows that if $\sup\{\text{injdim}_R \mathcal{M}(v) < \infty : v \in V\} \leq n$ then, $\text{injdim}_Q \mathcal{M} \leq n + 1$. The converse clearly is not true.

When $(Q, R\text{-Mod})$ is a Gorenstein category, we can define the *Gorenstein injective dimension* of a representation \mathcal{M} as the least natural number k such that the n -th cosyzygy of \mathcal{M} is Gorenstein injective, that is, k is the least integer such that there exists an exact sequence

$$0 \rightarrow \mathcal{M} \rightarrow \mathcal{E}_0 \rightarrow \mathcal{E}_1 \rightarrow \dots \rightarrow \mathcal{E}_{k-1} \rightarrow \mathcal{G}_k \rightarrow 0$$

with \mathcal{E}_i injective $0 \leq i \leq k - 1$ and \mathcal{G}_k Gorenstein injective ($k = \infty$ if there exists no such a natural number). We denote the Gorenstein injective dimension of a representation by $\text{Ginjdim}_Q \mathcal{M}$.

Proposition 6.6 *Let $(Q, R\text{-Mod})$ be a Gorenstein category and let \mathcal{M} be a representation. Suppose that $\mathcal{M}(v)$ is a Gorenstein injective R -module for all $v \in V$, then $\text{Ginjdim}_Q \mathcal{M} \leq 1$.*

Proof We consider a short exact sequence $0 \rightarrow \mathcal{M} \rightarrow \mathcal{G} \rightarrow \mathcal{C} \rightarrow 0$ with \mathcal{G} a Gorenstein injective representation and $\text{injdim}_Q \mathcal{C} < \infty$. Let $\mathcal{E} \rightarrow \mathcal{G} \rightarrow 0$ be exact with \mathcal{E} injective. We have the following pullback diagram.

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 & & \mathcal{K} & \xlongequal{\quad} & \mathcal{K} & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & \mathcal{P} & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{C} \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \parallel \\
 0 & \longrightarrow & \mathcal{M} & \longrightarrow & \mathcal{G} & \longrightarrow & \mathcal{C} \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
 \end{array}$$

Now \mathcal{K} is a Gorenstein injective representation, so by Corollary 6.3 and the hypothesis, we have that $\mathcal{P}(v)$ is a Gorenstein injective R -module. Since $\text{injdim}_Q \mathcal{C} < \infty$ it follows that $\text{injdim}_Q \mathcal{P} < \infty$, so by Proposition 6.5 we conclude that $\mathcal{P}(v)$ is an injective R -module. But then the previous remark says that $\text{injdim}_Q \mathcal{P} \leq 1$. Now we form the pushout diagram of $0 \rightarrow \mathcal{P} \rightarrow \mathcal{E}(\mathcal{P})$ (the injective hull) and $\mathcal{P} \rightarrow \mathcal{M} \rightarrow 0$:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 & & \mathcal{K} & \equiv & \mathcal{K} & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & \mathcal{P} & \longrightarrow & \mathcal{E}(\mathcal{P}) & \longrightarrow & \mathcal{I} \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \parallel \\
 0 & \longrightarrow & \mathcal{M} & \longrightarrow & \mathcal{Q} & \longrightarrow & \mathcal{I} \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
 \end{array}$$

Since \mathcal{K} is Gorenstein injective, \mathcal{Q} will be Gorenstein injective, and since $\text{injdim}_Q \mathcal{P} \leq 1$, \mathcal{I} will be injective, so $\text{Ginjdim}_Q \mathcal{M} \leq 1$. ■

Remark. The argument of the previous proposition can be easily extended and shows that if $\text{Ginjdim}_Q \mathcal{M}(v) \leq k$, for all $v \in V$ then $\text{Ginjdim}_Q \mathcal{M} \leq k + 1$.

In [9], the authors characterize flat representations of left rooted quivers. We may use this result with Theorem 4.2 to relate flat and injective representations. We need to introduce the following notation. We denote by $(Q^{\text{op}}, R\text{-Mod})$ the category of representations over the quiver $Q^{\text{op}} = (V, \Gamma^{\text{op}})$, with the same set of vertices as Q and where the arrows are the arrows of Γ reversed (so the arrows from v to w in Γ^{op} are in one-to-one correspondence with the arrows from w to v in Γ). It is obvious that if Q is left rooted then Q^{op} is right rooted.

Corollary 6.7 *Let Q be a left rooted quiver. Then \mathcal{F} is a flat representation over $(Q, R\text{-Mod})$ if and only if \mathcal{F}^+ is an injective representation over the quiver $(Q^{\text{op}}, \text{Mod-}R)$ (where \mathcal{F}^+ is the representation given by $\mathcal{F}^+(v) = \text{Hom}_Z(\mathcal{F}(v), \mathbb{Q}/\mathbb{Z})$ and, for an arrow $a: v \rightarrow w$ and the morphism $f_a: \mathcal{F}(v) \rightarrow \mathcal{F}(w)$, $f_a^+ = \text{Hom}_Z(f_a, \mathbb{Q}/\mathbb{Z}): \mathcal{F}^+(w) \rightarrow \mathcal{F}^+(v)$)*

Proof This corollary follows from [9, Theorem 3.7] and Theorem 4.2 by noticing that

$$0 \rightarrow \bigoplus_{t(a)=v} \mathcal{F}(s(a)) \xrightarrow{f_v} \mathcal{F}(v) \xrightarrow{p_v} \mathcal{C}(v) \rightarrow 0$$

is a pure exact sequence of left R -modules if and only if

$$0 \rightarrow \mathcal{C}^+(v) \xrightarrow{p_v^+} \mathcal{F}^+(v) \xrightarrow{f_v^+} \prod_{s(a)=v} \mathcal{F}^+(t(a)) \rightarrow 0$$

is a split short exact sequence of right R -modules. ■

Definition 6.8 An object M of a monoidal Grothendieck category \mathcal{A} (see, for example, [15, p. 157] for the definition of a monoidal category) is said to be *Gorenstein flat* if there is an exact sequence

$$\cdots \rightarrow F_{-3} \rightarrow F_{-2} \rightarrow F_{-1} \rightarrow F_0 \rightarrow F_1 \rightarrow F_2 \rightarrow \cdots$$

of flat objects such that $M = \text{Ker}(F_0 \rightarrow F_1)$ and such that the sequence is $E \otimes -$ exact for every injective object E (i.e., the functor $E \otimes -$ leaves the sequence exact).

Lemma 6.9 Let Q be a left rooted quiver and R be a Gorenstein ring. Then if \mathcal{M} is a Gorenstein flat representation of Q , the representation \mathcal{M}^+ of the quiver Q^{op} is Gorenstein injective.

Proof Since \mathcal{M} is Gorenstein flat, there exists an exact sequence

$$(6.2) \quad \cdots \rightarrow \mathcal{F}_{-3} \rightarrow \mathcal{F}_{-2} \rightarrow \mathcal{F}_{-1} \rightarrow \mathcal{F}_0 \rightarrow \mathcal{F}_1 \rightarrow \mathcal{F}_2 \rightarrow \cdots$$

of flat representations with $\mathcal{M} = \text{Ker}(\mathcal{F}_0 \rightarrow \mathcal{F}_1)$ which remains exact when applying $\mathcal{E} \otimes -$, for all injective representation in $(Q, \text{Mod-}R)$, but then by Corollary 6.7 we have the exact sequence

$$\cdots \rightarrow \mathcal{F}_2^+ \rightarrow \mathcal{F}_1^+ \rightarrow \mathcal{F}_0^+ \rightarrow \mathcal{F}_{-1}^+ \rightarrow \mathcal{F}_{-2}^+ \rightarrow \cdots$$

of injective representations, such that $\mathcal{M} = \text{Ker}(\mathcal{F}_{-1}^+ \rightarrow \mathcal{F}_{-2}^+)$. We see that the last exact sequence is $\text{Hom}(\mathcal{E}, -)$ exact, for all injective representation \mathcal{E} . So let \mathcal{E} be an injective representation. From (6.2) we have an exact

$$\cdots \rightarrow \mathcal{E} \otimes \mathcal{F}_{-2} \rightarrow \mathcal{E} \otimes \mathcal{F}_{-1} \rightarrow \mathcal{E} \otimes \mathcal{F}_0 \rightarrow \mathcal{E} \otimes \mathcal{F}_1 \rightarrow \mathcal{E} \otimes \mathcal{F}_2 \rightarrow \cdots,$$

so we have an exact sequence

$$\cdots \rightarrow (\mathcal{E} \otimes \mathcal{F}_2)^+ \rightarrow (\mathcal{E} \otimes \mathcal{F}_1)^+ \rightarrow (\mathcal{E} \otimes \mathcal{F}_0)^+ \rightarrow (\mathcal{E} \otimes \mathcal{F}_{-1})^+ \rightarrow (\mathcal{E} \otimes \mathcal{F}_{-2})^+ \rightarrow \cdots$$

in Q^{op} . But the canonical isomorphism

$$(\mathcal{E} \otimes_{RQ} \mathcal{F}_i)^+ \cong \text{Hom}_{RQ}(\mathcal{E}, \mathcal{F}_i^+), \quad i \in \mathbb{Z}$$

still holds for rings without unit (as the case of RQ whenever Q has an infinite number of vertices) so we have the exact sequence

$$\begin{aligned} \cdots \rightarrow \text{Hom}(\mathcal{E}, \mathcal{F}_2^+) \rightarrow \text{Hom}(\mathcal{E}, \mathcal{F}_1^+) \rightarrow \text{Hom}(\mathcal{E}, \mathcal{F}_0^+) \rightarrow \text{Hom}(\mathcal{E}, \mathcal{F}_{-1}^+) \\ \rightarrow \text{Hom}(\mathcal{E}, \mathcal{F}_{-2}^+) \rightarrow \cdots \quad \blacksquare \end{aligned}$$

Lemma 6.10 Let Q be a forest whose connected components are barren trees and ${}_R R$ be noetherian. Then a representation \mathcal{E} of Q is injective if and only if \mathcal{E}^+ is flat (over Q^{op})

Proof We will use the characterization of flat representations given in [9, Theorem 3.7] and Corollary 5.5. Notice that, for all $v \in V$, $\mathcal{E}(v)$ is an injective R -module if and only if $\mathcal{E}^+(v)$ is a flat R -module (see for example [8, Corollary 3.2.17]) and furthermore the sequence

$$0 \rightarrow \mathcal{K}(v) \rightarrow \mathcal{E}(v) \rightarrow \prod_{s(a)=v} \mathcal{E}(t(a)) \rightarrow 0$$

is exact, with $\mathcal{K}(v)$ injective if and only if

$$0 \rightarrow \bigoplus_{t(a)=v} \mathcal{E}^+(s(a)) \rightarrow \mathcal{E}^+(v) \rightarrow \mathcal{K}^+(v) \rightarrow 0$$

is an exact sequence with $\mathcal{K}^+(v)$ flat (so hence pure exact sequence) for all $v \in V$. Notice that, since the connected components of Q are barren trees, then, for every vertex $v \in V$, the set $\{a \in \Gamma : s(a) = v\}$ is finite. ■

Proposition 6.11 *Let Q be a forest whose connected components are barren trees and let R be Gorenstein. A representation \mathcal{M} of the quiver Q is Gorenstein flat if and only if for each vertex v , $\mathcal{M}(v)$ is a Gorenstein flat R -module and the sequence*

$$0 \rightarrow \bigoplus_{t(a)=v} \mathcal{M}(s(a)) \xrightarrow{f_v} \mathcal{M}(v) \rightarrow \mathcal{C}(v) \rightarrow 0$$

is exact and $\mathcal{C}(v)$ is Gorenstein flat (where $f_v: \bigoplus_{t(a)=v} \mathcal{M}(s(a)) \rightarrow \mathcal{M}(v)$ is a morphism induced by $\mathcal{M}(s(a)) \rightarrow \mathcal{M}(v)$).

Proof Necessity. By Lemma 6.9, \mathcal{M}^+ is Gorenstein injective, so by Corollary 6.3 $\mathcal{M}^+(v)$ is a Gorenstein injective R -module for all $v \in V$, but then $\mathcal{M}(v)$ is a Gorenstein flat R -module. Furthermore since

$$0 \rightarrow \mathcal{C}^+(v) \rightarrow \mathcal{M}^+(v) \rightarrow \prod_{s(a)=v} \mathcal{M}^+(t(a)) \rightarrow 0$$

is exact with $\mathcal{C}^+(v)$ Gorenstein injective, then

$$0 \rightarrow \bigoplus_{t(a)=v} \mathcal{M}(s(a)) \rightarrow \mathcal{M}(v) \rightarrow \mathcal{C}(v) \rightarrow 0$$

is exact with $\mathcal{C}(v)$ Gorenstein flat, for every $v \in V$.

Sufficiency. Since R is Gorenstein, the hypothesis is equivalent to saying that $\mathcal{M}^+(v)$ is Gorenstein injective and that

$$0 \rightarrow \mathcal{C}^+(v) \rightarrow \mathcal{M}^+(v) \rightarrow \prod_{s(a)=v} \mathcal{M}^+(t(a)) \rightarrow 0$$

is exact with $\mathcal{C}^+(v)$ Gorenstein injective. But, by Corollary 6.3, this implies that \mathcal{M}^+ is a Gorenstein injective representation. So there exists an exact sequence

$$\mathcal{E}_n \rightarrow \mathcal{E}_{n-1} \rightarrow \dots \mathcal{E}_0 \rightarrow \mathcal{M}^+ \rightarrow 0$$

with \mathcal{E}_i injective representations for all i such that $1 \leq i \leq n$. Now, by Lemma 6.10,

$$0 \rightarrow \mathcal{M}^{++} \rightarrow \mathcal{E}_0^+ \rightarrow \cdots \rightarrow \mathcal{E}_{n-1}^+ \rightarrow \mathcal{E}_n^+$$

is an exact sequence of flat representations. So $\text{Tor}^i(\mathcal{L}, \mathcal{M}^{++}) = 0$, for all $i \geq 1$ and for all representation \mathcal{L} such that $\text{injdim}_Q \mathcal{L} < \infty$. But now there exists a pure exact sequence $0 \rightarrow \mathcal{M} \rightarrow \mathcal{M}^{++} \rightarrow \mathcal{M}^{++}/\mathcal{M} \rightarrow 0$ of representations, so $\text{Tor}^1(\mathcal{L}, \mathcal{M}^{++}/\mathcal{M}) = 0$ for all such \mathcal{L} . But then $\text{Tor}^i(\mathcal{L}, \mathcal{M}^{++}/\mathcal{M}) = 0$, for every $i \geq 1$. So then $\text{Tor}^i(\mathcal{L}, \mathcal{M}) = 0$ for each $i \geq 1$ and for all representations \mathcal{L} such that $\text{injdim}_Q \mathcal{L} < \infty$. But this means that if we take a complex

$$\cdots \rightarrow \mathcal{F}_{-2} \rightarrow \mathcal{F}_{-1} \rightarrow \mathcal{F}_0 \rightarrow \mathcal{F}_1 \rightarrow \mathcal{F}_2 \rightarrow \cdots$$

such that $\mathcal{M} = \text{Ker}(\mathcal{F}_0 \rightarrow \mathcal{F}_1)$ with \mathcal{F}_i a flat representation for all $i \in \mathbb{Z}$, then this will be $\mathcal{E} \otimes -$ exact for all injective representation \mathcal{E} . Furthermore, by [5, Theorem 3.7], $\text{injdim}_{RQ} RQ < \infty$ so then the complex will be exact. ■

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References

- [1] I. Assem, D. Simson and A. Skowroński, *Elements of the representation theory of associative algebras. Vol. 1. Techniques of representation theory*. London Mathematical Society Student Texts 65, Cambridge University Press, Cambridge, 2006.
- [2] E. Enochs and S. Estrada, *Projective representations of quivers*. *Comm. Algebra*. **33**(2005), no. 10, 3467–3478.
- [3] ———, *Relative homological algebra in the category of quasi-coherent sheaves*. *Adv. Math.* **194**(2005), no. 2, 284–295.
- [4] E. Enochs, S. Estrada, and J. R. García Rozas, *Gorenstein categories and Tate cohomology on projective schemes*. *Math. Nachr.* **281**(2008), no. 4, 525–540.
- [5] E. Enochs, S. Estrada, J. R. García Rozas, and A. Iacob, *Gorenstein quivers*. *Arch. Math (Basel)* **88**(2007), no. 3, 199–206.
- [6] E. Enochs, J. R. García Rozas, L. Oyonarte, and S. Park, *Noetherian quivers*. *Quaest. Math.* **25**(2002), no. 4, 531–538.
- [7] E. Enochs and I. Herzog, *A homotopy of quiver morphisms with applications to representations*. *Canad. J. Math.* **51**(1999), no. 2, 294–308.
- [8] E. Enochs and O. Jenda, *Relative homological algebra*. de Gruyter Expositions in Mathematics 30, Walter de Gruyter, Berlin, 2000.
- [9] E. Enochs, L. Oyonarte, and B. Torrecillas, *Flat covers and flat representations of quivers*. *Comm. Algebra*. **32**(2004), no. 4, 1319–1338.
- [10] S. Estrada, *Monomial algebras over infinite quivers. Applications to N-complexes of modules*. *Comm. Algebra* **35**(2007), no. 10, 3214–3225.
- [11] P. Gabriel, *Unzerlegbare Darstellungen I*. *Manuscripta Math.* **6**(1972), 71–103.
- [12] M. Hovey, *Model categories*. Mathematical Surveys and Monographs 63, American Mathematical Society, Providence, RI, 1999.
- [13] Y. Iwanaga, *On rings with finite self-injective dimension*. *Comm. Algebra* **7**(1979), no. 4, 393–414.
- [14] L. Le Bruyn and C. Procesi, *Semisimple representations of quivers*. *Trans. Amer. Math. Soc.* **317**(1990), no. 2, 585–598.

- [15] S. Mac Lane, *Categories for the working mathematician*. Second ed., Graduate Texts in Mathematics 5, Springer-Verlag, New York, 1998.

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