9
World-volume curvature couplings

We've now seen that we can construct D-branes which, in superstring theory, have important extra properties. Much of what we have learned about them in the bosonic theory is still true here of course, a key result being that the world-volume dynamics is governed by the dynamics of open strings, etc. Still relevant is the Dirac–Born–Infeld action (equation (5.21)) for the coupling to the background NS–NS fields,

$$S_{\rm DBI} = -\tau_p \int_{\mathcal{M}_{p+1}} d^{p+1}\xi\, e^{-\Phi} \det{}^{1/2}(G_{ab} + B_{ab} + 2\pi\alpha' F_{ab}), \tag{9.1}$$

and the non-Abelian extensions mentioned later in chapter 5.

As we have seen in the previous chapter, for the R–R sector, they are sources of $C_{(p+1)}$. We therefore also have the Wess–Zumino-like term

$$S_{\rm WZ} = \mu_p \int_{\mathcal{M}_{p+1}} C_{(p+1)}. \tag{9.2}$$

Perhaps not surprisingly, there are other terms of great importance, and this chapter will uncover a number of them. In fact, there are many ways of deducing that there *must* be other terms, and one way is to use the fact that D-branes turn into each other under T-duality.

9.1 Tilted D-branes and branes within branes

There are additional terms in the action involving the D-brane gauge field. Again these can be determined from T-duality. Consider, as an example, a D1-brane in the 1–2 plane. The action is

$$\mu_1 \int dx^0\, dx^1 \,\Big(C_{01} + \partial_1 X^2 C_{02}\Big). \tag{9.3}$$

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Under a T-duality in the x^2-direction this becomes

$$\mu_2 \int dx^0 dx^1 dx^2 \ (C_{012} + 2\pi\alpha' F_{12} C_0). \tag{9.4}$$

We have used the T-transformation of the C fields as discussed in section 8.1.1, and also the recursion relation (5.11) between D-brane tensions.

This has an interesting interpretation. As we saw before in section 5.2.1, a Dp-brane tilted at an angle θ is equivalent to a D$(p+1)$-brane with a constant gauge field of strength $F = (1/2\pi\alpha')\tan\theta$. Now we see that there is additional structure: the flux of the gauge field couples to the R–R potential $C^{(p)}$. In other words, the flux acts as a source for a D$(p-1)$-brane living in the world-volume of the D$(p+1)$-brane. In fact, given that the flux comes from an integral over the whole world-volume, we cannot localise the smaller brane at a particular place in the world-volume: it is 'smeared' or 'dissolved' in the world-volume.

In fact, we shall see when we come to study supersymmetric combinations of D-branes that supersymmetry requires the D0-brane to be completely smeared inside the D2-brane. It is clear here how it manages this, by being simply T-dual to a tilted D1-brane. We shall see many consequences of this later.

9.2 Anomalous gauge couplings

The T-duality argument of the previous section can be generalised to discover more terms in the action, but we shall take another route to discover such terms, exploiting some important physics in which we already have invested considerable time.

Let us return to the type I string theory, and the curious fact that we had to employ the Green–Schwarz mechanism (see section 7.1.4, where we mixed a classical and a quantum anomaly in order to achieve consistency). Focusing on the gauge sector alone for the moment, the classical coupling which we wrote in equation (7.35) implies a mixture of the two-form $C_{(2)}$ with gauge field strengths:

$$S = \frac{1}{3\times 2^6 (2\pi)^5 \alpha'} \int C_{(2)} \left(\frac{\mathrm{Tr}_{\mathrm{adj}}(F^4)}{3} - \frac{[\mathrm{Tr}_{\mathrm{adj}}(F^2)]^2}{900} \right). \tag{9.5}$$

We can think of this as an interaction on the world-volume of the D9-branes showing a coupling to a D1-brane, analogous to that which we saw for a D0-brane inside a D2-brane in equation (9.4). This might seem a bit

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of a stretch, but let us write it in a different way:

$$S = \mu_9 \int \frac{(2\pi\alpha')^4}{3\times 2^6} C_{(2)} \left(\frac{\mathrm{Tr}_{\mathrm{adj}}(F^4)}{3} - \frac{[\mathrm{Tr}_{\mathrm{adj}}(F^2)]^2}{900} \right)$$
$$= \mu_9 \int \frac{(2\pi\alpha')^4}{4!} C_{(2)} \mathrm{Tr}(F^4), \tag{9.6}$$

where, crucially, in the last line we have used the properties (7.39) of the traces for $SO(32)$ to rewrite things in terms of the trace in the fundamental.

Another exhibit we would like to consider is the kinetic term for the modified three-form field strength, $\widetilde{G}_{(3)}$, which is

$$S = -\frac{1}{4\kappa_0^2} \int \widetilde{G}_{(3)} \wedge^* \widetilde{G}_{(3)}. \tag{9.7}$$

Since $d\omega_{3\mathrm{Y}} = \mathrm{Tr}(F\wedge F)$ and $d\omega_{3\mathrm{L}} = \mathrm{Tr}(R\wedge R)$, this gives, after integrating by parts and, dropping the parts with R for now:

$$S = \frac{\alpha'}{4\kappa^2} \int C_{(6)} \wedge \left(\frac{1}{30} \mathrm{Tr}_{\mathrm{adj}}(F\wedge F) \right)$$
$$= \mu_9 \int \frac{(2\pi\alpha')^2}{2} C_{(6)} \wedge (\mathrm{Tr}(F\wedge F)) \tag{9.8}$$

again, we have converted the traces using (7.39), we've used the relation (7.44) for κ_0 and we've recalled the definition (7.38).

Upon consideration of the three examples (9.4), (9.6), and (9.8), it should be apparent that a pattern is forming. The full answer for the gauge sector is the result[118, 119]

$$\mu_p \int_{\mathcal{M}_{p+1}} \left[\sum_p C_{(p+1)} \right] \wedge \mathrm{Tr}\, e^{2\pi\alpha' F + B}, \tag{9.9}$$

(We have included non-trivial B on the basis of the argument given in section 5.2.) So far, the gauge trace (which is in the fundamental) has the obvious meaning. We note that there is the possibility that in the full non-Abelian situation, the C can depend on *non-commuting* transverse fields X^i, and so we need something more general. We will return to this later. The expansion of the integrand (9.9) involves forms of various rank; the notation means that the integral picks out precisely the terms whose rank is $(p+1)$, the dimension of the Dp-brane's world-volume.

Looking at the first non-trivial term in the expansion of the exponential in the action we see that there is the term that we studied above corresponding to the dissolution of a D$(p-2)$-brane into the sub two-plane

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in the Dp-brane's world volume formed by the axes X^i and X^j, if field strength components F_{ij} are turned on.

At the next order, we have a term which is quadratic in F which we could rewrite as:

$$S = \frac{\mu_{p-4}}{8\pi^2}\int C_{(p-3)} \wedge \mathrm{Tr}(F \wedge F). \tag{9.10}$$

We have used the fact that $\mu_{p-4}/\mu_p = (2\pi\sqrt{\alpha'})^4$. Recall that there are non-Abelian field configurations called 'instantons' for which the quantity $\int \mathrm{Tr}(F \wedge F)/8\pi^2$ gives integer values. (See, for example, insert 9.4.) Interestingly, we see that if we excite an instanton configuration on a four dimensional sub-space of the Dp-brane's world-volume, it is equivalent to precisely one unit of D$(p-4)$-brane charge, which is remarkable.

In trying to understand what might be the justification (other than T-duality) for writing the full result (9.9) for all branes so readily, the reader might recognise something familiar about the object we built the action out of. The quantity $\exp(iF/(2\pi))$, using a perhaps more familiar normalisation, generates polynomials of the Chern classes of the gauge bundle of which F is the curvature. It is called the Chern character. In the Abelian case we first studied, we had non-vanishing first Chern class $\mathrm{Tr}F/(2\pi)$, which after integrating over the manifold, gives a number which is in fact quantised. For the non-Abelian case, the second Chern class $\mathrm{Tr}(F\wedge F)/(8\pi^2)$ computes the integer known as the instanton number, and so on.

These numbers, being integers, are topological invariants of the gauge bundle. By the latter, we mean the fibre bundle of the gauge group over the world-volume, for which the gauge field A is a connection.

A fibre bundle is a rule for assigning a copy of a certain space (the fibre: in this case, the gauge group G) to every point of another space (the base: here, the world-volume). The most obvious case of this is simply a product of two manifolds (since one can be taken as the base and then the product places a copy of the other at every point of the base), but this is awfully trivial. More interesting is to have only a product space locally. Then, the whole structure of the bundle is given by a collection of such local products glued together in an overlapping way, together with a set of transition functions which tell one how to translate from one local patch to another on the overlap. In the case of a gauge theory, this is all familiar. The transition rule is simply a G gauge transformation, and we are allowed to use the term 'vector bundle' in this case. For the connection or gauge field this is: $A \to gAg^{-1} + gdg^{-1}$. So the gauge field is not globally defined. Perhaps the most familiar gauge bundle is the monopole bundle corresponding to a Dirac monopole. See insert 9.1.

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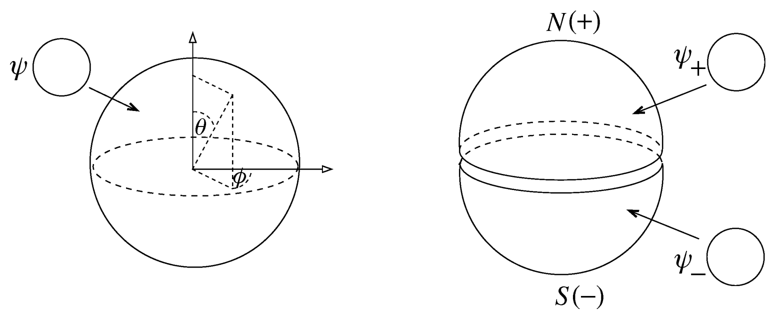
Insert 9.1. The Dirac monopole as a gauge bundle

A gauge bundle is sometimes called a principal fibre bundle. Perhaps everybody's favourite gauge bundle is the Dirac monopole. Take a sphere S^2 as our base. We will fibre a circle over it. Recall that S^2 cannot be described by a global set of coordinates, but we can use two, the Northern and the Southern hemisphere, with overlap in the vicinity of the Equator. Put standard polar coordinates (θ, ϕ) on S^2, where $\theta = \pi/2$ is the Equator. Put an angular coordinate $e^{i\psi}$ on the circle. We will use ϕ_+ in the North and ψ_- in the South.
So our bundle is a copy of two patches which are locally $S^2 \times S^1$,

$$+\text{Patch}: \quad \{\theta, \phi, e^{i\psi_+}\}; \qquad -\text{Patch}: \quad \{\theta, \phi, e^{i\psi_-}\},$$

together with a transition function which relates them.



The relation between the two can be chosen to be

$$e^{i\psi_-} = e^{in\phi} e^{i\psi_+},$$

where n is an integer, since as we go around the equator, $\phi \to \phi + 2\pi$, the gluing together of the fibres must still make sense.
The boring case $n = 0$ is sensible, but it simply gives the trivial bundle $S^2 \times S^1$. The case $n = 1$ is the familiar Hopf fibration, which describes the manifold S^3 as a circle bundle over S^2. It is a Dirac monopole of unit charge. Higher values of n give charge n monopoles. The integer n is characteristic of the bundle. It is in fact (minus) the integral of the first Chern class.

The reader who found this a little dry might turn straight to insert 9.2 where we describe the connection on the bundle and compute the first Chern class explicitly.

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Insert 9.2. The first Chern class of the Dirac bundle

Following what we did in insert 9.1, we can uncover more features, which will be useful later on. A natural choice for the connection one-form (gauge potential) in each patch is simply

$$+\text{Patch}: \quad A_+ + d\psi_+; \qquad -\text{Patch}: \quad A_- + d\psi_-,$$

so that the transition function defined in insert 9.1 allows us to connect the two patches, defining the standard $U(1)$ gauge transformation

$$A_+ = A_- + n\phi.$$

Here are the gauge potentials which are standard in this example:

$$A_\pm = n\frac{(\pm 1 - \cos\theta)}{2}d\phi,$$

which, while being regular almost everywhere, clearly have a singularity (the famous Dirac string) in the $\mp$ patch. The curvature two-form is simply

$$F = dA = \frac{n}{2}\sin\theta d\theta \wedge d\phi.$$

This is a closed form, but it is not exact, since there is not a unique answer to what A can be over the whole manifold. According to what we describe in the text, we can compute the first Chern number by integrating the first Chern class to get:

$$\int_{S^2}\frac{F}{2\pi} = \int_+\frac{F}{2\pi} + \int_-\frac{F}{2\pi} = n.$$

9.3 Characteristic classes and invariant polynomials

The topology of a particular fibration can be computed by working out just the right information about its collection of transition functions. For a gauge bundle, the field strength or curvature two-form $F = dA + A\wedge A$ is a nice object with which to go and count, since it is globally defined over the whole base manifold. When the group is Abelian, $F = dA$ and so $dF = 0$. If the bundle is not trivial, then we can't write F as dA everywhere and so F is closed but not exact. Then F is said to be an element of the cohomology group $H^2(\mathcal{B},\mathbb{R})$ of the base, which we'll call $\mathcal{B}$. The first Chern class $F/2\pi$ defines an integer when integrated over $\mathcal{B}$,

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Insert 9.3. The Yang–Mills instanton as a gauge bundle

A favourite non-Abelian example[120] is the $SU(2)$ Yang–Mills instanton. The base is S^4, with coordinates (r, θ, ϕ, ψ), which is $\mathbb{R}^4$ with the point at infinity added. A metric on it for radius $\rho/2$ is:

$$ds^2 = \left(1 + \frac{r^2}{\rho^2}\right)^{-2} \left(dr^2 + r^2(\sigma_1^2 + \sigma_2^2 + \sigma_3^2)\right).$$

The gauge group (fibre) is $G = SU(2)$, which also happens to be the manifold S^3. By analogy with what we saw in insert 9.1, we can divide the S^4 into Northern and Southern hemispheres. The equator is in fact an S^3 and that is where we define the overlap region. Recall that there is a natural $SU(2)$ favoured writing of the coordinates, defining an element $h(\theta, \phi, \psi) \in SU(2)$ as in insert 7.4. We can define similar Euler angles (α, β, γ) as coordinates on the fibre g, for the North $(+)$ and South $(-)$ patches, giving:

$$+\text{Patch}: \quad \{\theta, \phi, \psi, \alpha_+, \beta_+, \gamma_+\}; \quad -\text{Patch}: \quad \{\theta, \phi, \psi, \alpha_-, \beta_-, \gamma_-\}.$$

Our transition functions at the equator, taking us from the North to the South fibres are again parametrised by an integer, k:

$$g(\alpha_+, \beta_+, \gamma_+) = h^k(\theta, \phi, \psi) g(\alpha_-, \beta_-, \gamma_-).$$

Again $k = 0$ is trivial. The case $k = 1$ is the Hopf fibration of S^7 as an S^3 over S^4. It is the one instanton solution. Other k are the multi-instantons. Also k will give the second Chern class of the bundle.

telling us to which topological class F belongs; this integer is a topological invariant.

For the non-Abelian case, F is no longer closed, and so we don't have the first Chern class. However, the quantity $\mathrm{Tr}(F \wedge F)$ is closed, since as we know from insert 7.3 (p. 167), it is actually $d\omega_{3\mathrm{Y}}$. So if the Chern–Simons three-form $\omega_{3\mathrm{Y}}$ is not globally defined, we have a non-trivial bundle, and $\mathrm{Tr}(F \wedge F)$, being closed but not exact, defines an element of the cohomology group $H^4(\mathcal{B}, \mathbb{R})$. The second Chern class $\mathrm{Tr}(F \wedge F)/8\pi^2$ integrated over $\mathcal{B}$ gives an integer which says to which topological class F belongs. See insert 9.4.

As we have said above, D-branes appear to compute certain topological features of the gauge bundle on their world-volumes, corresponding here

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Insert 9.4. The BPST one-instanton connection

Just as with the Dirac monopole case, we can write the connection 1-form for each patch:

$$+\text{Patch}: \quad g_+^{-1}A_+g_+ + g_+^{-1}dg_+; \qquad -\text{Patch}: \quad g_-^{-1}A_-g_- + g_-^{-1}dg_-,$$

so that the transition function defined in insert 9.3 allows us to connect the two patches with a gauge transformation

$$A_+ = h^k A_- h^{-k} + h^k dh^{-k}.$$

The $k = 1$ solution can be written quite simply:

$$A_+ = \frac{r^2}{r^2+\rho^2} h^{-1}dh = \frac{r^2}{r^2+\rho^2} i\tau_n\sigma_n,$$

where the σ_n are the left-invariant one-forms. This solution is smooth everywhere except at a singularity at $r = 0$. The South pole solution is obtained by gauge transformation:

$$A_- = hA_+h^{-1} + hdh^{-1} = -\frac{\rho^2}{\rho^2+r^2} dhh^{-1} = \frac{\rho^2}{\rho^2+r^2} i\tau_n\bar{\sigma}_n,$$

where the $\bar{\sigma}_n$ are the right-invariant one-forms. This solution is singular at $r = \infty$. The curvature two-form is best described using the veilbiens $\{e^0, e^1, e^2, e^3\} = (1 + r/\rho)^{-2}\{dr, r\sigma_1, r\sigma_2, r\sigma_3\}$:

$$F_+ = dA + A \wedge A = i\tau_k \frac{2}{\rho^2}\left(e^0 \wedge e^k + \frac{1}{2}\epsilon_{kij} e^i \wedge e^j\right).$$

Of course, $F_- = hF_+h^{-1}$. It is worth checking that this solution is self dual, i.e. ${}^*F = F$, with anti-self duality made by $\sigma_n \leftrightarrow \bar{\sigma}_n$. The instanton number is (minus) the second Chern class integrated over the S^4:

$$k = -\frac{1}{8\pi^2}\int_{S_4} \text{Tr}(F \wedge F) = \frac{1}{8\pi^2}\left(\frac{48}{\rho^4}\right)\int_{S_4} e^0 \wedge e^1 \wedge e^2 \wedge e^3 = 1,$$

where in the latter we have used that the volume of the S^4 is $\pi^2\rho^4/6$. Here, ρ has the interpretation as the 'core size' of the instanton.

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to the Chern classes of the cohomology. As we shall see, they compute other topological numbers as well, and so let us pause to appreciate a little of the tools that they employ, in order to better be able to put them to work for us.

The first and second Chern classes shall be denoted $c_1(F)$ and $c_2(F)$ and so on, $c_j(F)$ for the jth Chern class. Let us call the gauge group G, and keep in mind $U(n)$ (we will make appropriate modifications to our statements to include $O(n)$ later). We'd like to know how to compute the $c_j(F)$. The remarkable thing is that they arise from forming polynomials in F which are invariant under G. Forget that F is a two-form for now, and just think of it as an $n\times n$ matrix. The $c_j(F)$ are found by expanding a natural invariant expression in F as a series in t:

$$\det\left(t\mathbf{I}+\frac{iF}{2\pi}\right)=\sum_{j=0}^{n}c_{n-j}(F)t^j. \tag{9.11}$$

(Here, we use the i in F to keep the expression real, since $U(N)$ generators are anti-Hermitean.) The great thing about this is that there is an excellent trick for finding explicit expressions for the c_js which will allow us to manipulate them and relate them to other quantities. Assume that the matrix $iF/2\pi$ has been diagonalised. Call this diagonal matrix X, with n distinct non-vanishing eigenvalues x_i, $i=1,\ldots,n$. Then we have

$$\det(t\mathbf{I}+X)=\prod_{i=1}^{n}(t+x_i)=\sum_{j=0}^{n}c_{n-j}(x)t^j, \tag{9.12}$$

and we find by explicit computation that the c_js are symmetric polynomials:

$$\begin{aligned}
c_0 &= 1,\quad c_1=\sum_{i}^{n}x_i,\quad c_2=\sum_{i_1<i_2}^{n}x_{i_1}x_{i_2},\ \ldots\\
c_j &= \sum_{i_1<i_2<\cdots i_j}^{n}x_{i_1}x_{i_2}\ldots x_{i_j},\quad c_n=x_1x_2\ldots x_n.
\end{aligned} \tag{9.13}$$

Now rewrite the expressions on the eigenvalues back as matrix expressions in terms of X, and then replace X by $iF/2\pi$, to get:

$$\begin{aligned}
c_0(F) &= 1,\quad c_1(F)=\frac{i}{2\pi}\mathrm{Tr}F,\\
c_2(F) &= \frac{1}{2}\left(\frac{i}{2\pi}\right)^2[\mathrm{Tr}F\wedge\mathrm{Tr}F-\mathrm{Tr}(F\wedge F)],\\
c_n(F) &= \left(\frac{i}{2\pi}\right)^n\det F.
\end{aligned} \tag{9.14}$$

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In the case of $SU(N)$, the generators are traceless, and so

$$c_2(F) = \frac{1}{8\pi^2}\mathrm{Tr}(F \wedge F),$$

the expression we saw before. The $c_j(F)$ are rank $2j$ forms, so of course, the largest one that gives a meaningful quantity is the one for which $\dim(\mathcal{B}) = 2j$.

The natural object which D-branes seem to have on their world-volume is in fact the Chern character, $ch(F) = \mathrm{Tr}\exp(iF/2\pi)$. This computes a specific combination of the Chern classes, and we can compute this by using our symmetric polynomial expressions in (9.13). Working with the diagonal X again we have

$$\begin{aligned} ch(x) &= \sum_i e^{x_i} = \sum_i \left(1 + x_i + \frac{x_i^2}{2} + \cdots\right) \\ &= n + c_1 + \frac{1}{2}(c_1^2 - 2c_2) + \cdots, \quad \text{and so we have:} \\ ch(F) &= n + c_1(F) + \frac{1}{2}(c_1^2(F) - 2c_2(F)) + \cdots. \end{aligned} \tag{9.15}$$

The Chern character is to be thought of as an important generating function of the Chern classes and in fact it is a powerful tool, in that it is well behaved in the sense that for bundle E and a bundle F, the relations

$$ch(E \oplus F) = ch(E) + ch(F) \quad \text{and} \quad ch(E \otimes F) = ch(E) \wedge ch(F) \tag{9.16}$$

are true. This is part of an important technology to doing 'algebra' on bundles allowing one to perform operations which compare them to each other, etc.

For the case $G = O(n)$, the characteristic classes are called Pontryagin classes. We may think of the bundle as a real vector bundle. Now we have

$$\det\left(t\mathbf{I} + \frac{F}{2\pi}\right) = \sum_{j=0}^{n} p_{n-j}(F)t^j. \tag{9.17}$$

Again, consider having diagonalised to X. We can't quite diagonalise, but can get it into the block diagonal form:

$$X = \begin{pmatrix} 0 & x_1 & & & \\ -x_1 & 0 & & & \\ & & 0 & x_2 & \\ & & -x_2 & 0 & \\ & & & & \ddots \end{pmatrix}. \tag{9.18}$$

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Now we have the relation:

$$\det(t\mathbf{I} + X) = \det\left(t\mathbf{I} + X^T\right) = \det(t\mathbf{I} - X),$$

and so we see that the $p_j(F)$ must be even in F. A bit of work similar to that which we did above for the Chern classes gives:

$$\begin{aligned} p_1(F) &= -\frac{1}{2}\left(\frac{1}{2\pi}\right)^2 \mathrm{Tr}F^2, \\ p_2(F) &= \frac{1}{8}\left(\frac{1}{2\pi}\right)^4 \left[(\mathrm{Tr}F^2)^2 - 2\mathrm{Tr}F^4\right], \ldots, \text{etc.}, \\ p_{[\frac{n}{2}]}(F) &= \left(\frac{1}{2\pi}\right)^n \det F, \end{aligned} \tag{9.19}$$

where $[n/2] = n/2$ if n is even or $(n-1)/2$ otherwise.

Now an important case of orthogonal groups is of course the tangent bundle to a manifold of dimension n. Using the veilbiens formalism of section 2.8, the structure group is $O(n)$. The two-form to use is the curvature two-form R. Then we have, e.g.

$$p_1(R) = -\frac{1}{8\pi^2}\mathrm{Tr}R \wedge R. \tag{9.20}$$

The Euler class is naturally defined here too. For an orientable even dimensional $n = 2k$ manifold M, the Euler class class $e(M)$ is defined via

$$e(X)e(X) = p_k(X).$$

We write X here and not the two-form R, since we would have a $4k$-form which vanishes on M. However, $e(R)$ makes sense as a form since its rank is n, which is the dimension of M. For an example, see insert 9.5.

Two other remarkable generating functions of importance are the $\hat{\mathcal{A}}$ ('A-roof') or Dirac genus:

$$\begin{aligned} \hat{\mathcal{A}} &= \prod_{j=1}^{n} \frac{x_j/2}{\sinh x_j/2} = \prod_{j=1}^{n}\left(1 + \sum_{n\geq 1}(-1)^n \frac{2^{2n}-2}{(2n)!} B_n x_j^{2n}\right) \\ &= 1 - \frac{1}{24}p_1 + \frac{1}{5760}(7p_1^2 - 4p_2) + \cdots, \end{aligned} \tag{9.21}$$

and the Hirzebruch $\hat{\mathcal{L}}$-polynomial

$$\begin{aligned} \hat{\mathcal{L}} &= \prod_{j=1}^{n} \frac{x_j}{\tanh x_j} = \prod_{j=1}^{n}\left(1 + \sum_{n\geq 1}(-1)^{n-1} \frac{2^{2n}}{(2n)!} B_n x_j^{2n}\right) \\ &= 1 + \frac{1}{3}p_1 + \frac{1}{45}(7p_2 - p_1^2) + \cdots, \end{aligned} \tag{9.22}$$

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Insert 9.5. The Euler number of the sphere

Lets test this out for the two-sphere S^2. Using the formalism of section 2.8, the curvature two-form can be computed as $R_{\theta\phi} = \sin\theta d\theta \wedge d\phi$. Then we can compute

$$p_1(S^2) = -\frac{1}{8\pi^2}\mathrm{Tr}R \wedge R = \left(\frac{1}{2\pi}\sin\theta d\theta \wedge d\phi\right)^2.$$

So we see that

$$e(S^2) = \frac{1}{2\pi}\sin\theta d\theta \wedge d\phi.$$

The integral of this from over the manifold given the Euler number:

$$\chi = \int_{S^2} e(S^2) = 2,$$

a result we know well and have used extensively.

where the B_n are the Bernoulli numbers, $B_1 = 1/6, B_2 = 1/30, B_3 = 1/42, \ldots$. These are very important characteristics as well, and again have useful algebraic properties for facilitating the calculus of vector bundles along the lines given by equations (9.16). As we shall see, they also play a very natural role in our story here.

9.4 Anomalous curvature couplings

So we seem to have wandered away from our story somewhat, but in fact we are getting closer to a big part of the answer. If the above formula (9.9) is true, then D-branes evidently know how to compute the topological properties of the gauge bundle associated to their world-volumes. This is in fact a hint of a deeper mathematical structure underlying the structure of D-branes and their charge, and we shall see it again later.

There is another strong hint of what is going on based on the fact that we began to deduce much of this structure using the terms we discovered were needed to cancel anomalies. So far we have only looked at the terms involving the curvature of the gauge bundle, and not the geometry of the brane itself which might have non-trivial R associated to its tangent bundle. Indeed, since the gauge curvature terms came from anomalies, we might expect that the tangent bundle curvatures do too. Since these are so closely related, one might expect that there is a very succinct formula for

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those couplings as well. Let us look at the anomaly terms again. The key terms are the curvature terms in (7.35) and the curvature terms arising from the modification (7.38) of the field strength of $C_{(3)}$ to achieve gauge invariance. The same deduction we made to arrive at (9.8) will lead us to $\mathrm{Tr}R^2$ terms coupling to $C_{(6)}$. Also, if we convert to the fundamental representation, we can see that there is a term

$$-\frac{1}{3\times 2^6(2\pi)^5}\int C_{(2)}\mathrm{Tr}F^2\mathrm{tr}R^2.$$

This mixed anomaly type term can be generated in a number of ways, but a natural guess[110, 111, 112] (motivated by remarks we shall make shortly) is that there is a $\sqrt{\hat{\mathcal{A}}}$ term on the world volume, multiplying the Chern characteristic. In fact, the precise term, written for all branes, is:

$$\mu_p\int_{\mathcal{M}_{p+1}}\sum_i C_{(i)}\left[e^{2\pi\alpha' F+B}\right]\sqrt{\hat{\mathcal{A}}(4\pi^2\alpha' R)}. \tag{9.23}$$

Working with this expression, using the precise form given in (9.21) will get the mixed term precisely right, but the $C_{(6)}\mathrm{tr}R^2$ will not have the right coefficient, and also the remaining fourth order terms coupling to $C_{(2)}$ are incorrect, after comparing the result to (7.35).

The reason why they are not correct is because there is another crucial contribution which we have not included. There is an orientifold O9-plane of charge $-32\mu_9$ as well. As we saw, it is crucial in the anomaly cancellation story of the previous chapter and it must be included here for precisely the same reasons. While it does not couple to the $SO(32)$ gauge fields (open strings), it certainly has every right to couple to gravity, and hence source curvature terms involving R. Again, as will be clear shortly, the precise term for Op-planes of this type is [125]:

$$\tilde{\mu}_p\int_{\mathcal{M}_{p+1}}\sum_i C_{(i)}\sqrt{\hat{\mathcal{L}}(\pi^2\alpha' R)}, \tag{9.24}$$

where $\hat{\mathcal{L}}(R)$ is defined above in equation (9.22). Remarkably, expanding this out will repair the pure curvature terms so as to give all of the correct terms in X_8 to reproduce (7.35), and the $C_{(6)}$ coupling is precisely:

$$\begin{aligned}S &= \mu_9\int\frac{(2\pi\alpha')^2}{2}C_{(6)}\wedge(\mathrm{Tr}(F\wedge F)-\mathrm{Tr}R\wedge R)\\ &= \mu_9\int\frac{(2\pi\alpha')^2}{2}C_{(6)}\wedge Y_4.\end{aligned} \tag{9.25}$$

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Beyond just type I, it is worth noting that the $R \wedge R$ term will play an important role on the world volumes of branes. It can be written in the form:

$$\frac{\mu_p(4\pi^2\alpha')^2}{48}\int_{\mathcal{M}_{p+1}} C_{(p-3)} \wedge p_1(R). \tag{9.26}$$

By straightforward analogy with what we have already observed about instantons, another way to get a D$(p-4)$-brane inside the world-volume of a Dp-brane is to wrap the brane on a four dimensional surface of non-zero $p_1(R)$. Indeed, as we saw in equation (7.54), the K3 surface has $p_1 = -2\chi = -48$, and so wrapping a Dp-brane on K3 gives D$(p-4)$-brane charge of precisely -1. This will be important to us later[115, 121].

9.5 A relation to anomalies

There is one last amusing fact that we should notice, which will make it very clear that the curvature couplings that we have written above are natural for branes and O-planes of all dimensionalities. The point is that the curvature terms just don't accidentally resemble the anomaly polynomials we saw before, but are built out of the very objects which can be used to generate the anomaly polynomials that we listed in insert 7.2.

In fact, we can use them to generate anomaly polynomials for dimension $D = 4k+2$. We can pick out the appropriate powers of the curvature two forms by using the substitution

$$\sum_{i=1}^{2k+1} x_i^{2m} = \frac{1}{2}(-1)^m \mathrm{tr} R^{2m}.$$

Then in fact the polynomial $\hat{I}_{1/2}$ is given by the $\hat{\mathcal{A}}$ genus:

$$\begin{aligned}
\hat{I}_{1/2} = \hat{\mathcal{A}} &= \prod_{j=1}^{2k+1} \frac{x_j/2}{\sinh x_j/2} \\
&= \prod_{j=1}^{2k+1}\left(1 + \frac{y_j^2}{3!} + \frac{y_j^4}{5!} + \cdots\right)^{-1} \\
&= \prod_{j=1}^{2k+1}\left(1 - \frac{1}{6}y_j^2 + \frac{7}{360}y_j^4 - \frac{31}{15120}y_j^6 + \cdots\right) \\
&= 1 - \frac{1}{6}\mathcal{Y}_2 + \frac{1}{180}\mathcal{Y}_4 + \frac{1}{72}\mathcal{Y}_2^2 \\
&\quad - \frac{1}{2835}\mathcal{Y}_6 - \frac{1}{1080}\mathcal{Y}_2\mathcal{Y}_4 - \frac{1}{1296}\mathcal{Y}_2^3 + \cdots
\end{aligned} \tag{9.27}$$

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where

$$\mathcal{Y}_{2m} = \sum_{i=1}^{2k+1} y_i^{2m} = \frac{1}{2}\left(-\frac{1}{4}\right)^m \mathrm{tr} R^{2m}.$$

The trick is then to simply pick out the piece of the expansion which fits the dimension of interest, remembering that the desired polynomial is of rank $D+2$. So for example, picking out the order 12 terms will give precisely the 12-form polynomial in insert 7.2, etc.

The gravitino polynomials come about in a similar way. In fact,

$$\begin{aligned} I_{3/2} &= I_{1/2}\left(-1 + 2\sum_{j=1}^{2k+1}\cosh x_i\right) \\ &= I_{1/2}\left(D - 1 + 4\mathcal{Y}_2 + \frac{4}{3}\mathcal{Y}_4 + \frac{8}{45}\mathcal{Y}_6 + \cdots\right). \end{aligned} \tag{9.28}$$

Also, the polynomials for the antisymmetric tensor come from

$$\begin{aligned} I_A &= -\frac{1}{8}\hat{\mathcal{L}}(R) = -\frac{1}{8}\sum_{j=1}^{2k+1}\frac{x_j}{\tanh x_j} \\ &= -\frac{1}{8} - \frac{1}{6}\mathcal{Y}_2 + \left(\frac{7}{45} - \frac{1}{9}\mathcal{Y}_2^2\right) \\ &\quad + \frac{1}{2835}\left(-496\mathcal{Y}_6 + 588\mathcal{Y}_2\mathcal{Y}_4 - 140\mathcal{Y}_2^3 + \cdots\right). \end{aligned} \tag{9.29}$$

Finally, it is easy to work out the anomaly polynomial for a charged fermion. One simply multiplies by the Chern character:

$$I_{1/2}(F,R) = \mathrm{Tr} e^{iF} I_{1/2}(R). \tag{9.30}$$

Now it is perhaps clearer what must be going on[111, 112]. The D-branes and O-planes, and any intersections between them all define sub-spacetimes of the ten dimensional spacetime, where potentially anomalous theories live. This is natural, since as we have already learned, and shall explore much more, there are massless fields of various sorts living on them, possibly charged under any gauge group they might carry.

As the world-volume intersections may be thought of as embedded in the full ten dimensional theory, there is a mechanism for cancelling the anomaly which generalises that which we have already encountered. For example, since the Dp-brane is also a source for the R–R sector field $G^{(p+2)}$, it modifies it according to

$$G_{(p+2)} = dC_{(p+1)} - \mu_p\delta(x_0)\ldots\delta(x_p)dx_0\wedge\ldots dx_p\mathcal{F}(R,F), \tag{9.31}$$

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where the delta functions are chosen to localise the contribution to the world-volume of the brane, extended in the directions $x_0, x_1, \ldots, x_p$. Also μ_p is the Dp-brane (or Op-plane) charge, and the polynomial $\mathcal{F}$ must be chosen so that the classically anomalous variation $\delta C_{(p+1)}$ required to keep $G^{(p+2)}$ gauge invariant can cancel the anomaly on the branes' intersection. Following this argument to its logical conclusion, and using the previously mentioned fact that the possible anomalies are described in terms of the characteristic classes $\exp(iF)$, $\hat{\mathcal{A}}(R)$ and $\hat{\mathcal{L}}(R)$, reveals that $\mathcal{F}$ takes the form of the couplings that we have already written. We see that the Green–Schwarz mechanism from type I is an example of something much more general, involving the various geometrical objects which can appear embedded in the theory, and not just the fundamental string itself.

Arguments along these lines also uncover the feature that the normal bundle also contributes to the curvature couplings as well. The full expressions, for completeness, are:

$$\mu_p \int_{\mathcal{M}_{p+1}} \sum_i C_{(i)} \left[e^{2\pi\alpha' F + B} \right] \sqrt{\frac{\hat{\mathcal{A}}(4\pi^2\alpha' R_{\rm T})}{\hat{\mathcal{A}}(4\pi^2\alpha' R_{\rm N})}}, \tag{9.32}$$

and

$$\tilde{\mu}_p \int_{\mathcal{M}_{p+1}} \sum_i C_{(i)} \sqrt{\frac{\hat{\mathcal{L}}(\pi^2\alpha' R_{\rm T})}{\hat{\mathcal{L}}(\pi^2\alpha' R_{\rm N})}}, \tag{9.33}$$

where the subscripts 'T','N' denote curvatures of the tangent and the normal frame, respectively.

9.6 D-branes and K-theory

In fact, the sort of argument above is an independent check on the precise normalisation of the D-brane charges, which we worked out by direct computation in previous sections. As already said before, the close relation to the topology of the gauge and tangent bundles of the branes suggests a connection with tools which might uncover a deeper classification. This tool is called *'K-theory'*. K-theory should be thought of as a calculus for working out subtle topological differences between vector bundles, and as such makes a natural physical appearance here[113, 18].

This is because there is a means of constructing a D-brane by a mechanism known as 'tachyon condensation' on the world-volume of higher dimensional branes. Recall that in chapter 8 we observed that a Dp-brane and an anti-Dp-brane will annihilate. Indeed, there is a tachyon in the spectrum of $p\bar{p}$ strings. Let us make the number of branes be N, and the

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number of anti-branes be $\bar{N}$. Then the tachyon is charged under the gauge group $U(N)\times U(\bar{N})$. The idea is that a suitable choice of the tachyon can give rise to topology which must survive even if all of the parent branes annihilate away. For example, if the tachyon field is given a topologically stable kink (see insert 1.4, p. 18) as a function of one of the dimensions inside the brane, then there will be a $p-1$ dimensional structure left over, to be identified with a D$(p-1)$-brane. This idea is the key to seeing how to classify D-branes, by constructing all branes in this way.

Most importantly, we have two gauge bundles, that of the Dp-branes, which we might call E, and that of the D$\bar{p}$-branes, called F. To classify the possible D-branes which can exist in the world volume, one must classify all such bundles, defining as equivalent all pairs which can be reached by brane creation or annihilation: If some number of Dp-branes annihilate with D$\bar{p}$-branes (or if the reverse happens, i.e. creation of Dp-D$\bar{p}$ pairs), the pair (E,F) changes to $(E\oplus G, F\oplus G)$, where G is the gauge bundle associated to the new branes, identical in each set. These two pairs of bundles are equivalent. The group of distinct such pairs is (roughly) the object called $K(X)$, where X is the spacetime that the branes fill (the base of the gauge bundles). Physically distinct pairs have non-trivial differences in their tachyon configurations which would correspond to different D-branes after complete annihilation had taken place. So K-theory, defined in this way, is a sort of more subtle or advanced cohomology which goes beyond the more familiar sort of cohomology we encounter daily.

The technology of K-theory is beyond that which we have room for here, but it should be clear from what we have seen in this chapter that it is quite natural, since the world-volume couplings of the charge of the branes is via the most natural objects with which one would want to perform sensible operations on the gauge bundles of the branes like addition and subtraction: the characteristic classes, $\exp(iF)$, $\hat{\mathcal{A}}(R)$ and $\hat{\mathcal{L}}(R)$. Actually, this might have been enough to simply get the result that D-brane charges were classified by cohomology. That it is in fact K-theory (which can compute differences between bundles that cohomology alone would miss) is probably related to a very important physical fact about the underlying theory which will be more manifest one day.

9.7 Further non-Abelian extensions

One can use T-duality to go a bit further and deduce a number of non-Abelian extensions of the action, being mindful of the sort of complications mentioned at the beginning of section 5.5. In the absence of geometrical

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curvature terms it turns out to be[51, 52]:

$$\mu_p \int_{p-\text{brane}} \text{Tr}\left(\left[e^{2\pi\alpha' i_\Phi i_\Phi}\sum_p C_{(p+1)}\right] e^{2\pi\alpha' F+B}\right). \tag{9.34}$$

Here, we ascribe the same meaning to the gauge trace as we did previously (see section 5.5). The meaning of $\mathbf{i}_X$ is as the 'interior product' in the direction given by the vector Φ^i, which produces a form of one degree fewer in rank. For example, on a two form $C_{(2)}(\Phi) = (1/2)C_{ij}(\Phi)dX^i dX^j$, we have

$$\mathbf{i}_\Phi C_{(2)} = \Phi^i C_{ij}(\Phi)dX^j; \quad \mathbf{i}_\Phi \mathbf{i}_\Phi C_{(2)}(\Phi) = \Phi^j \Phi^i C_{ij}(\Phi) = \frac{1}{2}[\Phi^i, \Phi^j]C_{ij}(\Phi), \tag{9.35}$$

where we see that the result of acting twice is non-vanishing when we allow for the non-Abelian case, with C having a non-trivial dependence on Φ. We shall see this action work for us to produce interesting physics later.

9.8 Further curvature couplings

We deduced geometrical curvature couplings to the R–R potentials a few subsections ago. In particular, such couplings induce the charge of lower p branes by wrapping larger branes on topologically non-trivial surfaces.

In fact, as we saw before, if we wrap a Dp-brane on K3, there is induced precisely -1 units of charge of a D$(p-4)$-brane. This means that the charge of the effective $(p-4)$-dimensional object is

$$\mu = \mu_p V_{\text{K3}} - \mu_{p-4}, \tag{9.36}$$

where V_{K3} is the volume of the K3. However, we can go further and notice that since this is a BPS object of the six dimensional $\mathcal{N} = 2$ string theory obtained by compactifying on K3, we should expect that it has a tension which is

$$\tau = \tau_p V_{\text{K3}} - \tau_{p-4} = g_{\text{s}}^{-1}\mu. \tag{9.37}$$

If this is indeed so, then there must be a means by which the curvature of K3 induces a shift in the tension in the world-volume action. Since the part of the action which refers to the tension is the Dirac–Born–Infeld action, we deduce that there must be a set of curvature couplings for that part of the action as well. Some of them are given by the

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following[122, 128]:

$$S = -\tau_p \int d^{p+1}\xi \; e^{-\Phi} \det{}^{1/2}(G_{ab} + \mathcal{F}_{ab}) \Bigg(1 - \frac{1}{3 \times 2^8 \pi^2} \times \\ \Big(\mathcal{R}_{abcd}\mathcal{R}^{abcd} - \mathcal{R}_{\alpha\beta ab}R^{\alpha\beta ab} + 2\hat{\mathcal{R}}_{\alpha\beta}\hat{\mathcal{R}}^{\alpha\beta} - 2\hat{\mathcal{R}}_{ab}\hat{\mathcal{R}}^{ab} \Big) + O(\alpha'^4) \Bigg), \tag{9.38}$$

where $\mathcal{R}_{abcd} = (4\pi^2\alpha')R_{abcd}$, etc., and a, b, c, d are the usual tangent space indices running along the brane's world-volume, while α, β are normal indices, running transverse to the world-volume.

Some explanation is needed. Recall that the embedding of the brane into D-dimensional spacetime is achieved with the functions $X^\mu(\xi^a)$, $(a = 0, \ldots, p; \mu = 0, \ldots, D-1)$ and the pull-back of a spacetime field F_μ is performed by soaking up spacetime indices μ with the local 'tangent frame' vectors $\partial_a X^\mu$, to give $F_a = F_\mu \partial_a X^\mu$. There is another frame, the 'normal frame', with basis vectors ζ^μ_α, $(\alpha = p+1, \ldots, D-1)$. Orthonormality gives $\zeta^\mu_\alpha \zeta^\nu_\beta G_{\mu\nu} = \delta_{\alpha\beta}$ and also we have $\zeta^\mu_\alpha \partial_a X^\nu G_{\mu\nu} = 0$.

We can pull back the spacetime Riemann tensor $R_{\mu\nu\kappa\lambda}$ in a number of ways, using these different frames, as can be seen in the action. $\hat{R}$ with two indices are objects which were constructed by contraction of the *pulled-back* fields. They are *not* the pull-back of the bulk Ricci tensor, which vanishes at this order of string perturbation theory anyway.

In fact, for the case of K3, it is Ricci flat and everything with normal space indices vanishes and so we get only $R_{abcd}R^{abcd}$ appearing, which alone computes the result (7.54) for us, and so after integrating over K3, the action becomes:

$$S = -\int d^{p-3}\xi \; e^{-\Phi} \left[\tau_p V_{\rm K3} - \tau_{p-4}\right] \det{}^{1/2}(G_{ab} + \mathcal{F}_{ab}), \tag{9.39}$$

where again we have used the recursion relation between the D-brane tensions. So we see that we have correctly reproduced the shift in the tension that we expected on general grounds for the effective D$(p-4)$-brane. We will use this action later.

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