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EXISTENCE AND NONEXISTENCE OF REGULAR GENERATORS

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ABSTRACT. A total category is constructed which has no regular generator although it has an object C such that every object is a regular quotient of a copower of C.

Introduction. Given a class $\mathcal{E}$ of epimorphisms in a category $\mathbf{K}$, by a (*weak*) $\mathcal{E}$-*generator* is meant a small collection $\mathbf{G}$ of objects such that every object X is an $\mathcal{E}$-quotient of its canonical coproduct w.r.t. $\mathbf{G}$ (or any coproduct of $\mathbf{G}$-objects, respectively). The former means that the canonical coproduct $\coprod_{G\in\mathbf{G}}\coprod_{f\in\hom(G,X)} G_f$, $G_f = G$, exists and the canonical morphism into X is in $\mathcal{E}$. Whereas the two concepts of $\mathcal{E}$-generator and weak $\mathcal{E}$-generator coincide for $\mathcal{E}$ = epis, strong epis, extremal epis, we will prove that they do not coincide for $\mathcal{E}$ = regular epis. The example we present is very "well-behaved": it is a total category $\mathbf{K}$, *i.e.*, the Yoneda embedding $\mathbf{K} \longrightarrow \mathrm{Set}^{\mathbf{K}^{\mathrm{op}}}$ is a right adjoint; thus, $\mathbf{K}$ is complete, cocomplete, compact, *etc.*

Let us remark that, on the other hand, the existence of a weak regular generator does imply the existence of a regular generator provided that $\mathbf{K}$ has regular factorizations or, more generally, has the cancellation property for regular epimorphisms.

The counterexample. We define a category Γ as follows: Γ-*objects* are quadruples (X, X_0, X_1, α) where $X \supseteq X_0 \supseteq X_1$ are sets and $\alpha\colon \exp X_1 \longrightarrow X_0$ is a function such that

$$\alpha(\emptyset) = \alpha(\{x\}) \in X_1$$

for all $x \in X_1$.

Elements of X_0 are called *internal*, those of $X - X_0$ are called *external*; an element $x \in X_0$ is called *special* in case $x = \alpha(\emptyset)$ or $x = \alpha(M)$ for some infinite set $M \subseteq X_1$ with $\alpha(\{m_1, m_2\}) = \alpha(M)$ for all $m_1 \neq m_2$ in M.

Γ-*morphisms* from (X, X_0, X_1, α) to (Y, Y_1, Y_0, β) are functions $f\colon X \longrightarrow Y$ such that

(1) $f[X_0] \subseteq Y_0$

(2) $f[X_1] \subseteq Y_1$

(3) $f\big(\alpha(M)\big) = \beta(f[M])$ for each $M \subseteq X_1$

(4) for each $x \in X - X_0$ either $f(x) \in Y - Y_0$ or $f(x)$ is special.

Composition and identities are defined on the level of Set. We have to verify that morphisms are indeed closed under set-theoretical composition. This follows easily from:

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LEMMA 1. *Γ-morphisms preserve special points.*

PROOF. Lef $f\colon (X, X_0, X_1, \alpha) \longrightarrow (Y, Y_0, Y_1, \beta)$ be a morphism, and let $x \in X_0$ be special. If $f(x) = \beta(\emptyset)$, then $f(x)$ is special. Suppose $f(x) \neq \beta(\emptyset)$. Since x is special, there is $M \subseteq X_1$ infinite with $x = \alpha(M) = \alpha(\{x_1, x_2\})$ for all $x_1 \neq x_2$ in M. Then for $x_1 \neq x_2$ in M

$$\beta\big(\{f(x_1), f(x_2)\}\big) = \beta(f[\{x_1, x_2\}]) = f(x) \neq \beta(\emptyset)$$

implies $f(x_1) \neq f(x_2)$. Thus, $f[M]$ is an infinite set with $f(x) = \beta(f[M]) = \beta(\{y_1, y_2\})$ for all $y_1 \neq y_2$ in $f[M]$.

LEMMA 2 (DESCRIPTION OF COLIMITS IN Γ). *Let $D\colon \mathbf{D} \longrightarrow \Gamma$ be a small diagram with objects $Dd = (X_d, X_{d0}, X_{d1}, \alpha_d)$, and let $\approx$ be the smallest equivalence on $\coprod_d X_d$ with the following properties:*

(1) $x \approx D\delta(x)$ for each $x \in X_d$ and each $\mathbf{D}$-morphism $\delta\colon d \longrightarrow d'$;

(2) $\alpha_d(M) \approx \alpha_{d'}(M')$ for any $M \subseteq D_{d1}$, $M' \subseteq X_{d'1}$ with $c[M] = c[M']$ where $c\colon \coprod_d X_d \longrightarrow \bar{X} = \coprod_d X_d / \approx$ is the canonical map.

Then a colimit of D is formed by the sink of canonical maps from Dd to

$$\operatorname{colim} D = (\bar{X} + Z, \bar{X}_0 + Z, \bar{X}_1, \bar{\alpha})$$

where $\bar{X}_1 = c(\coprod_d X_{d1})$, $\bar{X}_0 = c(\coprod_d X_{d0})$, $Z = \{M \subseteq \bar{X}_0; M \not\subseteq c[X_{d0}]$ for any $d\}$, and for each $M \subseteq \bar{X}_1$

$$\bar{\alpha}(M) = \begin{cases} c(\alpha_d[M']) & \text{whenever } M = c[M'] \text{ for some } M' \subseteq X_{d0} \\ M & \text{whenever } M \in Z. \end{cases}$$

PROOF. Let us first verify that for each d the restriction c_d of the canonical map c is a morphism $c_d\colon Dd \longrightarrow \operatorname{colim} D$. It is obvious that $c_d[X_{d1}] \subseteq \bar{X}_1$ and $c_d[X_{d0}] \subseteq \bar{X}_0$, furthermore, $c\big(\alpha_d(M')\big) = \bar{\alpha}(c[M'])$ for each $M' \subseteq X_{d1}$. Let us verify that, for each $x \in X_d - X_{d0}$, whenever $c_d(x)$ is not special in $\operatorname{colim} D$, then $c_d(x) \in \bar{X} - \bar{X}_0$. If there would exist $y \approx x$, $y \in X_{d'} - X_{d'0}$ with $D\delta(y) \in X_{d''0}$ for some $\delta\colon d' \longrightarrow d''$ in $\mathbf{D}$, then the point $D\delta(y) \approx x$ would be special in Dd'', and arguing as in Lemma 1, we would conclude that $c_d\big(D\delta(y)\big) = c_d(x)$ is special in $\operatorname{colim} D$. Consequently, all external points in the equivalence class of x are mapped to external points by all morphisms of the diagram D. Since the condition (2) above concerns non-external points only (recall that $\alpha(M)$ is never external), it follows that the whole equivalence class of x contains external points only. Thus, $c(x) = c_d(x) \in \bar{X} - \bar{X}_0$.

The sink of all $c_d\colon Dd \longrightarrow \operatorname{colim} D$ is, obviously, compatible with D. Let $f_d\colon Dd \longrightarrow B = (Y, Y_0, Y_1, \beta)$ be another compatible sink. Then the equivalence $\approx$ is clearly contained in the equivalence merging $x \in X_d$ with $x' \in X_{d'}$ iff $f_d(x) = f_{d'}(x')$. Thus, we have a unique mapping $\bar{f}\colon \bar{X} \longrightarrow Y$ with $\bar{f}\big(c_d(x)\big) = f_d(x)$ for all d, x. Define $f\colon \bar{X} + Z \longrightarrow Y$ to be the extension of $\bar{f}$ with $f(M) = \beta(\bar{f}[M])$ for each $M \in Z$. Then $f_d = f \cdot c_d$ and $f\big(\alpha(M)\big) = \beta(f[M])$ for all $M \subseteq \bar{X}_1$, and f is the unique function with these properties. Moreover, for each $x \in \bar{X} - \bar{X}_0$ we have $x' \in X_d - X_{d0}$ with $x = c(x')$; thus $f(x) = f_d(x')$ is either external or special. Thus, $f\colon \operatorname{colim} D \longrightarrow B$ is a morphism.

COROLLARY. *Γ is cocomplete, and for each coproduct in Γ the only special points are the special points of the individual summands.*

In fact, if the diagram in Lemma 2 is discrete and a set $M \subseteq \bar{X}_0$ fulfils $\bar{\alpha}(M) = \bar{\alpha}(\{m_1, m_2\})$ for all $m_1 \neq m_2$ in M, then, obviously, $M \in Z$ (else there exist $m_1 \neq m_2$ in M with $\{m_1, m_2\} \in Z$). Thus, $M = c(M')$ for some $M' \subseteq X_{d0}$, and $\alpha_d(M')$ is a special point of Dd.

THEOREM. *The category Γ has no regular generator, although each object is a regular quotient of a copower of the object*

$$C = (\{a, b, b', c, c'\}, \{b, b', c, c'\}, \{c, c'\}, \alpha)$$

where

$$\alpha(\emptyset) = c \text{ and } \alpha(\{c, c'\}) = b.$$

REMARK. Since the category Γ is cocomplete (by Lemma 2), the existence of a weak regular generator $\{C\}$ implies that Γ is total—this has been proved in [BT$_2$].

PROOF. Denote by Γ_0 the full subcategory of Γ consisting of all objects without external points, and define a functor

$$U: \Gamma \longrightarrow \Gamma_0$$

by

$$U(X, X_0, X_1, \alpha) = (X_0, X_0, X_1, \alpha); \quad Uf = f/X_0.$$

I. Let us prove that each object $K = (X, X_0, X_1, \alpha)$ is a regular quotient of a copower of C. It is obvious that

$$K = K_0 + K_1$$

where

$$K_0 = UK$$

and

$$K_1 = \big(X, \{\alpha(\emptyset)\}, \{\alpha(\emptyset)\}, \emptyset \longmapsto \alpha(\emptyset)\big).$$

Thus, it is obviously sufficient to prove that both K_0 and K_1 are regular quotients of copowers of C.

(a) Let us first verify that a morphism $f: A \longrightarrow A'$ in Γ with $A' \in \Gamma_0$ is a regular epimorphism whenever both f and the domain-codomain restriction of f to "X_1-type" points are onto. More precisely, put $A = (X, X_0, X_1, \alpha)$ and $A' = (X', X_0', X_1', \alpha')$. Suppose $X' = X_0'$ and $f(X) = X', f(X_1) = X_1'$. Then f is a regular epimorphism in Γ. In fact, the kernel set $\ker f \subseteq X \times X$ of f defines a subobject of $A \times A$:

$$E = \big(\ker f, (X_0 \times X_0) \cap \ker f, (X_1 \times X_1) \cap \ker f, \alpha^*\big)$$

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where for the projections π_1, π_2 of $X \times X$ we have

$$\alpha^*(M) = (\alpha \cdot \pi_1[M], \alpha \cdot \pi_2[M]) \quad \text{for all } M.$$

It is easy to see that $\pi_1, \pi_2 \colon E \longrightarrow A$ are morphisms of Γ, and of course $f \cdot \pi_1 = f \cdot \pi_2$. Now for each morphism

$$\bar{f} \colon A \longrightarrow \bar{A} = (\bar{X}, \bar{X}_0, \bar{X}_1, \bar{\alpha})$$

with $\bar{f} \cdot \pi_1 = \bar{f} \cdot \pi_2$ we have (since f is onto) a unique map $g \colon \ker f \longrightarrow \bar{X}$ with $\bar{f} = g \cdot f$. This is a morphism $g \colon A' \longrightarrow \bar{A}$ of Γ because

$$g[X_i'] = g \cdot f[X_i] = \bar{f}[X_i] \subseteq \bar{X}_i \quad \text{for } i = 0, 1$$

and given $M' \subseteq X_0' = f[X_0]$ there exists $M \subseteq X_0$ with $M' = f[M]$ and then

$$\begin{aligned} g\big(\alpha'(M')\big) &= g\big(\alpha'(f[M])\big) \\ &= g\big(f(\alpha[M])\big) \\ &= \bar{f}\big(\alpha(M)\big) \\ &= \bar{\alpha}(\bar{f}[M]) \\ &= \bar{\alpha}(g[M']). \end{aligned}$$

Consequently, f is a regular epimorphism in Γ.

We now prove that K_0 is a regular quotient of a copower of C. In fact, the canonical morphism

$$f \colon \coprod_{h \in \hom(C, K_0)} C \longrightarrow K_0$$

is onto, since for each $x \in X_0$ we have a morphism

$$h \colon C \longrightarrow K_0, \quad h(b') = x, \quad h(-) = \alpha(\emptyset) \text{ otherwise.}$$

Its restriction to X_1-type points is also onto, since for each $x \in X_1$ we have a morphism $h \colon C \longrightarrow K_0$ defined by

$$\begin{aligned} h(c') &= x \\ h(b) &= \alpha\big(\{x, \alpha(\emptyset)\}\big) \\ h(-) &= \alpha(\emptyset) \text{ otherwise.} \end{aligned}$$

Consequently, f is a regular epimorphism.

II. Γ DOES NOT HAVE A REGULAR GENERATOR. Suppose that, to the contrary, $\mathbf{G}$ is a regular generator of Γ. We derive a contradiction by exhibiting, for each cardinal n, an object in $\mathbf{G}$ of cardinality at least n. If all $\mathbf{G}$-objects would lie in Γ_0, then their coproducts would also lie in Γ_0, and this is impossible. Thus, some $G_0 \in \mathbf{G}$ has external points.

For each infinite cardinal n define an object

$$D_n = (n, n, n, \delta_n)$$

$$\delta_n(M) = \begin{cases} 0 & \text{if card } M \neq n, \text{ card } M \neq 2 \\ 1 & \text{if card } M = n \text{ or card } M = 2. \end{cases}$$

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Then 1 is a special point of D_n. Thus, we have the following morphism $f\colon G_0 \longrightarrow D_n$: f maps all external points to 1 and all internal ones to 0. Consider the canonical morphism c of the canonical coproduct of D_n w.r.t. $\mathbf{G}$ which, by assumption, is a coequalizer of some pair g, g':

$$B \underset{g'}{\overset{g}{\rightrightarrows}} \coprod_{i \in I} G_i \xrightarrow{\ c\ } D_n, \qquad G_i = (X_i, X_{i0}, X_{i1}, \alpha_i).$$

Since $f\colon G_0 \longrightarrow D_n$ is one of the components of c, there exists $j \in J$ with $c(x) = 1$ for some external point x of G_j. Since 1 is internal in D_n, we claim that there exists y in B such that $c(x) = c\big(g(y)\big) = c\big(g(y')\big)$ and one of the points $g(y)$, $g'(y)$ is external whereas the other one is internal. (In fact, suppose that no such y exists. By the description of colimits in Lemma 2 it then follows that the $\approx$-class of x contains only external points; thus, $c(x)$ is an external point of the colimit, *i.e.*, of D_n—a contradiction.) It follows that y is an external point of B. Consequently, one of the points $g(y)$, $g'(y)$ is special in $\coprod_{i\in I} G_i$. By the above Corollary, we thus have a special point $z \in G_{i_0}$ for some $i_0 \in I$ with $c(z) = 1$. Since $1 \neq \alpha(\emptyset)$, it follows that there exists an infinite set $M \subseteq X_{i_0 1}$ with $\alpha(M) = \alpha(\{m_1, m_2\}) = z$ for all $m_1, m_2 \in M$, $m_1 \neq m_2$. Consequently, in D_n we have

$$1 = \delta_n(c[M]) = \delta_n\big(\{c(m_1), c(m_2)\}\big).$$

Since $\delta(\{x\}) = 0$, we conclude $c(m_1) \neq c(m_2)$, *i.e.*, c is one-to-one when restricted to M. Thus, $c[M]$ is an infinite set which is mapped by δ_n to 1—consequently, card $c[M] = n$. This proves that G_{i_0} has at least n points, which concludes the proof.

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