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AN ASYMPTOTIC FORMULA FOR RECIPROCALS OF LOGARITHMS OF CERTAIN MULTIPLICATIVE FUNCTIONS

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Sums of the form $\sum'_{n\le x} 1/\log f(n)$, where $f(n)$ is a multiplicative arithmetical function and $\sum'$ denotes summation over those values of n for which $f(n)>0$ and $f(n)\neq 1$, were studied by De Koninck [2], De Koninck and Galambos [3], Brinitzer [1] and Ivić [5]. The aim of this note is to give an asymptotic formula for $\sum'_{n\le x} 1/\log f(n)$ for a certain class of multiplicative, positive, prime-independent functions (an arithmetical function is prime-independent if $f(p^\nu)=g(\nu)$ *for all primes* p and $\nu=1,2,\ldots$). This class of functions includes, among others, the functions $a(n)$ and $\tau^{(e)}(n)$, which represent the number of non-isomorphic abelian groups of order n and the number of exponential divisors of n respectively, and none of the estimates of the above-mentioned papers may be applied to this class of functions. We prove the following.

THEOREM. *Let $f(n)$ be a multiplicative arithmetical function such that for all primes p and $\nu=1,2,\ldots$ we have $f(p^\nu)=g(\nu)$, where $g(1)=1$, $g(\nu)>1$ for $\nu\ge 2$ and* $\liminf_{\nu\to\infty} g(\nu)>1$. *Then we have*

$$(1)\qquad \sum{}' 1/\log f(n) = x\int_{-\infty}^{0} (C(t)-6/\pi^2)\,dt + 0(x^{1/2}\log^{1/2} x),$$

where $C(t)=\Pi_p(1+\sum_{k=2}^{\infty}(g^t(k)-g^t(k-1))p^{-k})$, *and $\sum'$ denotes summation over those values of n for which $f(n)>1$.*

Proof. First of all $f(n)\ge 1$, and $f(n)=1$ if and only if n is square-free, or equivalently if and only if $1-\mu^2(n)=0$, where $\mu(n)$ is the Möbius function.

Let us define

$$(2)\qquad \sum_{n\le x}{}' f^t(n) = \sum_{n\le x, f(n)>1} f^t(n).$$

Then we have

$$(3)\qquad \sum_{n\le x}{}' f^t(n) = \sum_{n\le x}(1-\mu^2(n))f^t(n) = \sum_{n\le x} f^t(n) - \sum_{n\le x}\mu^2(n)$$
$$= \sum_{n\le x} f^t(n) - \frac{6}{\pi^2}x + 0(x^{1/2}\exp(-C\log^{3/5} x(\log\log x)^{-1/5})),$$

where C is a positive constant (see [9]).

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We now proceed to estimate $\sum_{n\le x} f^t(n)$ for $t\le 0$. For Re $s>1$ we clearly have

$$\sum_{n=1}^{\infty} f^t(n)n^{-s}=\prod_p(1+p^{-s}+g^t(2)p^{-2s}+g^t(3)p^{-3s}+\cdots)$$

$$=\zeta(s)\prod_p\left(1+\sum_{k=2}^{\infty}(g^t(k)-g^t(k-1))p^{-ks}\right)=\zeta(s)g(s,t),$$

where $g(s,t)=\sum_{n=1}^{\infty}h(n,t)n^{-s}$, $h(n)$ is multiplicative and

$$h(p^j,t)=\begin{cases}0 & j=1,\\ g^t(j)-g^t(j-1) & j\ge 2.\end{cases}$$

Since $t\le 0$ we have $|h(n,t)|\le u(n)$, where

$$u(n)=\begin{cases}0 & \text{if there is a } p \text{ such that } p\,\|\,n,\\ 1 & \text{otherwise,}\end{cases}$$

so that $\sum_{n\le x}|h(n,t)|\le\sum_{n\le x}u(n)=0(x^{1/2})$. Denoting $w=g^t(2)$ for shortness, further factoring yields

$$g(s,t)=\zeta^{w-1}(2s)u(s,t), \tag{4}$$

where for $t\le 0$,

$$u(s,t)=\prod_p(1-p^{-2s})^{w-1}(1+(w-1)p^{-2s}+(g^t(3)-w)p^{-3s}+\cdots)$$

$$=\prod_p(1+(g^t(3)-w)(1-p^{-2s})^{w-1}p^{-3s}+0(p^{-4s})),$$

so that if $\sum_{n=1}^{\infty}c(n,t)n^{-s}=u(s,t)$, then for every $\varepsilon>0$ and uniformly in $t\le 0$,

$$\sum_{n\le x}|c(n,t)|=0(x^{1/3+\varepsilon}),$$

and partial summation gives

$$\sum_{n>x}|c(n,t)|\,n^{-1/2}=0(x^{-1/6+\varepsilon}). \tag{5}$$

If we set $\sum_{n=1}^{\infty}b(n,t)n^{-s}=\zeta^{w-1}(2s)$, then we have by a result of A. Selberg [7]

$$\sum_{n\le x}b(n,t)=\Gamma^{-1}(w-1)2^{2-w}x^{1/2}\log^{w-2}x+0(x^{1/2}\log^{w-3}x),$$

which gives uniformly in $t\le 0$

$$\sum_{n\le x}b(n,t)=0(x^{1/2}\log^{w-2}2x). \tag{6}$$

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From (4) it follows that

$$\sum_{n\le x} h(n, t) = \sum_{n\le x} c(n, t) \sum_{m\le x/n} b(m, t) = 0(x^{1/2} \sum_{n\le x} |c(n, t)|\, n^{-1/2} \log^{w-2} 2x/n).$$

Since $\sum_{n=1}^{\infty} |c(n, t)|\, n^{-1/2}$ converges we have

$$\sum_{n\le x} |c(n, t)|\, n^{-1/2} \log^{w-2} 2x/n = \sum_{n\le x^{1/2}} + \sum_{x^{1/2}<n\le x}$$

$$= 0(\log^{w-2} x) + 0\Big(\sum_{n>x^{1/2}} |c(n, t)|\, n^{-1/2}\Big) = 0(\log^{w-2} x) + 0(x^{-1/12+\varepsilon}) = 0(\log^{w-2} x)$$

so that

$$\sum_{n\le x} h(n, t) = 0(x^{1/2} \log^{w-2} x) = 0(x^{1/2} \log^{-1} x),$$

since for $t \le 0$ we have $w \le 1$, and partial summation gives

$$\sum_{n>x} h(n, t) n^{-1} = 0(x^{-1/2} \log^{-1} x).$$

Now take $y = x/\log x$, $z = \log x$. From $\sum_{n=1}^{\infty} f^t(n) n^{-s} = \zeta(s) g(s, t)$ we get

$$\sum_{n\le x} f^t(n) = \sum_{mn\le x} h(n, t) = \sum_{n\le y} h(n, t)[x/n] + \sum_{m\le z} 1 \sum_{n\le x/m} h(n, t)$$

$$- \sum_{m\le z} 1 \sum_{n\le y} h(n, t) = S_1 + S_2 - S_3.$$

$$S_3 = 0(zy^{1/2} \log^{-1} x) = 0(x^{1/2} \log^{-1/2} x).$$

$$S_2 = 0(x^{1/2} \sum_{m\le z} m^{-1/2} \log^{-1} x/m) = 0(x^{1/2} \log^{-1} y \sum_{m\le z} m^{-1/2}) = 0(x^{1/2} \log^{-1/2} x).$$

$$S_1 = \sum_{n\le y} h(n, t)(x/n + 0(1)) = C(t)x + x \sum_{n>y} h(n, t) n^{-1} + 0\Big(\sum_{n\le y} |h(n, t)|\Big)$$

$$= C(t)x + 0(xy^{-1/2} \log^{-1} y) + 0(y^{1/2}) = C(t)x + 0(x^{1/2} \log^{-1/2} x),$$

so that we obtain uniformly in t

(7) $$\sum_{n\le x} f^t(n) = C(t)x + 0(x^{1/2} \log^{-1/2} x),$$

where

$$C(t) = g(1, t) = \sum_{n=1}^{\infty} h(n, t) n^{-1} = \prod_p \Big(1 + \sum_{k=2}^{\infty} (g^t(k) - g^t(k-1)) p^{-k}\Big).$$

Putting (7) into (3) and integrating from $-T$ to $0(T>0)$ we get

$$(8)\quad \sum_{n\le x}{}' 1/\log f(n) = x\int_{-T}^{0} (C(t)-6/\pi^2)\,dt + 0(x^{1/2}\log^{-1/2} x\cdot T)$$
$$+0(Tx^{1/2}\exp(-C\log^{3/5} x(\log\log x)^{-1/5})) + \sum_{n\le x}{}' f^{-T}(n)/\log f(n).$$

To estimate $C(t)-6/\pi^2$ for $t\le 0$, let $C(t)=\prod_p(1-p^2+u(p,t))$, where

$$\begin{aligned}0<u(p,t) &= g^t(2)p^{-2}+(g^t(3)-g^t(2))p^{-3}+(g^t(4)-g^t(3))p^{-4}+\cdots\\ &= g^t(2)(p^{-2}-p^{-3})+g^t(3)(p^{-3}-p^{-4})+g^t(4)(p^{-4}-p^{-5})+\cdots\\ &\le g^t(r)(p^{-2}-p^{-3}+p^{-3}-p^{-4}+p^{-4}-\cdots)=g^t(r)p^{-2},\end{aligned}$$

where r is an integer such that $g(\nu)\ge g(r)>1$ for $\nu=2,3,\ldots$. Such an integer certainly exists, since $\liminf_{\nu\to\infty} g(\nu)>1$.

Using the inequality $\log(x+y)\le\log x+y/x$ $(x, y>0)$ we get

$$\begin{aligned}C(t) &= \exp\left(\log\prod_p(1-p^{-2}+u(p,t)\right)=\exp\left(\sum_p\log(1-p^{-2}+u(p,t)\right)\\ &\le\exp\left(\sum_p\log(1-p^{-2})+\sum_p(1-p^{-2})^{-1}u(p,t)\right)\\ &\le\frac{6}{\pi^2}\exp\left(g^t(r)\sum_p(p^2-1)^{-1}\right)\le 6\exp(g^t(r))/\pi^2.\end{aligned}$$

If $t<0$ is small enough we get

$$(9)\qquad 0\le C(t)-6/\pi^2\le(6/\pi^2)(\exp(g^t(r))-1)=0(g^t(r)),$$

$$(10)\qquad \int_{-\infty}^{-T}(C(t)-6/\pi^2)\,dt=0\left(\int_{-\infty}^{-T}g^t(r)\,dt\right)=0(g^{-T}(r)).$$

If $n=p_1^{\nu_1}\cdots p_i^{\nu_i}$, then $f(n)=g(\nu_1)\cdots g(\nu_i)\ge g(r)>1$ if $f(n)>1$, so that $f^T(n)\log f(n)\ge g^T(r)\log g(r)$ if $f(n)>1$, and we obtain

$$(11)\quad \sum_{n\le x}{}' f^{-T}(n)/\log f(n)\le\sum_{n\le x}{}' g^{-T}(r)/\log g(r)=0\left(g^{-T}(r)\sum_{n\le x}1\right)=0(g^{-T}(r)x).$$

Writing $\int_{-T}^{0}(C(t)-6/\pi^2)\,dt=\int_{-\infty}^{0}-\int_{-\infty}^{-T}$ and using (10) and (11) we get from (8)

$$(12)\quad \sum_{n\le x}{}' 1/\log f(n)=x\int_{-\infty}^{0}(C(t)-6/\pi^2)\,dt+0(g^{-T}(r)x)+0(x^{1/2}\log^{-1/2}x.T)$$
$$+0(Tx^{1/2}\exp(-C\log^{3/5}x(\log\log x)^{-1/5})).$$

Now take $T=\log x/2\log g(r)$. Then we have

$$g^{-T}(r)x=\exp(-T\log g(r)+\log x)=\exp(\tfrac{1}{2}\log x)=x^{1/2},$$

and so the theorem is proved.

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As a first example, let us take $a(n)$, the number of non-isomorphic abelian groups of order n. It is well-known (see [4]) that $a(n)$ is multiplicative, and that $a(p^\nu)=P(\nu)$ for any prime p and $\nu=1,2,\ldots$, where $P(\nu)$ is the number of unrestricted partitions of the integers ν, so that $P(\nu)=1$ if $\nu=1$ and $P(\nu)$ is strictly increasing with ν. Therefore the conditions of our theorem are satisfied, and (1) holds with $f(n)=a(n)$, $g(k)=P(k)$. Note that in this case we have $\liminf_{\nu\to\infty} g(\nu)=+\infty$ and $r=2$, $g(r)=2$.

Examples of other multiplicative, prime-independent functions that satisfy the conditions of our theorem may be readily found among enumerative functions of certain algebraic structures. Such is for example (see [6] for a detailed discussion) $S(n)$, the number of non-isomorphic semisimple finite rings of order n.

Finally let us consider $\tau^{(e)}(n)$, the number of exponential divisors of n. A divisor $d=p_1^{b_1}\cdots p_i^{b_i}$ is called an exponential divisor of $n=p_1^{\nu_1}\cdots p_i^{\nu_i}$ if $b_1\mid \nu_1,\ldots,b_i\mid \nu_i$ (see [8]). It follows that $\tau^{(e)}(n)$ is a multiplicative, prime-independent arithmetical function for which $\tau^{(e)}(p^\nu)=\tau(\nu)$, where $\tau(\nu)$ is the ordinary number of divisors function. Since $\tau(1)=1$ and $\tau(\nu)\geq 2$ if $\nu\geq 2$, the conditions of our theorem are satisfied and (1) holds with $f(n)=\tau^{(e)}(n)$ and $g(k)=\tau(k)$. Again it is of interest to note that $\liminf_{\nu\to\infty} g(\nu)=2$ and $r=2$, $g(r)=2$ also.

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