



Weighted L^p Boundedness of Pseudodifferential Operators and Applications

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Abstract. In this paper we prove weighted norm inequalities with weights in the A_p classes, for pseudodifferential operators with symbols in the class $S_{\rho,\delta}^{m(\rho-1)}$ that fall outside the scope of Calderón–Zygmund theory. This is accomplished by controlling the sharp function of the pseudodifferential operator by Hardy–Littlewood type maximal functions. Our weighted norm inequalities also yield L^p boundedness of commutators of functions of bounded mean oscillation with a wide class of operators in $OPS_{\rho,\delta}^m$.

1 Introduction

Recall that, given $u \in C_0^\infty(\mathbb{R}^n)$, a *pseudodifferential operator* is an operator defined by

$$a(x, D)u(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} a(x, \xi) \hat{u}(\xi) e^{i\langle x, \xi \rangle} d\xi,$$

where the *symbol* $a(x, \xi)$ is assumed to be smooth in both the spatial variable x and the frequency variable ξ , and satisfies certain growth conditions. An example of symbols one might consider is the class $S_{\rho,\delta}^m$, introduced by L. Hörmander [8], consisting of $a(x, \xi) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ with

$$|\partial_\xi^\alpha \partial_x^\beta a(x, \xi)| \leq C_{\alpha,\beta} \langle \xi \rangle^{m-\rho|\alpha|+\delta|\beta|},$$

where $\langle \xi \rangle = (1 + |\xi|^2)^{\frac{1}{2}}$, $m \in \mathbb{R}$, $0 \leq \delta, \rho \leq 1$.

An important problem in partial differential equations and harmonic analysis is the question of L^p boundedness of pseudodifferential operators, which has been extensively studied, but the problem of boundedness of these operators in weighted L^p spaces is a bit more involved. A pioneering investigation in this context was the paper by N. Miller [10], where he showed that for symbols in $S_{1,0}^0$ one has the weighted boundedness

$$(1.1) \quad \|a(x, D)u\|_{L_w^p} \leq C \|u\|_{L_w^p} \quad \text{for all } w \in A_p.$$

Here, of course, L_w^p denotes the weighted L^p space with weight w (see (2.1)). Later S. Chanillo and A. Torchinsky ([4]) considered symbols in the class $S_{\rho,\delta}^{\frac{m}{2}(\rho-1)}$ and showed (1.1) for $2 \leq p < \infty$ and $w \in A_{\frac{p}{2}}$.

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In this paper we obtain weighted boundedness results as well as commutator estimates for pseudodifferential operators. In the case of weighted boundedness results, we complete the study for operators in $S_{\rho,\delta}^{n(\rho-1)}$, with $0 \leq \delta < 1$, $0 < \rho \leq 1$ and show in Theorem 3.4 the L_w^p boundedness of these operators for all $w \in A_p$, $1 < p < \infty$. This is done by proving the pointwise estimate $(a(x, D)u)^\sharp(x_0) \leq M_p(u)(x_0)$. This is formulated as Theorem 3.3. As far as we are aware, the best previous result concerning weighed boundedness for these operators was the result of J. Álvarez and J. Hounie ([1]), where they showed weighted boundedness of pseudodifferential operators in $OPS_{\rho,\delta}^{n(\rho-1)}$ with $0 \leq \delta \leq \rho \leq \frac{1}{2}$. Our results are more natural in the sense that they remove the restriction $\delta \leq \rho \leq \frac{1}{2}$. While our methods are similar to those of [1] and [4] we are able to obtain a more favorable estimate on the Kernel in certain situations (see Lemma 3.2), which allows us to remove this restriction.

Our weighted norm inequalities also enable us to prove boundedness of the commutators of pseudodifferential operators with functions in BMO, the set of functions of *bounded mean oscillations* (see Definition 2.2). More precisely, we show that if $p \in (1, \infty)$, $f \in \text{BMO}$, and $a \in S_{\rho,\delta}^m$, with either $0 \leq \delta < \rho < 1$ and $m < n(\rho - 1)|\frac{1}{p} - \frac{1}{2}|$, or $\rho = 1$, and $m \leq 0$, then the commutator $[f, a(x, D)]$ is bounded on L^p . This is formulated as Theorem 4.2 and it extends the L^p boundedness of BMO commutators with $OPS_{1,0}^0$ due to R. Coifman, R. Rochberg, and G. Weiss [5], and the L^2 boundedness result of Chanillo [3] for commutators between BMO functions and operators in $OPS_{\rho,\delta}^m$, $m < 0$. In the case that $\rho = 1$ (even including $\delta = 1$) these were also extended by P. Auscher and M. Taylor [2].

Boundedness results for commutators arise in several places in connection to partial differential equations. For example they arise in the study of elliptic systems with BMO coefficients in [14] and the study of regularity for the Navier–Stokes equation in [6].

2 Basic Notions of Weights, Weighted Norm Inequalities, and BMO

Given $u \in L_{\text{loc}}^p$, the p -th maximal function $M_p u$ is defined by

$$M_p u(x) = \sup_{B \ni x} \left\{ \frac{1}{|B|} \int_B |u(y)|^p dy \right\}^{\frac{1}{p}},$$

where the supremum is taken over balls B in \mathbb{R}^n containing x . We shall use the notation $u_B := \frac{1}{|B|} \int_B |u(y)| dy$ for the average of the function u over B , so the standard Hardy–Littlewood maximal function is given by

$$Mu(x) := M_1 u(x) = \sup_{B \ni x} u_B.$$

Given the above notation one defines the class of Muckenhoupt A_p weights as follows.

Definition 2.1 Let w be a positive function in L^1_{loc} . One says $w \in A_1$ if $Mw(x) \leq Cw(x)$, for almost all $x \in \mathbb{R}^n$.

One says $w \in A_p$ for $p \in (1, \infty)$ if

$$\sup_{B \text{ balls in } \mathbb{R}^n} w_B(w_B^{-\frac{1}{p-1}})_B^{p-1} < \infty.$$

The A_p constants of a weight $w \in A_p$ for $p \in [1, \infty)$ are defined by

$$[w]_{A_1} := \sup_{B \text{ balls in } \mathbb{R}^n} w_B \|w^{-1}\|_{L^\infty(B)},$$

and

$$[w]_{A_p} := \sup_{B \text{ balls in } \mathbb{R}^n} w_B(w_B^{-\frac{1}{p-1}})_B^{p-1} \quad \text{for } p \in (1, \infty).$$

It is a well-known fact that for $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$, $w \in A_p$, if and only if $w^{-1/(p-1)} \in A_{p'}$; see [11]. In particular $w \in A_2$ if and only if $w^{-1} \in A_2$, a fact that will be used later on. Also $p_1 < p_2$ implies that $A_{p_1} \subset A_{p_2}$.

Given $w \in A_p$ the weighted L^p norm is defined by

$$(2.1) \quad \|u\|_{L^p_w} := \left\{ \int |u(y)|^p w(y) dy \right\}^{\frac{1}{p}}.$$

Furthermore for $u \in L^1_{\text{loc}}$, the Fefferman–Stein sharp function u^\sharp is defined by

$$u^\sharp(x) = \sup_{B \ni x} \inf_c \frac{1}{|B|} \int_B |u(y) - c| dy.$$

A fact about the weighted norm inequalities that will be crucial for our studies is that for a linear operator T a sharp function estimate of the form

$$(Tu)^\sharp(x) \leq C_p M_p u(x),$$

valid for all $1 < p < \infty$ and $u \in C^\infty_0$, implies the weighted L^p boundedness

$$\|Tu\|_{L^p_w} \lesssim \|u\|_{L^p_w},$$

for $1 < p < \infty$, $w \in A_p$ and $u \in L^p_w(\mathbb{R}^n)$; see for example [10, Theorem 2.12] for the details. In Section 4 of this paper we shall deal with functions of bounded mean oscillation (BMO) whose definition we recall.

Definition 2.2 A locally integrable function u belongs to BMO if

$$\sup_B \frac{1}{|B|} \int_B |u(x) - u_B| dx < \infty.$$

For $u \in \text{BMO}$ it is well known that $e^{r|u(x)|}$ is locally integrable for $r < 1$. This is a consequence of the John–Nirenberg theorem; see e.g., [7, p. 524]. Furthermore, for all $\gamma < \frac{1}{2^m e}$, there exists a constant $C_{n,\gamma}$ so that for $u \in \text{BMO}$ and all balls B ,

$$(2.2) \quad \frac{1}{|B|} \int_B e^{\gamma|u(x)-u_B|/\|u\|_{\text{BMO}}} dx \leq C_{n,\gamma}.$$

For this, see [7, p. 528]. This fact will be used in our commutator estimate, i.e., Theorem 4.2.

Notational convention Here and in the sequel, $a \lesssim b$ means $a \leq Cb$ for some constant C . The dependence of the constants on a and b will be obvious from the contexts. We will also denote all generic constants by C even though their value may differ from line to line.

3 Pseudodifferential Operators and their Weighted L^p Boundedness

In order to establish sharp function estimates we need a couple of lemmas that will handle balls of different sizes. In our proofs we will frequently use a standard Littlewood–Paley partition of unity $\{\varphi_k\}_{k \geq 0} \subset C_0^\infty(\mathbb{R}^n)$ with $\text{supp } \varphi_0 \subset \{\xi \mid |\xi| \leq 2\}$ and $\text{supp } \varphi_k \subset \{2^k \leq |\xi| \leq 2^{k+1}\}$ for $k \geq 1$.

One also has, for all multi-indices α and $N \geq 0$, $|\partial_\xi^\alpha \varphi_0(\xi)| \leq c_{\alpha,N} \langle \xi \rangle^{-N}$, $|\partial_\xi^\alpha \varphi_k(\xi)| \leq c_\alpha 2^{-k|\alpha|}$ for some $c_\alpha > 0$ and all $k \geq 1$, and

$$\varphi_0(\xi) + \sum_{k=1}^\infty \varphi_k(\xi) = 1, \quad \text{for all } \xi \in \mathbb{R}^n.$$

The first lemma, which is fairly simple, yields a classical kernel estimate. This implies the rapid decrease of the Schwartz kernel of the Hörmander class of pseudodifferential operators

Lemma 3.1 *Let $a \in S_{\rho,\delta}^m$ with $m \in \mathbb{R}$ and $\rho, \delta \in [0, 1]$. Let $a_k(x, \xi) = a(x, \xi)\varphi_k(\xi)$, for $k \geq 0$ with φ_k as in the above Littlewood–Paley decomposition. Then*

$$(3.1) \quad |z|^l \left| \int \partial_x^\beta a_k(x, \xi) e^{i\langle z, \xi \rangle} d\xi \right| \lesssim 2^{k(n+m-\rho l+\delta|\beta|)},$$

for all $x, z \in \mathbb{R}^n$ and $l \geq 0$

Proof Using the definition of the symbol class $S_{\rho,\delta}^m$ and the Leibniz rule one readily sees that $|\partial_\xi^\alpha \partial_x^\beta a_k(x, \xi)| \lesssim 2^{k(m-\rho|\alpha|+\delta|\beta|)}$. Integration by parts then yields

$$\begin{aligned} \left| z_i^l \int \partial_x^\beta a_k(x, \xi) e^{i\langle z, \xi \rangle} d\xi \right| &\simeq \left| \int \partial_x^\beta a_k(x, \xi) \partial_{\xi_i}^l (e^{i\langle z, \xi \rangle}) d\xi \right| \\ &\lesssim \left| \int \partial_{\xi_i}^l \partial_x^\beta a_k(x, \xi) e^{i\langle z, \xi \rangle} d\xi \right| \lesssim 2^{k(n+m-\rho l+\delta|\beta|)}. \end{aligned}$$

Summing in i proves (3.1) for all integers $l \geq 0$. For non-integer values of l , the result follows by interpolation of the inequality for k and $k + 1$, where $k < l < k + 1$. ■

Now since the Schwartz kernel of the pseudodifferential operator $a(x, D)$ is given by $K(x, y) = \frac{1}{(2\pi)^n} \int a(x, \xi) e^{i(y, \xi)} d\xi$, Lemma 3.1 yields

$$(3.2) \quad |K(x, y)| \lesssim |y|^{-N} \quad \text{for } N > 0, |x - y| \geq 1 \text{ and } \rho > 0.$$

Estimate (3.2) will be needed for the sharp function estimate over balls of radius larger than one. For the same reason we also need the following Hörmander-type kernel estimate, which corresponds to the kernel estimates of [9] for the case of multipliers.

Lemma 3.2 *Let $a \in S_{\rho, \delta}^m$, $0 \leq \delta \leq 1$, $0 < \rho \leq 1$ and let $K(x, z) = \int e^{i(z, \xi)} a(x, \xi) d\xi$. Then for*

$$|x - x_B| \leq r \leq 1, \quad \theta \in [0, 1], p \in [1, 2], \quad \frac{1}{p} + \frac{1}{p'} = 1,$$

$$\frac{m}{\rho} + \frac{n}{p\rho} < l < \frac{m}{\rho} + \frac{n}{p\rho} + \frac{1}{\rho}, \quad \frac{1}{2} < c_1 < 2c_2 < \infty \quad \text{and} \quad j \geq 1,$$

the following estimate holds:

$$(3.3) \quad \left\{ \int_{c_1 2^j r^\theta < |y - x_B| < c_2 2^{j+1} r^\theta} |K(x, x - y) - K(x_B, x_B - y)|^{p'} dy \right\}^{\frac{1}{p'}} \lesssim 2^{-jl} r^{l(\rho - \theta) - m - \frac{n}{p}}.$$

Proof Using the Littlewood-Paley partition of unity introduced previously, we set $K_k(x, z) = \int e^{i(z, \xi)} a_k(x, \xi) d\xi$ and observe that

$$(3.4) \quad \left\{ \int_{c_1 2^j r^\theta < |y - x_B| < c_2 2^{j+1} r^\theta} |K(x, x - y) - K(x_B, x_B - y)|^{p'} dy \right\}^{\frac{1}{p'}} \leq \sum_{k=0}^{\infty} \left\{ \int_{c_1 2^j r^\theta < |y - x_B| < c_2 2^{j+1} r^\theta} |K_k(x, x - y) - K_k(x_B, x_B - y)|^{p'} dy \right\}^{\frac{1}{p'}}.$$

We now set $A_j := \{y \mid c_1 2^j r^\theta < |y - x_B| < c_2 2^{j+1} r^\theta\}$, choose k_0 such that $2^{k_0} |x - x_B| \sim 1$, and split the sum on the left-hand side of (3.4) as

$$\sum_{k \geq k_0} \left\{ \int_{A_j} |K_k(x, x - y)|^{p'} dy \right\}^{\frac{1}{p'}} + \sum_{k \geq k_0} \left\{ \int_{A_j} |K_k(x_B, x_B - y)|^{p'} dy \right\}^{\frac{1}{p'}} + \sum_{k < k_0} \left\{ \int_{A_j} |K_k(x, x - y) - K_k(x_B, x_B - y)|^{p'} dy \right\}^{\frac{1}{p'}} =: I_1 + I_2 + I_3.$$

The estimates for I_1 and I_2 are similar, so we only carry out that of I_1 . To this end we observe that if $y \in A_j$ and $|x - x_B| \leq r$, then the triangle inequality yields that $|y - x| \geq 2^j r^\theta (c_1 - 2^{-j} r^{1-\theta})$, and since $r \leq 1$ and $j \geq 1$, one actually has $|y - x| \gtrsim 2^j r^\theta$. Therefore, using the Hausdorff–Young inequality we have

$$\begin{aligned}
 I_1 &\lesssim \sum_{k \geq k_0} 2^{-lj} r^{-l\theta} \left\{ \int_{A_j} |K_k(x, x - y)|^{p'} |x - y|^{p'l} dy \right\}^{\frac{1}{p'}} \\
 &\lesssim \sum_{k \geq k_0} 2^{-lj} r^{-l\theta} \sum_{|\alpha|=l} \left\{ \int |\partial_\xi^\alpha a_k(x, \xi)|^p d\xi \right\}^{\frac{1}{p}} \lesssim \sum_{k \geq k_0} 2^{-lj} r^{-l\theta} 2^{k(m-l\rho+\frac{n}{p})}.
 \end{aligned}$$

Hence, taking $l > \frac{m}{\rho} + \frac{n}{\rho p}$ and recalling that $2^{k_0} |x - x_B| \sim 1$, we obtain the estimate

$$(3.5) \quad I_1 \lesssim 2^{-lj} r^{-l\theta} 2^{k_0(m-l\rho+\frac{n}{p})} \lesssim 2^{-lj} r^{-l\theta} |x - x_B|^{l\rho-m-\frac{n}{p}} \leq 2^{-jl} r^{l(\rho-\theta)-m-n/p}.$$

To estimate I_3 we observe that

$$\begin{aligned}
 I_3 &\leq \sum_{k < k_0} \left\{ \int_{A_j} |K_k(x, x - y) - K_k(x_B, x - y)|^{p'} dy \right\}^{\frac{1}{p'}} \\
 &\quad + \sum_{k < k_0} \left\{ \int_{A_j} |K_k(x_B, x - y) - K_k(x_B, x_B - y)|^{p'} dy \right\}^{\frac{1}{p'}} := I_{3,1} + I_{3,2}.
 \end{aligned}$$

Reasoning in the same way as we did for I_1 , the Hausdorff–Young inequality and the mean-value theorem yield

$$\begin{aligned}
 (3.6) \quad I_{3,1} &\lesssim \sum_{k < k_0} 2^{-lj} r^{-l\theta} \left\{ \int_{A_j} |K_k(x, x - y) - K_k(x_B, x - y)|^{p'} |x - y|^{p'l} dy \right\}^{\frac{1}{p'}} \\
 &\lesssim \sum_{k < k_0} 2^{-lj} r^{-l\theta} \sum_{|\alpha|=l} \left\{ \int \sup_x |\partial_x \partial_\xi^\alpha a_k(x, \xi)|^p d\xi \right\}^{\frac{1}{p}} |x - x_B| \\
 &\lesssim \sum_{k < k_0} 2^{-lj} r^{-l\theta} 2^{k(m-l\rho+\frac{n}{p}+1)} |x - x_B| \\
 &\lesssim 2^{-lj} r^{-l\theta} |x - x_B|^{l\rho-m-\frac{n}{p}} \leq 2^{-jl} r^{l(\rho-\theta)-m-n/p},
 \end{aligned}$$

provided that $l < \frac{m}{\rho} + \frac{n}{\rho p} + \frac{1}{\rho}$.

Now, to estimate $I_{3,2}$ we observe that

$$\begin{aligned} & (y - x_B)^\alpha (K_k(x_B, x - y) - K_k(x_B, x_B - y)) \\ &= (y - x_B)^\alpha \int (a_k(x_B, \xi) e^{i\langle x-y, \xi \rangle} - a_k(x_B, \xi) e^{i\langle x_B-y, \xi \rangle}) d\xi \\ &= (y - x_B)^\alpha \int a_k(x_B, \xi) e^{i\langle x_B-y, \xi \rangle} (e^{i\langle x-x_B, \xi \rangle} - 1) d\xi \\ &= (-1)^{|\alpha|} \int e^{i\langle x_B-y, \xi \rangle} \partial_\xi^\alpha \{ a_k(x_B, \xi) (e^{i\langle x-x_B, \xi \rangle} - 1) \} d\xi \\ &= (-1)^{|\alpha|} \left[\partial_\xi^\alpha \{ a_k(x_B, \xi) (e^{i\langle x-x_B, \xi \rangle} - 1) \} \right]^\wedge (y - x_B), \end{aligned}$$

where $[\cdot]^\wedge$ denotes the Fourier transform in the ξ variable.

Therefore,

$$\begin{aligned} & |y - x_B|^l |K_k(x_B, x - y) - K_k(x_B, x_B - y)| \\ & \lesssim \left| \left[\sum_{|\beta+\gamma|=l, |\gamma|>0} \partial_\xi^\beta a_k(x_B, \xi) \partial_\xi^\gamma (e^{i\langle x-x_B, \xi \rangle} - 1) \right. \right. \\ & \qquad \qquad \qquad \left. \left. + \sum_{|\beta|=l} \partial_\xi^\beta a_k(x_B, \xi) (e^{i\langle x-x_B, \xi \rangle} - 1) \right]^\wedge (y - x_B) \right|. \end{aligned}$$

So once again, the Hausdorff–Young inequality, mean value theorem, and using again the fact that $2^{k_0} |x - x_B| \sim 1$ yield

$$\begin{aligned} (3.7) \quad I_{3,2} & \lesssim \sum_{k < k_0} 2^{-lj} r^{-l\theta} \left\{ \int_{A_j} |y - x_B|^{p'l} |K_k(x_B, x - y) - K_k(x_B, x_B - y)|^{p'} \right\}^{\frac{1}{p'}} \\ & \lesssim \sum_{k < k_0} 2^{-lj} r^{-l\theta} \left\{ \int \left| \left[\sum_{\substack{|\beta+\gamma|=l \\ |\gamma|>0}} \partial_\xi^\beta a_k(x_B, \xi) \partial_\xi^\gamma (e^{i\langle x-x_B, \xi \rangle} - 1) \right. \right. \right. \\ & \qquad \qquad \qquad \left. \left. \left. + \sum_{|\beta|=l} \partial_\xi^\beta a_k(x_B, \xi) (e^{i\langle x-x_B, \xi \rangle} - 1) \right] \right|^p d\xi \right\}^{\frac{1}{p}} \\ & \lesssim \sum_{k < k_0} 2^{-lj} r^{-l\theta} \sum_{|\beta+\gamma|=l, |\gamma|>0} 2^{k(m-\rho|\beta|+\frac{n}{p})} |x - x_B|^{|\gamma|-1} |x - x_B| \\ & \qquad \qquad \qquad + \sum_{k < k_0} 2^{-lj} r^{-l\theta} \sum_{|\beta|=l} 2^{k(m-\rho|\beta|+\frac{n}{p}+1)} |x - x_B| \end{aligned}$$

$$\begin{aligned} &\lesssim \sum_{k < k_0} 2^{-lj} r^{-l\theta} \sum_{|\beta+\gamma|=l, |\gamma|>0} 2^{k(m-\rho|\beta|+\frac{n}{p})} 2^{-k\rho(|\gamma|-1)} |x - x_B| \\ &\quad + \sum_{k < k_0} 2^{-lj} r^{-l\theta} 2^{k(m-\rho l+\frac{n}{p}+1)} |x - x_B| \\ &\lesssim \sum_{k < k_0} 2^{-lj} r^{-l\theta} 2^{k(m-\rho l+\frac{n}{p}+1)} |x - x_B| \\ &\lesssim 2^{-lj} r^{-l\theta} |x - x_B|^{l\rho-m-\frac{n}{p}} \leq 2^{-lj} r^{l(\rho-\theta)-m-n/p}, \end{aligned}$$

provided that we take $l < \frac{m}{\rho} + \frac{n}{\rho p} + \frac{1}{\rho}$. Thus putting the estimates (3.5), (3.6), and (3.7) together yields (3.3). ■

Theorem 3.3 *Let $a \in S_{\rho,\delta}^m$, with $0 < \rho \leq 1, 0 \leq \delta < 1$ and $m = n(\rho - 1)$. Then for $u \in C_0^\infty$ and $p \in (1, \infty)$ one has that*

$$(3.8) \quad (a(x, D)u)^\sharp(x_0) \lesssim M_p u(x_0).$$

Proof Given $x_0 \in \mathbb{R}^n$, we let B be a ball containing x_0 , with centre x_B and radius r , and we let B' be the ball concentric to B with radius $2r$. Let $u \in C_0^\infty(\mathbb{R}^n)$ and decompose u as

$$u = u\chi_{B'} + u(1 - \chi_{B'}) =: u_1 + u_2.$$

This yields

$$\begin{aligned} &\frac{1}{|B|} \int_B |a(x, D)u(x) - (a(\cdot, D)u)_B| dx \\ &\leq \frac{2}{|B|} \int_B |a(x, D)u_1| dx + \frac{1}{|B|} \int_B |a(x, D)u_2(x) - (a(\cdot, D)u_2)_B| dx \\ &:= \mathbf{I} + \mathbf{II}. \end{aligned}$$

To estimate **I** we just use Hölder’s inequality and the L^p boundedness of pseudo-differential operators of order $m = n(\rho - 1)$, which yields

$$\begin{aligned} \mathbf{I} &\leq 2 \left\{ \frac{1}{|B|} \int_B |a(x, D)u_1(x)|^p dx \right\}^{\frac{1}{p}} \lesssim \left\{ \frac{1}{|B|} \int_{\mathbb{R}^n} |u_1|^p dx \right\}^{\frac{1}{p}} \\ &\lesssim \left\{ \frac{|B'|}{|B|} \right\}^{\frac{1}{p}} \left\{ \frac{1}{|B'|} \int_{B'} |u|^p dx \right\}^{\frac{1}{p}} \lesssim M_p u(x_0). \end{aligned}$$

Here we would like to emphasize the fact that the estimate of **I** is completely independent of the radius of B .

In order to estimate \mathbf{II} we consider first the case of balls B with radius $r \geq 1$. We aim to show

$$(3.9) \quad \mathbf{II} \lesssim Mu(x_0),$$

which in turn can be bounded by $M_p u(x_0)$, because of the fact that the L^p maximal functions M_p are increasing in p (this follows from a simple application of Hölder’s inequality).

To show (3.9), let B_k be the ball with centre x_B and radius $2^{k+1}r$. Then \mathbf{II} can be estimated by

$$\begin{aligned} & \frac{1}{|B|} \int_B |a(x, D)u_2(x) - (a(\cdot, D)u_2)_B| dx \\ &= \frac{1}{|B|} \int_B \left| \frac{1}{|B|} \int_B a(x, D)u_2(x) - a(z, D)u_2(z) dz \right| dx \\ &= \frac{1}{|B|} \int_B \left| \frac{1}{|B|} \int_B \int_{\mathbb{R}^n} u_2(y) \int_{\mathbb{R}^n} (a(x, \xi)e^{i\langle x-y, \xi \rangle} - a(z, \xi)e^{i\langle z-y, \xi \rangle}) d\xi dy dz \right| dx \\ & \quad \times \int_{\mathbb{R}^n} |u_2(y)| \left\{ \frac{1}{|B|} \int_B \left| \int_{\mathbb{R}^n} a(x, \xi)e^{i\langle x-y, \xi \rangle} d\xi \right| dx \right. \\ & \quad \left. + \frac{1}{|B|} \int_B \left| \int_{\mathbb{R}^n} a(z, \xi)e^{i\langle z-y, \xi \rangle} d\xi \right| dz \right\} dy \\ & \leq 2 \sum_{k=1}^{\infty} \frac{1}{|B|} \int_B \int_{2^k r \leq |y-x_B| < 2^{k+1} r} |u(y)| \left| \int_{\mathbb{R}^n} a(x, \xi)e^{i\langle x-y, \xi \rangle} d\xi \right| dy dx \\ & \lesssim \sum_{k=1}^{\infty} \int_B \frac{2^{nk}}{|B_k|} \int_{2^k r \leq |y-x_B| < 2^{k+1} r} \frac{|u(y)|}{|x-y|^{n+1}} |x-y|^{n+1} |K(x, x-y)| dy dx \\ & \lesssim \sum_{k=1}^{\infty} r^n 2^{nk} (2^k r)^{-n-1} \frac{1}{|B_k|} \int_{B_k} |u(y)| dy \lesssim r^{-1} \sum_{k=1}^{\infty} 2^{-k} Mu(x_0) \lesssim Mu(x_0), \end{aligned}$$

where we have used that $\text{supp } u_2(y) \subset \{y; |y - x_B| > 2r\}$, and also (3.2), which holds here because the radius $r \geq 1$.

Therefore, for balls B containing x_0 with radius $r \geq 1$ we have

$$(3.10) \quad \frac{1}{|B|} \int_B |a(x, D)u(x) - (a(x, D)u)_B| dx \lesssim M_p u(x_0).$$

Here we remark that (3.10) is actually valid for all balls of radius $r \geq 1$ and all $a \in S_{\rho, \delta}^m$ with $0 \leq \delta < 1$, $0 < \rho \leq 1$ and $m \leq n(\rho - 1)|\frac{1}{p} - \frac{1}{2}| + \min(0, \frac{n(\rho - \delta)}{2})$ for which the L^p boundedness is known (see [1]).

It remains to estimate \mathbf{II} in the case that the radius of B is less than one. We set $C_B := \frac{1}{|B|} \int K(x_B, x_B - y)u_2(y)dy$. Since $\text{supp } u_2 \subset \{y \mid |y - x_B| > 2r\}$, applying

Lemma 3.2, with $c_1 = c_2 = 1, \theta = 1$ and $m = n(\rho - 1)$ yields

$$\begin{aligned}
 (3.11) \quad & \frac{1}{|B|} \int_B |a(x, D)u_2(x) - C_B| dx \\
 &= \frac{1}{|B|} \int_B \left| \int (K(x, x - y) - K(x_B, x_B - y)) u_2(y) dy \right| dx \\
 &\leq \sum_{j=1}^{\infty} \frac{1}{|B|} \int_B \left\{ \int_{2^j r < |y-x_B| < 2^{j+1} r} |K(x, x - y) - K(x_B, x_B - y)|^{p'} dy \right\}^{\frac{1}{p'}} \\
 &\quad \times \left\{ \int |u_2(y)|^p dy \right\}^{\frac{1}{p}} dx \\
 &\lesssim \sup_x \sum_{j=1}^{\infty} \left\{ \int_{2^j r < |y-x_B| < 2^{j+1} r} |K(x, x - y) - K(x_B, x_B - y)|^{p'} dy \right\}^{\frac{1}{p'}} \\
 &\quad \times \left\{ \int_{2^j r < |y-x_B| < 2^{j+1} r} |u_2(y)|^p dy \right\}^{\frac{1}{p}} \\
 &\lesssim \sum_{j=1}^{\infty} 2^{-jl} r^{l(\rho-1) - n(\rho-1) - \frac{n}{p}} (2^j r)^{\frac{n}{p}} M_p u(x_0) \lesssim \sum_{j=1}^{\infty} 2^{j(\frac{n}{p} - l)} M_p u(x_0),
 \end{aligned}$$

provided that $l < n$. Furthermore, in order to be able to sum the series $\sum_{j \geq 1} 2^{j(\frac{n}{p} - l)}$ we need to take $l > \frac{n}{p}$. The main problem here is that we need to combine this condition on l with that of the Lemma 3.2, which in the case of $m = n(\rho - 1)$ is $n - \frac{n}{\rho} + \frac{n}{p\rho} < l < n - \frac{n}{\rho} + \frac{n}{p\rho} + \frac{1}{\rho}$.

Combining these conditions means requiring

$$\max \left\{ \frac{n}{p}, n - \frac{n}{\rho} + \frac{n}{p\rho} \right\} < l < \min \left\{ n, n - \frac{n}{\rho} + \frac{n}{p\rho} + \frac{1}{\rho} \right\}.$$

Since $n - \frac{n}{\rho} + \frac{n}{p\rho} \leq \frac{n}{p}$ and $\frac{n}{p} < n$ for $p > 1$, this is possible provided

$$\frac{n}{p} < n - \frac{n}{\rho} + \frac{n}{p\rho} + \frac{1}{\rho}.$$

Rearranging this, we see that this is equivalent to

$$0 < n \left(1 - \frac{1}{\rho} \right) \left(1 - \frac{1}{p} \right) + \frac{1}{\rho},$$

which is the case if we take p sufficiently close to one.

Hence choosing an admissible l , summing the series in (3.11), and once again recalling the increasing behaviour of M_p in p , we obtain

$$\frac{1}{|B|} \int |a(x, D)u_2 - C_B| dx \lesssim M_p u(x_0),$$

for all balls B with radius less than one. Now taking the supremum over balls of various sizes that all contain x_0 yields (3.8) for all $p \in (1, \infty)$. ■

It is well known that sharp function estimates of the above form imply L^p_w estimates for all weights $w \in A_p$; see e.g., [10]. Therefore Theorem 3.3 yields the following theorem, whose proof we omit.

Theorem 3.4 *Let $a \in S^m_{\rho,\delta}$, with $0 < \rho \leq 1$, $0 \leq \delta < 1$ and $m \leq n(\rho - 1)$. Then for each $p \in (1, \infty)$ and each Muckenhoupt weight $w \in A_p$ the following weighted norm inequality holds:*

$$\|a(x, D)u\|_{L^p_w} \lesssim \|u\|_{L^p_w}.$$

We note that in some cases the ranges of ρ , δ , and m in Theorem 3.4 cannot be extended. The case $\delta = 1$ includes the class $S^0_{1,1}$ that fails to be L^2 bounded. In the case $\rho = 1$, the range of $m \leq 0$ is clearly sharp by considering $a(x, \xi) = \langle \xi \rangle^m$. Finally $\rho = 0$ is excluded purely as a technicality to avoid dividing by 0 in Lemma 3.2.

Using complex interpolation, we can show weighted boundedness for a rather large class of operators, but only for a subclass of all weights. This turns out to be sufficient in establishing L^p -boundedness of commutators of pseudodifferential operators with BMO functions.

Theorem 3.5 *Let $p \in [1, 2]$, $0 \leq \rho \leq 1$, $0 \leq \delta < 1$ be fixed numbers. Suppose that there exists $M_1 \in \mathbb{R}$ such that for all $a_1(x, \xi) \in S^m_{\rho,\delta}$ with $m < M_1$, the operators $a_1(x, D)$ are bounded from L^p to itself. Furthermore, assume that there exists $M_2 < M_1$ such that for all $a_2(x, \xi) \in S^m_{\rho,\delta}$ with $m < M_2$ and all $w \in A_p$, the operators $a_2(x, D)$ are bounded from L^p_w to itself. Then for each $m < M_1$ there exists $\alpha \in (0, 1)$ depending on p, ρ, δ , and m , such that the operator $a(x, D)$ is a bounded on L^p_w for all $\varepsilon \in [0, \alpha]$, all $w \in A_p$ and $a(x, D) \in OPS^m_{\rho,\delta}$.*

Remark 3.6 Here in the assumptions of the theorem we implicitly assume that the boundedness of the various operators require only finitely many derivatives of the corresponding symbols. Since for a generic pseudodifferential operator the L^p and weighted L^p boundedness only require differentiability in the symbol up to a finite order depending on the dimension n and p , for the purpose of the proof of this theorem the symbol estimates in the proof of the theorem can be restricted to only a finite number of derivatives in ξ and x .

Proof If $m < M_1$, then there exists M' such that $m < M' < M_1$. We put $m_\varepsilon := M' - m$, which is obviously a positive number. Also, let M'' be a number less than $\min(m, M_2)$. Then we put $m_b := M'' - m$, and therefore m_b is a negative number. Now for $a \in S^m_{\rho,\delta}$ we introduce a family of symbols $a_z(x, \xi) := \langle \xi \rangle^z a(x, \xi)$, where $z \in \Omega := \{z \in \mathbb{C}; m_b \leq \operatorname{Re} z \leq m_\varepsilon\}$. It is easy to see that for $|\alpha + \beta| \leq C_1$ with C_1 large enough and $z \in \Omega$,

$$|\partial_\xi^\alpha \partial_x^\beta a_z(x, \xi)| \lesssim (1 + |\operatorname{Im} z|)^{C_2} \langle \xi \rangle^{\operatorname{Re} z + m - \rho|\alpha| + \delta|\beta|}.$$

Given $u \in C^\infty_0$ and $w \in A_p$, we also introduce the operator

$$T_z u := w^{\frac{z - m_\varepsilon}{p(m_b - m_\varepsilon)}} a_z(x, D) \left(w^{-\frac{z - m_\varepsilon}{p(m_b - m_\varepsilon)}} u \right).$$

Since $p \in [1, 2]$, $A_p \subset A_2$, which in turn implies that both $w^{\frac{1}{p}}$ and $w^{-\frac{1}{p}}$ belong to L^p_{loc} . Therefore, for $z \in \Omega$, T_z is an analytic family of operators in the sense of Stein and Weiss [13]. Now we claim that for $z_b \in \mathbb{C}$ with $\operatorname{Re} z_b = m_b$, the operator $(1 + |\operatorname{Im} z_b|)^{-C_2} a_{z_b}(x, D)$ is bounded on L^p_w . Indeed the symbol of this operator is $(1 + |\operatorname{Im} z_b|)^{-C_2} a_{z_b}(x, \xi)$, which belongs to $S^{m_b+m}_{\rho, \delta}$ uniformly in z , and since by construction $m_b + m = M'' < M_2$ and $w \in A_p$, our assumption about the L^p_w -boundedness of the operators of order less than M_2 yields

$$\begin{aligned} \|T_{z_b} u\|_{L^p}^p &= (1 + |\operatorname{Im} z_b|)^{pC_2} \left\| (1 + |\operatorname{Im} z_b|)^{-C_2} w^{\frac{z_b - m_e}{p(m_b - m_e)}} a_{z_b}(x, D) (w^{-\frac{z_b - m_e}{p(m_b - m_e)}} u) \right\|_{L^p}^p \\ &\lesssim (1 + |\operatorname{Im} z_b|)^{pC_2} \left\| w^{-\frac{z_b - m_e}{p(m_b - m_e)}} u \right\|_{L^p_w}^p = (1 + |\operatorname{Im} z_b|)^{pC_2} \left\| w^{-\frac{z_b - m_e}{p(m_b - m_e)}} u \right\|_{L^p_w}^p \\ &= (1 + |\operatorname{Im} z_b|)^{pC_2} \|u\|_{L^p}^p, \end{aligned}$$

where we have used the fact that $|w^{\pm \frac{z_b - m_e}{p(m_b - m_e)}}| = w^{\pm \frac{1}{p}}$. Similarly if $z_e \in \mathbb{C}$ with $\operatorname{Re} z_e = m_e$, then $|w^{\pm \frac{z_e - m_e}{p(m_b - m_e)}}| = 1$ and since the symbol of the operator $(1 + |\operatorname{Im} z_e|)^{-C_2} a_{z_e}(x, D)$ belongs to $S^{m_e+m}_{\rho, \delta}$ uniformly in z and $m_e + m = M' < M_1$, the L^p -boundedness assumption for operators of order $m < M_1$ yields

$$\begin{aligned} \|T_{z_e} u\|_{L^p}^p &\lesssim (1 + |\operatorname{Im} z_e|)^{pC_2} \|u\|_{L^p}^p, \\ \|T_0 u\|_{L^p}^p &= \left\| w^{\frac{m_e}{p(m_b - m_e)}} a(x, D) (w^{-\frac{m_e}{p(m_b - m_e)}} u) \right\|_{L^p}^p \leq C \|u\|_{L^p}^p. \end{aligned}$$

Hence, setting $\alpha = \frac{m_e}{m_e - m_b}$ we have $\alpha \in (0, 1)$ and $\|a(x, D)u\|_{L^{p\alpha}_w} \lesssim \|u\|_{L^{p\alpha}}$. Now interpolation of this estimate with $\|a(x, D)u\|_{L^p} \lesssim \|u\|_{L^p}$, using interpolation of operators with change of measure [12], we finally arrive at $\|a(x, D)u\|_{L^{p\varepsilon}_w} \lesssim \|u\|_{L^{p\varepsilon}}$ for all $\varepsilon \in [0, \alpha]$, $a \in S^m_{\rho, \delta}$ with $m < M_1$ and $w \in A_p$. ■

Corollary 3.7 *For pseudodifferential operators, one has the following weighted estimates:*

- (i) Let $p \in (1, 2]$, $0 < \rho < 1$, $0 \leq \delta < 1$, $m < n(\rho - 1)|\frac{1}{p} - \frac{1}{2}| + \min\{0, \frac{n(\rho - \delta)}{2}\}$ then there exists $\alpha \in (0, 1)$ such that for all $\varepsilon \in [0, \alpha]$, $a(x, \xi) \in S^m_{\rho, \delta}$, $a(x, D)$ is bounded on $L^{p\varepsilon}_w$, for all $w \in A_p$.
- (ii) If $a(x, \xi) \in S^m_{1, \delta}$ with $m \leq 0$ and $\delta < 1$, then $a(x, D)$ is bounded on $L^{p\varepsilon}_w$, for all $\varepsilon \in [0, 1]$, $p \in (1, \infty)$ and $w \in A_p$.

Proof (i) Let us take $M_1 = n(\rho - 1)|\frac{1}{p} - \frac{1}{2}| + \min\{0, \frac{n(\rho - \delta)}{2}\}$ and $M_2 = n(\rho - 1)$, in Theorem 3.5. Then for $m < n(\rho - 1)$, $p \in (1, \infty)$, $0 < \rho < 1$, $0 \leq \delta < 1$, $a(x, D) \in OPS^m_{\rho, \delta}$ is a bounded operator on L^p_w for all $w \in A_p$ (as a consequence of Theorem 3.4), and for

$$\begin{aligned} m &< n(\rho - 1) \left| \frac{1}{p} - \frac{1}{2} \right| + \min\left\{ 0, \frac{n(\rho - \delta)}{2} \right\}, \quad p \in (1, 2], \\ 0 &< \rho \leq 1, 0 \leq \delta < 1, \quad a(x, D) \in OPS^m_{\rho, \delta} \end{aligned}$$

is bounded on L^p (see [1]). Consequently, all the assumptions of Theorem 3.5 are fulfilled, and therefore we obtain the desired result.

(ii) If $a(x, \xi) \in S_{1,\delta}^m$ and $m \leq 0, \delta < 1$ then we have the L_w^p -boundedness of $a(x, D)$ from Theorem 3.3 and also, in particular just L^p -boundedness. Interpolation with change of measure between these two results yields the claim. ■

4 Applications to the Boundedness of Commutators

In this section we show how our weighted norm inequalities can be used to derive the L^p boundedness of commutators of these operators with functions of bounded mean oscillation. We start with the following lemma.

Lemma 4.1 *Let $a(x, D) \in OPS_{\rho,\delta}^m$ be an L^2 bounded operator. Given $f \in BMO$ with $\|f\|_{BMO} = 1$ and u and v in C_0^∞ , there exists $\lambda > 0$ such that the function $\Phi(z) := \int e^{zf(x)} a(x, D)(e^{-zf(x)}u)(x)v(x) dx$ is holomorphic in the disc $|z| < \lambda$.*

Proof First of all, we can assume that the symbol of $a(x, D)$ is compactly supported in the x variables. Indeed since the function $v(x)$ in the definition of Φ is compactly supported, by multiplying the integrand defining $\Phi(z)$ with a compactly supported function equal to 1 on the support of v , we can make the x -support of the symbol of $a(x, D)$ compact. Given $g \in C_0^\infty$, we now take a smooth cut-off function $\chi(y, \xi)$ equal to 1 in a neighbourhood of the origin, take $\varepsilon \in (0, 1]$, and define the function

$$a_\varepsilon(x, D)g(x) := \iint a(x, \xi) \chi(\varepsilon y, \varepsilon \xi) e^{i\langle x-y, \xi \rangle} g(y) dy d\xi.$$

It is well known (see [11, p. 258]) that $a_\varepsilon(x, D)g$ converges in the Schwartz class \mathcal{S} to $a(x, D)g$. Since convergence in \mathcal{S} also implies uniform convergence on compact sets, this together with the assumption of compact support in x of the symbol $a(x, \xi)$, yields that if $g \in C_0^\infty, \lim_{\varepsilon \rightarrow 0} \|a_\varepsilon(x, D)g - a(x, D)g\|_{L^2} = 0$.

Due to density of C_0^∞ in L^2 and the L^2 -boundedness assumption on $a(x, D)$ (and that of $a_\varepsilon(x, D)$ with uniform bounds in ε), it is easy to show that for all $g \in L^2$,

$$(4.1) \quad \lim_{\varepsilon \rightarrow 0} \|a_\varepsilon(x, D)g - a(x, D)g\|_{L^2} = 0.$$

Now recall the fact mentioned in Section 2 about functions $f \in BMO$, namely the local integrability of $e^{r|f(x)|}$ for $r < 1$. This means $e^{r|f|} \in L_{loc}^2$ if $r < \frac{1}{2}$, and hence for $u \in C_0^\infty$ and $|z| < \frac{1}{2}$, one has $ue^{\pm zf} \in L^2$. Now, if we define $\Phi_\varepsilon(z) = \int v(x) e^{zf(x)} a_\varepsilon(x, D)(e^{-zf(x)}u)(x) dx$, then since the integral defining Φ_ε is absolutely convergent and its integrand is holomorphic in z for $|z| < 1$, it follows that Φ_ε is a holomorphic function in $|z| < 1$. Now we claim that for γ , as in (2.2),

$$\lim_{\varepsilon \rightarrow 0} \sup_{|z| < \frac{\gamma}{2}} |\Phi_\varepsilon(z) - \Phi(z)| = 0.$$

Indeed, since $\frac{\gamma}{2} < \frac{1}{2}$, for $|z| < \frac{\gamma}{2}$,

$$\begin{aligned} |\Phi_\varepsilon(z) - \Phi(z)| &= \left| \int v(x)e^{zf(x)} [a_\varepsilon(x, D) - a(x, D)] (e^{-zf}u)(x) dx \right| \\ &\leq \|v e^{zf}\|_{L^2} \| [a_\varepsilon(x, D) - a(x, D)] (u e^{-zf}) \|_{L^2} \\ &\leq \|v\|_{L^\infty} \left\{ \int_{\text{supp } v} e^{2\text{Re } zf(x)} dx \right\}^{\frac{1}{2}} \| [a_\varepsilon(x, D) - a(x, D)] (u e^{-zf}) \|_{L^2}. \end{aligned}$$

Using the assumption $\|f\|_{\text{BMO}} = 1$, and (2.2), it follows that for any compact set K , $\int_K e^{\pm 2\text{Re } zf(x)} dx \leq C_\gamma(K)$, for $|z| < \frac{\gamma}{2}$. Therefore, (4.1) yields

$$\overline{\lim}_{\varepsilon \rightarrow 0} \sup_{|z| < \frac{\gamma}{2}} |\Phi_\varepsilon(z) - \Phi(z)| = 0,$$

and hence $\Phi(z)$ is a holomorphic function in $|z| < \frac{\gamma}{2}$. ■

An application of the above lemma yields our main result on the boundedness of BMO commutators.

Theorem 4.2 *Let $a \in S_{\rho,\delta}^m$, with either $0 \leq \delta < \rho < 1$ and $m < n(\rho - 1)|\frac{1}{p} - \frac{1}{2}|$ or $\rho = 1, 0 \leq \delta < 1$ and $m \leq 0$. Then for all $p \in (1, \infty)$ and all $f \in \text{BMO}$, the commutator $[f, a(x, D)]$ is bounded on L^p .*

Proof Without loss of generality we can assume that $\|f\|_{\text{BMO}} = 1$. Let us first consider the case $1 < p \leq 2, 0 \leq \delta < \rho < 1$ and take u and v in C_0^∞ with $\|u\|_{L^p} \leq 1$ and $\|v\|_{L^{p'}} \leq 1$. Define the function $\Phi(z) := \int e^{zf(x)} a(x, D)(e^{-zf(x)}u)(x)v(x)dx$. Since $a(x, D)$ is bounded on L^2 ([11]), Lemma 4.1 yields that $\Phi(z)$ is a holomorphic function in the disc $|z| < \lambda$, and the Hölder inequality applied to $\Phi(z)$ and the assumption on v yield

$$|\Phi(z)|^p \leq \int e^{p \text{Re } zf(x)} |a(x, D)(e^{-zf(x)}u)|^p dx.$$

In the proof of the L^p boundedness, we first consider the case $1 < p \leq 2$ and $0 \leq \delta < \rho < 1, m < n(\rho - 1)|\frac{1}{p} - \frac{1}{2}|$. Our first goal is to show that the function $\Phi(z)$ defined above is bounded on a disc with centre at the origin and sufficiently small radius. At this point we recall a lemma due to Chanillo ([3]), which states that if $\|f\|_{\text{BMO}} = 1$, then for $2 < q < \infty$, there is an r_0 depending on q such that for all $r \in [-r_0, r_0]$, $e^{rf(x)} \in A_{\frac{q}{2}}$.

For $p \in (1, 2]$, taking $q = 2p$ in Chanillo’s lemma, we see that there is some r_1 depending on p such that for $|r| < r_1, e^{rf(x)} \in A_p$. Now let α be the parameter from Corollary 3.7, for which $a \in \text{OPS}_{\rho,\delta}^m$ with $m < n(\rho - 1)(\frac{1}{p} - \frac{1}{2})$ is $L_{w^\alpha}^p$ bounded for $1 < p \leq 2$. Then with λ as in Lemma 4.1, we claim that if $R := \min(\lambda, \frac{\alpha r_1}{2})$ and $|z| < R$, then $|\Phi(z)| \lesssim 1$. Indeed since $R < \frac{\alpha r_1}{2}$ and $1 < p \leq 2$, we have $|\text{Re } z| < \frac{\alpha r_1}{p}$ and therefore $|\frac{p \text{Re } z}{\alpha}| < r_1$. Therefore Chanillo’s lemma yields that for

$|z| < R$, $w := e^{\frac{\rho \operatorname{Re} z}{\alpha} f(x)} \in A_p$ and since $e^{p \operatorname{Re} z f(x)} = w^\alpha$, Corollary 3.7 and the L^p bound on u imply that for $|z| < R$,

$$\begin{aligned} |\Phi(z)|^p &\leq \int e^{p \operatorname{Re} z f(x)} |a(x, D)(e^{-z f(x)} u)|^p dx = \int w^\alpha |a(x, D)(e^{-z f(x)} u)|^p dx \\ &\lesssim \int w^\alpha |e^{-z f(x)} u|^p dx = \int w^\alpha w^{-\alpha} |u|^p dx \lesssim 1, \end{aligned}$$

and therefore $|\Phi(z)| \lesssim 1$ for $|z| < R$. In the the case of $1 < p \leq 2$, $\rho = 1$ and $m \leq 0$, we just need to take $R = \min(\lambda, \frac{r}{2})$, and since by Chanillo’s lemma for $|z| < R$, $e^{p \operatorname{Re} z f(x)} \in A_p$, in a way similar to the above argument, Theorem 3.4 yields the boundedness of $\Phi(z)$, on the aforementioned disc.

Finally, using the holomorphicity of $\Phi(z)$ in the disc $|z| < R$, the Cauchy’s integral formula applied to the circle $|z| = R' < R$, and the estimate $|\Phi(z)| \lesssim 1$, we conclude that

$$|\Phi'(0)| \leq \frac{1}{2\pi} \int_{|z|=R'} \frac{|\Phi(\zeta)|}{|\zeta^2|} |d\zeta| \lesssim 1.$$

By construction of $\Phi(z)$, we actually have that $\Phi'(0) = \int v(x)[f, a(x, D)]u(x) dx$, and the definition of the L^p norm of the operator $[f, a(x, D)]$ together with the assumptions on u and v yield at once that $[f, a(x, D)]$ is a bounded operator from L^p to itself for $a(x, D) \in \operatorname{OPS}_{\rho, \delta}^m$ with $0 \leq \delta < \rho < 1$ and $m < n(\rho - 1)|\frac{1}{p} - \frac{1}{2}|$ and $a(x, D) \in \operatorname{OPS}_{1, \delta}^m$ with $m \leq 0$ and $0 \leq \delta < 1$ (in both cases $1 < p \leq 2$).

To handle the case $2 < p < \infty$, we just use duality in both cases and the fact that for $a \in S_{\rho, \delta}^m$ and $\delta < \rho$, the adjoint $a(x, D)^* \in \operatorname{OPS}_{\rho, \delta}^m$. Furthermore, since $\frac{1}{p} + \frac{1}{p'} = 1$, then for $p \in (2, \infty)$, $1 < p' < 2$ and $m < n(\rho - 1)|\frac{1}{p} - \frac{1}{2}|$, then $m < n(\rho - 1)|\frac{1}{p'} - \frac{1}{2}|$. Therefore the results of the previous case of $p \in (1, 2]$ yield that $\|[f, a^*(x, D)]\|_{L^{p'} \rightarrow L^{p'}} \lesssim 1$. Hence

$$\begin{aligned} \|[f, a(x, D)]\|_{L^p \rightarrow L^p} &= \sup_{\|u\|_{L^p} \leq 1} \sup_{\|v\|_{L^{p'}} \leq 1} |\langle [f, a(x, D)]u, v \rangle| \\ &= \sup_{\|u\|_{L^p} \leq 1} \sup_{\|v\|_{L^{p'}} \leq 1} |\langle u, -[f, a^*(x, D)]v \rangle| \\ &\leq \|[f, a^*(x, D)]\|_{L^{p'} \rightarrow L^{p'}} \|v\|_{L^{p'}} \|u\|_{L^p} \lesssim 1. \end{aligned}$$

This concludes the proof of the theorem. ■

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