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Invariants of degree 3 and torsion in the Chow group of a versal flag

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Abstract

We prove that the group of normalized cohomological invariants of degree 3 modulo the subgroup of semidecomposable invariants of a semisimple split linear algebraic group G is isomorphic to the torsion part of the Chow group of codimension-2 cycles of the respective versal G-flag. In particular, if G is simple, we show that this factor group is isomorphic to the group of indecomposable invariants of G. As an application, we construct nontrivial cohomological invariants for indecomposable central simple algebras.

1. Introduction

Let G be a split semisimple linear algebraic group over a field F. The purpose of the present paper is to elucidate the relationships between the *geometry* of twisted G-flag varieties, the theory of *cohomological invariants* of G and the *representation theory* of G.

For the first, let U/G be a *classifying space* of G in the sense of Totaro, i.e. U is an open G-invariant subset in some representation of G such that $U(F) \neq \emptyset$ and $U \to U/G$ is a G-torsor. Consider the generic fiber U^{gen} of U over U/G; it is a G-torsor over the quotient field K of U/G called the *versal* G-torsor [GMS03, pp. 11–12]. We denote by X^{gen} the corresponding flag variety U^{gen}/B over K, where B is a Borel subgroup of G, and call it the *versal* flag. The variety X^{gen} can be viewed as the 'most twisted' form of the 'most complicated' G-flag variety and hence is a natural object to study. In particular, understanding its geometry via studying the *Chow group* $\mathrm{CH}(X^{\mathrm{gen}})$ of algebraic cycles modulo the rational equivalence relation leads to understanding the geometry of all other G-flag varieties.

The group $\mathrm{CH}(X)$ of a (twisted) flag variety X has been a subject of intensive investigation for decades. Research on $\mathrm{CH}(X)$ started with the fundamental results of Grothendieck, Demazure, and Bernstein, Gelfand and Gelfand in the 1970s describing its *free part*, and then progressed due to the development of motivic cohomology theory in the 1990s, culminating in numerous results by Karpenko, Peyre, Vishik and others (including the authors), who aimed to understand its *torsion part*.

The second ingredient of this paper, the theory of cohomological invariants, was mainly inspired by the works of Serre and Rost. Given a field extension L/F and a positive integer d, we consider the Galois cohomology group $H^{d+1}(L, \mathbb{Q}/\mathbb{Z}(d))$, denoted simply by $H^{d+1}(L, d)$ (see

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[GMS03, p. 151]). Following [GMS03, p. 106], a degree-d *cohomological invariant* is a natural transformation of functors

$$a : H^1(—, G) \to H^d(—, d-1)$$

on the category of field extensions over F. We denote the group of degree-d invariants by $\mathrm{Inv}^d(G, d-1)$. An invariant a is said to be *normalized* if it sends the trivial torsor to zero. We denote the subgroup of normalized invariants by $\mathrm{Inv}^d(G, d-1)_{\mathrm{norm}}$.

A normalized invariant a is said to be *decomposable* if it is given by a cup product with an invariant of degree 2, i.e. there exist $b_i \in \mathrm{Inv}^2(G, 1)_{\mathrm{norm}}$ and $a_i \in F^\times$ such that for every field extension L/F and every G-torsor Y over L,

$$a(Y) = \sum_{i \text{ finite}} (a_i) \cup b_i(Y),$$

where (a_i) denotes the corresponding class in $H^1(L, \mathbb{Q}/\mathbb{Z})$. We denote the subgroup of decomposable invariants by $\mathrm{Inv}^3(G, 2)_{\mathrm{dec}}$.

The factor group $\mathrm{Inv}^3(G, 2)_{\mathrm{norm}}/\mathrm{Inv}^3(G, 2)_{\mathrm{dec}}$ is denoted by $\mathrm{Inv}^3(G, 2)_{\mathrm{ind}}$ and is called the group of *indecomposable* invariants. Observe that elements of this group are not invariants but, rather, their classes. This group has been studied by Garibaldi, Kahn, Levine, Rost, Serre and others in the simply connected case, and is closely related to the Rost invariant. Recent work of [Mer13a] has shown how to compute it in general using new results on motivic cohomology obtained in [Mer13b]; in particular, it was computed for all adjoint split groups in [Mer13a] and for split simple groups in [BR13].

As to the last ingredient, namely the representation theory of G, recall that the classical *character map* identifies the representation ring of G with the subring $\mathbb{Z}[T^*]^W$ of W-invariant elements of the integral group ring $\mathbb{Z}[T^*]$, where W is the Weyl group which acts naturally on the group of characters T^* of a split maximal torus T of G. In particular, one of the key objects of the paper, the ideal $(\widetilde{I}^W)$ generated by augmented W-invariant elements in $\mathbb{Z}[\Lambda]$ where Λ is the respective weight lattice, can be identified with the ideal generated by classes of augmented (i.e. virtual of dimension 0) representations of the simply connected cover of G.

We relate all these ingredients with each other by introducing a subgroup of *semidecomposable* invariants, $\mathrm{Inv}^3(G, 2)_{\mathrm{sdec}}$, which consists of invariants $a \in \mathrm{Inv}^3(G, 2)_{\mathrm{norm}}$ such that there exist $b_i \in \mathrm{Inv}^2(G, 1)_{\mathrm{norm}}$ and, for every field extension L/F and every G-torsor Y over L, $\phi_i = \phi_i(Y) \in L^\times$ with

$$a(Y) = \sum_{i \text{ finite}} (\phi_i) \cup b_i(Y). \tag{1}$$

Observe that, by definition, we have $\mathrm{Inv}^3(G, 2)_{\mathrm{dec}} \subseteq \mathrm{Inv}^3(G, 2)_{\mathrm{sdec}} \subseteq \mathrm{Inv}^3(G, 2)_{\mathrm{norm}}$.

Our main result then says the following.

Theorem. *Let G be a split semisimple linear algebraic group over a field F, and let X^{gen} denote the associated versal flag. Then there is a short exact sequence*

$$0 \to \frac{\mathrm{Inv}^3(G, 2)_{\mathrm{sdec}}}{\mathrm{Inv}^3(G, 2)_{\mathrm{dec}}} \to \mathrm{Inv}^3(G, 2)_{\mathrm{ind}} \to \mathrm{CH}^2(X^{\mathrm{gen}})_{\mathrm{tors}} \to 0,$$

together with a group isomorphism $\mathrm{Inv}^3(G, 2)_{\mathrm{sdec}}/\mathrm{Inv}^3(G, 2)_{\mathrm{dec}} \simeq c_2((\widetilde{I}^W) \cap \mathbb{Z}[T^*])/c_2(\mathbb{Z}[T^*]^W)$, *where c_2 is the second* Chern class *map (see e.g. [Mer13a, § 3c]).*

In addition, if G is simple, then $\mathrm{Inv}^3(G, 2)_{\mathrm{sdec}} = \mathrm{Inv}^3(G, 2)_{\mathrm{dec}}$, *so there is an isomorphism* $\mathrm{Inv}^3(G, 2)_{\mathrm{ind}} \simeq \mathrm{CH}^2(X^{\mathrm{gen}})_{\mathrm{tors}}$.

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If G is not simple, then $\mathrm{Inv}^3(G,2)_{\mathrm{sdec}}$ does not necessarily coincide with $\mathrm{Inv}^3(G,2)_{\mathrm{dec}}$ (see Example 3.1).

The nature of our result suggests that it should have applications in several directions, such as to cohomological invariants and algebraic cycles on twisted flag varieties. Later in this paper we will discuss a few of these appplications.

For instance, since the group $\mathrm{Inv}^3(G,2)_{\mathrm{ind}}$ has been computed for all simple split groups in [Mer13a, BR13], it immediately gives computation of the torsion part of $\mathrm{CH}^2(X^{\mathrm{gen}})$, thus extending previous results by [Kar98, Pey98]. As another straightforward consequence, using the equality $\mathrm{Inv}^3(G,2)_{\mathrm{sdec}} = \mathrm{Inv}^3(G,2)_{\mathrm{dec}}$ we can construct nontrivial cohomological classes for indecomposable central simple algebras, hence answering questions posed in [GPT09, Bar14].

The paper is organized as follows. In § 2 we construct an exact sequence relating the groups of invariants with the torsion part of the Chow group, proving the first part of the main theorem. In § 3 we compute this exact sequence case by case for all simple groups (the $\mathbf{PGO}_8$ case is treated separately in Appendix A), hence proving the second part of the theorem. In the final section we discuss some applications.

2. Semidecomposable invariants and the Chow group

Let G be a split semisimple linear algebraic group over a field F. We fix a split maximal torus T of G and a Borel subgroup B containing T. We consider the T-equivariant structure map $U \to \operatorname{Spec} F = pt$ where U is the open G-invariant subset from the introduction.

Characteristic maps and classes. By [EG98], the induced pullback on T-equivariant Chow groups $\mathrm{CH}_T(pt) \to \mathrm{CH}_T(U)$ is an isomorphism. Since $\mathrm{CH}_T(U) \simeq \mathrm{CH}(U/T) \simeq \mathrm{CH}(U/B)$ and $\mathrm{CH}_T(pt)$ can be identified with the symmetric algebra $\mathrm{Sym}(T^*)$ of the group of characters of T, this gives an isomorphism

$$\mathfrak{c}^{\mathrm{CH}} : \mathrm{Sym}(T^*) \xrightarrow{\simeq} \mathrm{CH}(U/B). \tag{2}$$

Similarly, by the homotopy invariance and localization property of the equivariant K-theory [Mer05, Theorems 8 and 11], the induced pullback on T-equivariant K-groups gives a surjection

$$\mathfrak{c}^{\mathrm{K}_0} : \mathbb{Z}[T^*] \twoheadrightarrow K_0(U/B),$$

where the group ring $\mathbb{Z}[T^*]$ can be identified with $K_T(pt)$ and $K_0(U/B) \simeq K_0(U/T) \simeq K_T(U)$.

Let $\tau^i(X)$ denote the ith term of the *topological filtration* on K_0 of a smooth variety X, and let $\tau^{i/i+1}$ ($i \geqslant 0$) denote its ith subsequent quotient. Let I denote the augmentation ideal of $\mathbb{Z}[T^*]$.

Lemma 2.1. *The map $\mathfrak{c}^{\mathrm{K}_0}$ induces isomorphisms on subsequent quotients*

$$I^i/I^{i+1} \xrightarrow{\simeq} \tau^{i/i+1}(U/B) \quad \text{for } 0 \leqslant i \leqslant 2,$$

and its restriction $\mathfrak{c}^{\mathrm{K}_0} : I^2 \to \tau^2(U/B)$ is surjective.

Proof. By [Ful98, Example 15.3.6], the Chern class maps induce isomorphisms $c_i : \tau^{i/i+1}(X) \xrightarrow{\simeq} \mathrm{CH}^i(X)$ for $0 \leqslant i \leqslant 2$. Since the Chern classes commute with pullbacks and $\mathrm{Sym}^i(T^*) \simeq I^i/I^{i+1}$, the isomorphisms follow from (2). Finally, since $I/I^2 \simeq \tau^{1/2}(U/B)$, $\mathfrak{c}^{\mathrm{K}_0}(x) \in \tau^2(U/B)$ implies that $x \in I^2$. □

Consider the natural inclusion of the versal flag $\imath : X^{\mathrm{gen}} = U^{\mathrm{gen}}/B \hookrightarrow U/B$. Since $\imath$ is a limit of open embeddings, by the *localization property* of Chow groups, the induced pullback gives surjections

$$\imath^{\mathrm{CH}} : \mathrm{CH}^i(U/B) \twoheadrightarrow \mathrm{CH}^i(X^{\mathrm{gen}}).$$

Moreover, the induced pullback in K-theory restricted to τ^i also gives surjections

$$\imath^{\mathrm{K}_0} : \tau^i(U/B) \twoheadrightarrow \tau^i(X^{\mathrm{gen}}).$$

Indeed, by definition $\tau^i(X^{\mathrm{gen}})$ is generated by the classes $[\mathcal{O}_Z]$ for closed subvarieties Z of X^{gen} with $\operatorname{codim} Z \geqslant i$ and each $[\mathcal{O}_Z]$ is the pullback of the element $[\mathcal{O}_{\bar{Z}}]$ in $\tau^i(U/B)$, where $\bar{Z}$ is the closure of Z inside U/B.

Let L be a splitting field of the versal torsor U^{gen}. According to [GZ12, Theorem 4.5], the compositions

$$\mathrm{Sym}(T^*) \xrightarrow{\mathfrak{c}^{\mathrm{CH}}} \mathrm{CH}(U/B) \xrightarrow{\imath^{\mathrm{CH}}} \mathrm{CH}(X^{\mathrm{gen}}) \xrightarrow{\mathrm{res}} \mathrm{CH}(X_L^{\mathrm{gen}})$$

and

$$\mathbb{Z}[T^*] \xrightarrow{\mathfrak{c}^{\mathrm{K}_0}} K_0(U/B) \xrightarrow{\imath^{\mathrm{K}_0}} K_0(X^{\mathrm{gen}}) \xrightarrow{\mathrm{res}} K_0(X_L^{\mathrm{gen}})$$

give the classical characteristic maps for the Chow groups and the K-groups, respectively (here we identify the rightmost groups with, respectively, the Chow group and the K-group of the split flag G/B). Restricting the latter to I^2 and τ^2, we obtain the map

$$\mathfrak{c} : I^2 \xrightarrow{\mathfrak{c}^{\mathrm{K}_0}} \tau^2(U/B) \xrightarrow{\imath^{\mathrm{K}_0}} \tau^2(X^{\mathrm{gen}}) \xrightarrow{\mathrm{res}} \tau^2(X_L^{\mathrm{gen}}) = \tau^2(G/B).$$

From this point on, we denote by $\mathfrak{c}^{\mathrm{CH}}$, $\imath^{\mathrm{CH}}$, $\mathfrak{c}^{\mathrm{K}_0}$ and $\imath^{\mathrm{K}_0}$ the restrictions to Sym^2, CH^2, I^2 and τ^2, respectively.

Let Λ be the weight lattice. Consider the integral group ring $\mathbb{Z}[\Lambda]$. Let $\widetilde{I}$ denote its augmentation ideal. The Weyl group W acts naturally on $\mathbb{Z}[\Lambda]$. Let $(\widetilde{I}^W)$ denote the ideal generated by W-invariant elements in $\widetilde{I}$.

LEMMA 2.2. *The kernel of the composition* $I^2 \xrightarrow{\mathfrak{c}^{\mathrm{K}_0}} \tau^2(U/B) \xrightarrow{\imath^{\mathrm{K}_0}} \tau^2(X^{\mathrm{gen}})$ *is* $(\widetilde{I}^W) \cap I^2$.

Proof. By the results of Panin [Pan94], $K_0(X^{\mathrm{gen}})$ is the direct sum of $K_0(A_i)$ for some central simple algebras A_i over K. So $K_0(X^{\mathrm{gen}})$ is a free abelian group, and hence the restriction $\tau^2(X^{\mathrm{gen}}) \to \tau^2(X_L^{\mathrm{gen}})$ is injective. Therefore, $\ker(\imath^{\mathrm{K}_0} \circ \mathfrak{c}^{\mathrm{K}_0})$ coincides with the kernel of the characteristic map $\mathfrak{c} : I^2 \to \tau^2(G/B)$.

By considering the inclusion $Z[T^*] \subseteq Z[\Lambda]$, we see that the kernel of $\mathfrak{c} : I^2 \to \tau^2(G/B)$ equals the intersection of I^2 with the kernel of the characteristic map $\widetilde{\mathfrak{c}} : Z[\Lambda] \to K_0(\widetilde{G}/\widetilde{B}) = K_0(G/B)$, where $\widetilde{G}$ is the simply connected cover of G and $\widetilde{B}$ is the corresponding Borel subgroup of $\widetilde{G}$. By Steinberg's theorem [Ste75], the kernel of the map $\widetilde{\mathfrak{c}}$ is equal to $\widetilde{I}^W$. Hence we get that $\ker(\imath^{\mathrm{K}_0} \circ \mathfrak{c}^{\mathrm{K}_0}) = (\widetilde{I}^W) \cap I^2$. □

Consider the second Chern class map $c_2 : \tau^2(U/B) \to \mathrm{CH}^2(U/B)$.

LEMMA 2.3. *We have* $c_2(\ker \imath^{\mathrm{K}_0}) = \ker \imath^{\mathrm{CH}}$.

Proof. Consider the following diagram.

$$\begin{tikzcd}
\tau^3(U/B) \arrow[r] \arrow[d, "\imath^{\mathrm{K}_0}|_{\tau^3}"'] & \tau^2(U/B) \arrow[r, "c_2"] \arrow[d, "\imath^{\mathrm{K}_0}"] & \mathrm{CH}^2(U/B) \arrow[r] \arrow[d, "\imath^{\mathrm{CH}}"] & 0 \\
\tau^3(X^{\mathrm{gen}}) \arrow[r] & \tau^2(X^{\mathrm{gen}}) \arrow[r, "c_2"] & \mathrm{CH}^2(X^{\mathrm{gen}}) \arrow[r] & 0
\end{tikzcd}$$

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Its vertical maps are surjective and the rows are exact by [Ful98, Example 15.3.6]. The result then follows by the diagram chase. □

Consider the composition $\mathfrak{c}_2 : I^2 \xrightarrow{\mathfrak{c}^{\mathrm{K}_0}} \tau^2(U/B) \xrightarrow{c_2} \mathrm{CH}^2(U/B)$. Observe that it coincides with the Chern class map defined in [Mer13a, § 3c].

Lemma 2.4. *We have* $\ker \imath^{\mathrm{CH}} = \mathfrak{c}_2((\widetilde{I}^W) \cap I^2)$.

Proof. Since $\mathfrak{c}^{\mathrm{K}_0}$ is surjective by Lemma 2.1, from Lemmas 2.2 and 2.3 we obtain

$$\mathfrak{c}_2((\widetilde{I}^W) \cap I^2) = \mathfrak{c}_2(\ker(\imath^{\mathrm{K}_0} \circ \mathfrak{c}^{\mathrm{K}_0})) = c_2(\ker \imath^{\mathrm{K}_0}) = \ker \imath^{\mathrm{CH}}. \qquad \square$$

Following [Mer13a], we write

$$\mathrm{Dec}(G) := (\mathfrak{c}^{\mathrm{CH}})^{-1} \circ \mathfrak{c}_2(\mathbb{Z}[T^*]^W)$$

and set

$$\mathrm{SDec}(G) := (\mathfrak{c}^{\mathrm{CH}})^{-1} \circ \mathfrak{c}_2((\widetilde{I}^W) \cap \mathbb{Z}[T^*]).$$

Since $\Lambda^W = 0$, we have $(\widetilde{I}^W) \subseteq \widetilde{I}^2$. Therefore, for any $x \in (\widetilde{I}^W)$ we have $x \equiv x' \bmod \widetilde{I}^3$ and hence $\mathfrak{c}_2(x) = \mathfrak{c}_2(x')$ for some $x' \in \mathbb{Z}[\Lambda]^W$, where $\mathbb{Z}[\Lambda]^W$ is the subring of W-invariants. So there are inclusions

$$\mathrm{Dec}(G) \subseteq \mathrm{SDec}(G) \subseteq \mathrm{Sym}^2(T^*)^W. \tag{3}$$

Lemma 2.5. *We have* $\mathrm{CH}^2(X^{\mathrm{gen}}) \simeq \mathrm{Sym}^2(T^*)/\mathrm{SDec}(G)$.

Proof. By (2) and Lemma 2.4,

$$\mathrm{CH}^2(X^{\mathrm{gen}}) \simeq \mathrm{CH}^2(U/B)/\mathfrak{c}_2((\widetilde{I}^W) \cap I^2) \simeq \mathrm{Sym}^2(T^*)/\mathrm{SDec}(G). \qquad \square$$

Corollary 2.6. *We have* $\mathrm{CH}^2(X^{\mathrm{gen}})_{\mathrm{tors}} \simeq \mathrm{Sym}^2(T^*)^W/\mathrm{SDec}(G)$.

Proof. By Lemma 2.5 it remains to show that

$$(\mathrm{Sym}^2(T^*)/\mathrm{SDec}(G))_{\mathrm{tors}} = \mathrm{Sym}^2(T^*)^W/\mathrm{SDec}(G).$$

Indeed, suppose that $x \in \mathrm{Sym}^2(T^*)$ and $nx \in \mathrm{SDec}(G)$. Then nx lies in $\mathrm{Sym}^2(T^*)^W$ by (3). So for every $w \in W$ we have $n(wx - x) = 0$. Since $\mathrm{Sym}^2(T^*)$ has no torsion, $x \in \mathrm{Sym}^2(T^*)^W$.

Conversely, let $x \in \mathrm{Sym}^2(T^*)^W$. Since the second Chern class map $c_2 : I^2 \to \mathrm{Sym}^2(T^*)$ is surjective, there is a preimage $y \in I^2$ of x. Take $y' = \sum_{w \in W} w \cdot y \in \mathbb{Z}[T^*]^W \subseteq (\widetilde{I}^W) \cap \mathbb{Z}[T^*]$. Since c_2 is W-equivariant and coincides with the composition $(\mathfrak{c}^{\mathrm{CH}})^{-1} \circ \mathfrak{c}_2$, we obtain

$$(\mathfrak{c}^{\mathrm{CH}})^{-1} \circ \mathfrak{c}_2(y') = |W| \cdot x \in \mathrm{SDec}(G). \qquad \square$$

Cohomological invariants. For a smooth F-scheme X, let $\mathcal{H}^3(2)$ denote the Zariski sheaf on X associated to a presheaf $W \mapsto H^3_{\text{ét}}(W, \mathbb{Q}/\mathbb{Z}(2))$. The Bloch–Ogus–Gabber theorem (see [CHK97, GS88]) implies that its group of global sections $H^0_{\mathrm{Zar}}(X, \mathcal{H}^3(2))$ is a subgroup in $H^3(F(X), 2)$.

Consider the versal G-torsor U^{gen} over the quotient field K of the classifying space U/G. By [BM13, Theorem A] the map $\Theta : \mathrm{Inv}^3(G, 2) \to H^3(K, 2)$ defined by $\Theta(a) := a(U^{\mathrm{gen}})$ gives an inclusion

$$\mathrm{Inv}^3(G, 2) \hookrightarrow H^0_{\mathrm{Zar}}(U/G, \mathcal{H}^3(2)).$$

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Lemma 2.7. *We have $a(U^{\mathrm{gen}}) \in \ker[H^3(K,2) \to H^3(K(X^{\mathrm{gen}}),2)]$ for any $a \in \mathrm{Inv}^3(G,2)_{\mathrm{norm}}$.*

Proof. Consider the composition $q : \operatorname{Spec} K(U^{\mathrm{gen}}) \to U^{\mathrm{gen}} \to U/G$. Observe that the pullback q^* factors as

$$q^* : H^0_{\mathrm{Zar}}(U/G, \mathcal{H}^3(2)) \to H^3(K(X^{\mathrm{gen}}),2) \to H^3(K(U^{\mathrm{gen}}),2).$$

Since $U^{\mathrm{gen}} \to X^{\mathrm{gen}}$ is a B-torsor, $K(U^{\mathrm{gen}})$ is purely transcendental over $K(X^{\mathrm{gen}})$, so the last map of the composition is injective. Since U^{gen} becomes trivial over $K(U^{\mathrm{gen}})$, we have

$$q^*(a(U^{\mathrm{gen}})) = a(U^{\mathrm{gen}} \times_K K(U^{\mathrm{gen}})) = 0.$$

Therefore, $a(U^{\mathrm{gen}}) \in \ker[H^3(K,2) \to H^3(K(X^{\mathrm{gen}}),2)]$. □

Lemma 2.8. *Let $Y \to \operatorname{Spec} L$ be a G-torsor and let $X = Y/B$. Denote by L^{sep} the separable closure of L and Γ_L its Galois group, and let $X^{\mathrm{sep}} = X \times_L L^{\mathrm{sep}}$. Then the Γ_L action on $\operatorname{Pic} X^{\mathrm{sep}}$ is trivial.*

Proof. This follows from [MT95, Proposition 2.2]. □

The Tits map. Consider a short exact sequence of F-group schemes

$$1 \to C \to \widetilde{G} \xrightarrow{\pi} G \to 1.$$

Given a character $\chi \in C^*$ of the center and a field extension L/F, consider the *Tits map*

$$\alpha_{\chi,L} : H^1(L,G) \xrightarrow{\partial} H^2(L,C) \xrightarrow{\chi_*} H^2(L,\mathbb{G}_m)$$

(see [Tit71, §§ 4 and 5]), where ∂ is the connecting homomorphism (if C is non-smooth, we replace it by $\mathbb{G}_m$ and replace G by the respective push-out as in [GMS03, p. 106, Example 2.1]). This gives rise to a cohomological invariant β_χ of degree 2,

$$\beta_\chi : Y \mapsto \alpha_{\chi,L}(Y) \quad \text{for every } G\text{-torsor } Y \in H^1(L,G).$$

By [BM13, Theorem 2.4], the assignment $\chi \to \beta_\chi$ provides an isomorphism $C^* \to \mathrm{Inv}^2(G,1)$.

For a G-torsor Y over L, there is the following exact sequence studied in [Mer95], [Pey98] and [GMS03, p. 126, Theorem 8.9]:

$$A^1((Y/B)^{\mathrm{sep}}, K_2)^\Gamma \xrightarrow{\rho} \ker[H^3(L,2) \to H^3(L(Y/B),2)] \xrightarrow{\delta_Y} \mathrm{CH}^2(Y/B).$$

The multiplication map $L^{\mathrm{sep}} \otimes \mathrm{CH}^1(Y/B)^{\mathrm{sep}} \to A^1((Y/B)^{\mathrm{sep}}, K_2)$ is an isomorphism. By Lemma 2.8 we obtain an exact sequence

$$L \otimes \Lambda \xrightarrow{\rho_Y} \ker[H^3(L,2) \to H^3(L(Y/B),2)] \xrightarrow{\delta_Y} \mathrm{CH}^2(Y/B). \tag{4}$$

According to [Mer95], the map ρ_Y acts as follows:

$$\rho_Y(\phi \otimes \lambda) = \phi \cup \beta_{\overline{\lambda}}(Y),$$

where $\phi \in L^\times$, $\lambda \in \Lambda$ and $\overline{\lambda}$ denotes the image of λ in $\Lambda/T^* = C^*$.

Recall that, by definition (1) of the group of semidecomposable invariants,

$$a \in \mathrm{Inv}^3(G,2)_{\mathrm{sdec}} \quad \text{if and only if} \quad a(Y) \in \mathrm{im}(\rho_Y) = \ker(\delta_Y) \text{ for every torsor } Y.$$

Lemma 2.9. *We have $a \in \mathrm{Inv}^3(G,2)_{\mathrm{sdec}}$ if and only if $a(U^{\mathrm{gen}}) \in \ker(\delta_{U^{\mathrm{gen}}})$.*

Proof. If a is a semidecomposable invariant, then $a(U^{\mathrm{gen}}) = \sum_{\chi\in C^*} \phi_\chi \cup \beta_\chi(U^{\mathrm{gen}})$ lies in the image of $\rho_{U^{\mathrm{gen}}}$; hence $\delta_{U^{\mathrm{gen}}}(a(U^{\mathrm{gen}})) = 0$. On the other hand, let a be a degree-3 invariant such that $\delta_{U^{\mathrm{gen}}}(a(U^{\mathrm{gen}})) = 0$, and let Y be a G-torsor over a field extension L/F. We show that $\delta_Y(a(Y)) = 0$.

We may assume that L is infinite (replacing L by $L(t)$ if needed). Choose a rational point $y \in (U/G)_L$ such that Y is isomorphic to the fiber of $U \to U/G$ over y. Let R be the completion of the regular local ring $\mathcal{O}_{(U/G)_L,y}$ and let $\hat{K}$ be its quotient field. The ring R is a regular local ring with residue field L. By a theorem of Grothendieck, Y_R is a pullback of Y via the projection $\operatorname{Spec} R \to \operatorname{Spec} L(y)$. Then the G-torsors $Y_{\hat{K}}$ and $U^{\mathrm{gen}}_{\hat{K}}$ over $\hat{K}$ are isomorphic. We have

$$\delta_Y(a(Y))_{\hat{K}} = \delta_{Y_{\hat{K}}}(a(Y_{\hat{K}})) = \delta_{U^{\mathrm{gen}}_{\hat{K}}}(a(U^{\mathrm{gen}}_{\hat{K}})) = \delta_{U^{\mathrm{gen}}}(a(U^{\mathrm{gen}}))_{\hat{K}} = 0.$$

The restriction $\mathrm{CH}^2(Y/B) \to \mathrm{CH}^2((Y/B)_{\hat{K}})$ is injective, since it is split by the specialization map with respect to a system of local parameters of R. Therefore $\delta_Y(a(Y)) = 0$ for every Y, and hence a is semidecomposable. $\square$

Now we are ready to prove the first part of the main theorem.

Theorem 2.10. *The map $\delta_{U^{\mathrm{gen}}}$ induces a short exact sequence*

$$0 \longrightarrow \frac{\mathrm{Inv}^3(G,2)_{\mathrm{sdec}}}{\mathrm{Inv}^3(G,2)_{\mathrm{dec}}} \longrightarrow \mathrm{Inv}^3(G,2)_{\mathrm{ind}} \xrightarrow{g} \mathrm{CH}^2(X^{\mathrm{gen}})_{\mathrm{tors}} \longrightarrow 0,$$

and there is a group isomorphism

$$\frac{\mathrm{Inv}^3(G,2)_{\mathrm{sdec}}}{\mathrm{Inv}^3(G,2)_{\mathrm{dec}}} \simeq \frac{c_2((\widetilde{I}^W) \cap \mathbb{Z}[T^*])}{c_2(\mathbb{Z}[T^*]^W)}.$$

Proof. Consider the following diagram, where the rows are exact sequences given by [Kah96, Theorem 1.1] and the vertical arrows are pullbacks.

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{CH}^2(U/G) \arrow[r] \arrow[d] & \mathbb{H}^4_{\text{ét}}(U/G,\mathbb{Z}(2)) \arrow[r] \arrow[d] & H^0_{\mathrm{Zar}}(U/G,\mathcal{H}^3(2)) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{CH}^2(U/B) \arrow[r] \arrow[d, "\imath^{\mathrm{CH}}"'] & \mathbb{H}^4_{\text{ét}}(U/B,\mathbb{Z}(2)) \arrow[r] \arrow[d] & H^0_{\mathrm{Zar}}(U/B,\mathcal{H}^3(2)) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{CH}^2(X^{\mathrm{gen}}) \arrow[r] & \mathbb{H}^4_{\text{ét}}(X^{\mathrm{gen}},\mathbb{Z}(2)) \arrow[r] & H^0_{\mathrm{Zar}}(X^{\mathrm{gen}},\mathcal{H}^3(2)) \arrow[r] & 0
\end{tikzcd}$$

Since $F(U/B) = K(X^{\mathrm{gen}})$, Lemma 2.7 implies that the composition

$$\mathrm{Inv}^3(G,2)_{\mathrm{norm}} \to H^0_{\mathrm{Zar}}(U/G,\mathcal{H}^3(2)) \to H^0_{\mathrm{Zar}}(U/B,\mathcal{H}^3(2))$$

is zero. By the diagram chase, there is a homomorphism

$$\mathrm{Inv}^3(G,2)_{\mathrm{norm}} \to \mathrm{CH}^2(U/B)/\mathrm{CH}^2(U/G).$$

The map $X^{\mathrm{gen}} \to U/B \to U/G$ factors as $X^{\mathrm{gen}} \to \operatorname{Spec} K \to U/G$; hence the composition of pullbacks $\mathrm{CH}^2(U/G) \to \mathrm{CH}^2(U/B) \xrightarrow{\imath^{\mathrm{CH}}} \mathrm{CH}^2(X^{\mathrm{gen}})$ coincides with the composition

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$\mathrm{CH}^2(U/G) \to \mathrm{CH}^2(\operatorname{Spec} K) \to \mathrm{CH}^2(X^{\mathrm{gen}})$, which is zero. This gives a homomorphism $g : \mathrm{Inv}^3(G,2)_{\mathrm{norm}} \to \mathrm{CH}^2(U/B)/\mathrm{CH}^2(U/G) \to \mathrm{CH}^2(X^{\mathrm{gen}})$, which by the proof of B. Kahn's theorem [GMS03, § 8, pp. 124–125] factors through the map $\delta_{U^{\mathrm{gen}}}$ of (4). By [Mer13a, 3.9], the map g also factors through the isomorphism $\mathrm{Inv}^3(G,2)_{\mathrm{ind}} \xrightarrow{\simeq} \mathrm{Sym}^2(T^*)^W/\mathrm{Dec}(G)$. So there is a commutative diagram as follows.

$$\begin{tikzcd} \mathrm{Inv}^3(G,2)_{\mathrm{norm}} \arrow[r, "g"] \arrow[d] & \mathrm{CH}^2(X^{\mathrm{gen}})_{\mathrm{tors}} \\ \frac{\mathrm{Sym}^2(T^*)^W}{\mathrm{Dec}(G)} \arrow[r] & \frac{\mathrm{Sym}^2(T^*)^W}{\mathrm{SDec}(G)} \arrow[u, "\simeq"', "\text{Cor. 2.6}"] \end{tikzcd} \tag{5}$$

The bottom row of (5) gives a short exact sequence

$$0 \to \frac{\mathrm{SDec}(G)}{\mathrm{Dec}(G)} \to \frac{\mathrm{Sym}^2(T^*)^W}{\mathrm{Dec}(G)} \to \mathrm{CH}^2(X^{\mathrm{gen}})_{\mathrm{tors}} \to 0.$$

Lemma 2.9 and the composition (4) give an exact sequence

$$0 \to \mathrm{Inv}^3(G,2)_{\mathrm{sdec}} \to \mathrm{Inv}^3(G,2)_{\mathrm{norm}} \xrightarrow{g} \mathrm{CH}^2(X^{\mathrm{gen}})_{\mathrm{tors}}.$$

Upon combining these and factoring modulo $\mathrm{Inv}^3(G,2)_{\mathrm{dec}}$, we obtain an isomorphism

$$\frac{\mathrm{Inv}^3(G,2)_{\mathrm{sdec}}}{\mathrm{Inv}^3(G,2)_{\mathrm{dec}}} \cong \frac{\mathrm{SDec}(G)}{\mathrm{Dec}(G)}. \qquad \square$$

3. Semidecomposable invariants versus decomposable invariants

In this section, we prove case by case that the groups of decomposable $\mathrm{Inv}^3(G,2)_{\mathrm{dec}}$ and semidecomposable $\mathrm{Inv}^3(G,2)_{\mathrm{sdec}}$ invariants coincide for all split simple G, hence establishing the second part of our main theorem. More precisely, we show that

$$\mathrm{Dec}(G) = \mathfrak{c}_2(\mathbb{Z}[T^*]^W) = \mathfrak{c}_2((\widetilde{I}^W) \cap \mathbb{Z}[T^*]) = \mathrm{SDec}(G) \;\text{ in }\; \mathrm{Sym}^2(T^*)^W$$

(here we denote $(\mathfrak{c}^{\mathrm{CH}})^{-1} \circ \mathfrak{c}_2$ simply by $\mathfrak{c}_2$). Observe that in the simply connected case we have $\mathrm{Sym}^2(\Lambda)^W = \mathbb{Z}q$, where q corresponds to the normalized Killing form from [GZ14, § 1B], and

$$\mathrm{Dec}(G) \subseteq \mathrm{SDec}(G) \subseteq \mathrm{SDec}(\widetilde{G}) = \mathrm{Dec}(\widetilde{G})$$

for any G (not necessarily simply connected).

Example 3.1. If G is not simple, then $\mathrm{Dec}(G) \neq \mathrm{SDec}(G)$ in general.

Indeed, consider a quadratic form q of degree 4 with trivial discriminant (it corresponds to a $\mathbf{SO}_4$-torsor). According to [GMS03, p. 46, Example 20.3], there is an invariant given by $q \mapsto \alpha \cup \beta \cup \gamma$, where α is represented by q and $\langle\langle \beta, \gamma \rangle\rangle = \langle \alpha \rangle q$ is the 2-Pfister form. By definition, this invariant is semidecomposable (this fact was pointed out to us by Vladimir Chernousov). Since it is nontrivial over an algebraic closure of F, it is not decomposable.

3a. Adjoint groups of type $A_n\,(n \geqslant 1)$, $B_n\,(n \geqslant 2)$, $C_n\,(n \geqslant 3,\, 4 \nmid n)$, $D_n\,(n \geqslant 5,\, 4 \nmid n)$, E_6, E_7 and special orthogonal groups of type $D_n\,(n \geqslant 4)$

For classical adjoint types, we have $\mathrm{Inv}^3(G,2)_{\mathrm{norm}} = \mathrm{Inv}^3(G,2)_{\mathrm{dec}}$ by [Mer13a, § 4b], so we immediately obtain $\mathrm{Inv}^3(G,2)_{\mathrm{dec}} = \mathrm{Inv}^3(G,2)_{\mathrm{sdec}}$. For exceptional types, by [GMS03, p. 135] and [Mer13a, § 4b] we have $\mathrm{Dec}(G) = \mathrm{Dec}(\widetilde{G}) = 6\mathbb{Z}q$ for E_6 and $\mathrm{Dec}(G) = \mathrm{Dec}(\widetilde{G}) = 12\mathbb{Z}q$ for E_7. For special orthogonal groups $G = \mathbf{SO}_{2n}$, by [GMS03, p. 145, § 15] we have $\mathrm{Dec}(\mathbf{SO}_{2n}) = \mathrm{Dec}(\mathbf{Spin}_{2n}) = 2\mathbb{Z}q$ (here $\tilde{G} = \mathbf{Spin}_{2n}$) and hence $\mathrm{Dec}(G) = \mathrm{SDec}(G)$.

3b. Non-adjoint groups of type A_{n-1} ($n \geqslant 4$)
Let p be a prime integer and let $G = \mathbf{SL}_{p^s}/\boldsymbol{\mu}_{p^r}$ for some integers $s \geqslant r > 0$. If p is odd, we set $k = \min\{r, s-r\}$; and if $p = 2$, we assume that $s \geqslant r+1$ and set $k = \min\{r, s-r-1\}$. It is shown in [BR13, § 4] that the group $\mathrm{Inv}^3(G,2)_{\mathrm{ind}}$ is cyclic of order p^k. On the other hand, by [Kar98, Example 4.15], if X is the Severi–Brauer variety of a generic algebra A^{gen}, then $\mathrm{CH}^2(X)_{\mathrm{tors}}$ is also a cyclic group of order p^k. The canonical morphism $X^{\mathrm{gen}} \to X$ is an iterated projective bundle, hence $\mathrm{CH}^2(X^{\mathrm{gen}})_{\mathrm{tors}} \simeq \mathrm{CH}^2(X)_{\mathrm{tors}}$ is a cyclic group of order p^k. It follows from the exact sequence of Theorem 2.10 that $\mathrm{Inv}^3(G,2)_{\mathrm{sdec}} = \mathrm{Inv}^3(G,2)_{\mathrm{dec}}$.

More generally, let $G = \mathbf{SL}_n/\boldsymbol{\mu}_m$ where $m \mid n$. Let p^s and p^r be the highest powers of a prime integer p dividing n and m, respectively. Consider the homomorphism $H = \mathbf{SL}_{p^s}/\boldsymbol{\mu}_{p^r} \to G$ (cf. § 4b). We claim that it induces an isomorphism between the p-primary component of $\mathrm{Inv}^3(G, 2)_{\mathrm{ind}}$ and the group $\mathrm{Inv}^3(H,2)_{\mathrm{ind}}$.

Indeed, let $H' = \mathbf{SL}_n/\boldsymbol{\mu}_{p^r}$. It follows from [BR13, Theorem 4.1] that the natural homomorphism $\mathrm{Inv}^3(H',2)_{\mathrm{ind}} \to \mathrm{Inv}^3(H,2)_{\mathrm{ind}}$ is an isomorphism. Thus, it suffices to show that the pullback map for the canonical surjective homomorphism $H' \to G$ with kernel $\boldsymbol{\mu}_t$, where $t := m/p^r$ is relatively prime to p, induces an isomorphism between the p-primary component of $\mathrm{Inv}^3(G,2)_{\mathrm{ind}}$ and $\mathrm{Inv}^3(H',2)_{\mathrm{ind}}$. Let $\Lambda \subset \Lambda'$ be the character groups of maximal tori of G and H', respectively. The factor group Λ'/Λ is isomorphic to $\boldsymbol{\mu}_t^* = \mathbb{Z}/t\mathbb{Z}$. Since the functor $\Lambda \mapsto \mathrm{Sym}^2(\Lambda)^W/\mathrm{Dec}(\Lambda)$ is quadratic in Λ, the kernel and cokernel of the homomorphism

$$\mathrm{Inv}^3(G,2)_{\mathrm{ind}} = \frac{\mathrm{Sym}^2(\Lambda)^W}{\mathrm{Dec}(\Lambda)} \to \frac{\mathrm{Sym}^2(\Lambda')^W}{\mathrm{Dec}(\Lambda')} = \mathrm{Inv}^3(H',2)_{\mathrm{ind}}$$

are killed by t^2. As t is relatively prime to p, the claim follows.

Since the p-primary component of $\mathrm{CH}(X^{\mathrm{gen}})_{\mathrm{tors}}$ and the group $\mathrm{CH}(X_H^{\mathrm{gen}})_{\mathrm{tors}}$ are isomorphic by [Kar98, Proposition 1.3] (here X_H^{gen} denotes the versal flag for H), we obtain that $\mathrm{Inv}^3(G, 2)_{\mathrm{ind}} \simeq \mathrm{CH}(X^{\mathrm{gen}})_{\mathrm{tors}}$ and therefore, by the exact sequence of Theorem 2.10, $\mathrm{Inv}^3(G,2)_{\mathrm{sdec}} = \mathrm{Inv}^3(G,2)_{\mathrm{dec}}$.

3c. Adjoint groups of type C_{4m} ($m \geqslant 1$)
By [Mer13a, § 4b] we have $\mathrm{Sym}^2(T^*)^W = \mathbb{Z}q$ and $\mathrm{Dec}(G) = \mathfrak{c}_2(\mathbb{Z}[T^*]^W) = 2\mathbb{Z}q$. We want to show that $\mathfrak{c}_2(x) \in 2\mathbb{Z}q$ for every element $x \in (\widetilde{I}^W) \cap \mathbb{Z}[T^*]$.

Given a weight $\chi \in \Lambda$, we denote by $W(\chi)$ its W-orbit and set $\widehat{e^{\chi}} := \sum_{\lambda \in W(\chi)}(1 - e^{-\lambda})$. By definition, the ideal $(\widetilde{I}^W)$ is generated by elements $\{\widehat{e^{\omega_i}}\}_{i=1,\dots,4m}$ corresponding to the fundamental weights ω_i. An element x can be written as

$$x = \sum_{i=1}^{4m} n_i \widehat{e^{\omega_i}} + \delta_i \widehat{e^{\omega_i}} \quad \text{where } n_i \in \mathbb{Z} \text{ and } \delta_i \in \widetilde{I}. \tag{6}$$

Similar to [Zai12, § 3], we consider a ring homomorphism $f : \mathbb{Z}[\Lambda] \to \mathbb{Z}[\Lambda/T^*]$ induced by taking the quotient $\Lambda \to \Lambda/T^* = C^*$. We have $\Lambda/T^* \simeq \mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}[\Lambda/T^*] = \mathbb{Z}[y]/(y^2 - 2y)$, where $y = f(e^{\omega_1} - 1)$. Observe that W acts trivially on C^*.

By definition, $f(I) = 0$, so $f(x) = 0$. Since $\omega_i \in T^*$ for all even i, $f(\widehat{e^{\omega_i}}) = y$ for all odd i and $f(\delta_i) \in f(\widetilde{I}) = (y)$, we get

$$0 = f(x) = \sum_{i \text{ is odd}} n_i d_i y + m_i d_i y^2 = \sum_{i \text{ is odd}} (n_i + 2m_i) d_i y,$$

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where $m_i \in \mathbb{Z}$ and $d_i = 2^i \binom{4m}{i}$ is the cardinality of $W(\omega_i)$; this implies that

$$\sum_{i \text{ is odd}} (n_i + 2m_i) d_i = 0.$$

Dividing this sum by the greatest common divisor of all the d_i and taking the result modulo 2 (here one uses the fact that $(n/\gcd(n,k)) \mid \binom{n}{k}$), we obtain that the coefficient n_1 in the expression (6) has to be even.

We now compute $\mathfrak{c}_2(x)$. Let $\Lambda = \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_{4m}$. Then $T^* = \{\sum a_i e_i \mid \sum a_i \text{ is even}\}$ and

$$\omega_1 = e_1, \quad \omega_2 = e_1 + e_2, \quad \omega_3 = e_1 + e_2 + e_3, \ \ldots, \quad \omega_{4m} = e_1 + \cdots + e_{4m}.$$

By [GZ14, § 2] we have $\mathfrak{c}_2(x) = \sum_{i=1}^{4m} n_i \mathfrak{c}_2(\widehat{e^{\omega_i}})$ and $\mathfrak{c}_2(\widehat{e^{\omega_i}}) = N(\widehat{e^{\omega_i}})q$, where

$$N\left(\sum a_j e^{\lambda_j}\right) = \frac{1}{2}\sum a_j \langle \lambda_j, \alpha^\vee\rangle^2 \quad \text{for a fixed long root } \alpha.$$

If we set $\alpha = 2e_{4m}$, then $\langle \lambda, \alpha^\vee\rangle = (\lambda, e_{4m})$ and

$$N(\widehat{e^{\omega_i}}) = \frac{1}{2}\sum_{\lambda \in W(\omega_i)} \langle \lambda, \alpha^\vee\rangle^2 = \frac{1}{2}\sum_{\lambda \in W(\omega_i)} (\lambda, e_{4m})^2 = 2^{i-1}\binom{4m-1}{i-1},$$

which is even for $i \geqslant 2$ (here we have used the fact that the Weyl group acts by permutations and sign changes on $\{e_1, \ldots, e_{4m}\}$). Since n_1 is even, we obtain that $\mathfrak{c}_2(x) \in 2\mathbb{Z}q$.

3d. Half-spin and adjoint groups of type D_{4m} ($m \geqslant 1$)

We first treat the half-spin group $G = \mathbf{HSpin}_{8m}$. As in the C_n case, all even fundamental weights are in T^* and all odd fundamental weights correspond to a generator of $\Lambda/T^* \simeq \mathbb{Z}/2\mathbb{Z}$. Therefore, the map $f : \mathbb{Z}[\Lambda] \to \mathbb{Z}[\Lambda/T^*]$ applied to the element $x = \sum_{i=1}^{4m} n_i \widehat{e^{\omega_i}} + \delta_i \widehat{e^{\omega_i}}$ gives the same equality $\sum_{i \text{ is odd}} (n_i + 2m_i) d_i = 0$, where $m_i \in \mathbb{Z}$, $d_i = 2^i \binom{4m}{i}$ for $i \leqslant 4m-2$ and $d_{4m-1} = 2^{4m-1}$. Upon dividing by the greatest common divisor of the d_i and taking modulo 2, we obtain that n_1 is even if $m > 1$ and $n_1 + n_3$ is even if $m = 1$.

We now compute $\mathfrak{c}_2(x)$. Take a long root $\alpha = e_{4m-1} + e_{4m}$. Then $(\alpha, \alpha) = 2$ and $\langle \lambda, \alpha^\vee \rangle = (\lambda, e_{4m-1}) + (\lambda, e_{4m})$. For $i \leqslant 4m-2$ we have

$$N(\widehat{e^{\omega_i}}) = \frac{1}{2}\sum_{\lambda \in W(\omega_i)} \langle \lambda, \alpha^\vee\rangle^2 = \sum_{\lambda \in W(\omega_i)} ((\lambda, e_{4m})^2 + (\lambda, e_{4m})(\lambda, e_{4m-1})).$$

Since $\sum_{\lambda \in W(\omega_i)} (\lambda, e_{4m})(\lambda, e_{4m-1}) = 0$, we get that

$$N(\widehat{e^{\omega_i}}) = \sum_{\lambda \in W(\omega_i)} (\lambda, e_{4m})^2 = 2^i \binom{4m-1}{i-1}$$

and $N(\widehat{e^{\omega_{4m-1}}}) = N(\widehat{e^{\omega_{4m}}}) = 2^{4m-3}$ (here we have used the fact that W acts by permutations and even sign changes).

Finally, if $m > 1$, we obtain $\mathfrak{c}_2(x) = \sum_i n_i N(\widehat{e^{\omega_i}})q \in 4\mathbb{Z}q$, where $4\mathbb{Z}q = \mathrm{Dec}(\mathbf{HSpin}_{8m})$ by [BR13, § 5]. If $m = 1$, then $N(\widehat{e^{\omega_8}}) = 2$; hence $\mathfrak{c}_2(x) \in 2\mathbb{Z}q$, where $2\mathbb{Z}q = \mathrm{Dec}(\mathbf{HSpin}_8)$ again by [BR13, § 5].

If $m > 1$, for the adjoint group $G = \mathbf{PGO}_{8m}$, by [Mer13a, § 4] and the corresponding half-spin case we obtain that

$$4\mathbb{Z}q = \mathrm{Dec}(\mathbf{PGO}_{8m}) \subseteq \mathrm{SDec}(\mathbf{PGO}_{8m}) \subseteq \mathrm{SDec}(\mathbf{HSpin}_{8m}) = 4\mathbb{Z}q.$$

If $G = \mathbf{PGO}_8$, direct computations (see Appendix A) show that $\mathrm{Dec}(G) = \mathrm{SDec}(G)$.

4. Applications

Observe that $H^3(F,\mathbb{Z}/n\mathbb{Z}(2))$ is the nth torsion part of $H^3(F,2)$ for every n and that $H^3(F,\mathbb{Z}/n\mathbb{Z}(2)) = H^3(F,\mu_n^{\otimes 2})$ if $\mathrm{char}(F)$ does not divide n.

4a. Type C_n

Let $G = \mathbf{PGSp}_{2n}$ be the split projective symplectic group. For a field extension L/F, the set $H^1(L,G)$ is identified with the set of isomorphism classes of central simple L-algebras A of degree $2n$ with a symplectic involution σ (see [KMRT98, § 29]). A decomposable invariant of G takes an algebra with involution (A,σ) to the cup product $\phi \cup [A]$ for a fixed element $\phi \in F^\times$. In particular, decomposable invariants of G are independent of the involution.

Suppose that $4 \mid n$. It is shown in [Mer13a, Theorem 4.6] that the group of indecomposable invariants $\mathrm{Inv}^3(G,2)_{\mathrm{ind}}$ is cyclic of order 2. If $\mathrm{char}(F) \neq 2$, Garibaldi *et al.* constructed in [GPT09, Theorem A] a degree-3 cohomological invariant Δ_{2n} of the group G with coefficients in $\mathbb{Z}/2\mathbb{Z}$. They showed that if $a \in A$ is a σ-symmetric element of $A^\times$ and $\sigma' = \mathrm{Int}(a) \circ \sigma$, then

$$\Delta_{2n}(A,\sigma') = \Delta_{2n}(A,\sigma) + \mathrm{Nrp}(a) \cup [A], \tag{7}$$

where Nrp is the pfaffian norm. In particular, Δ_{2n} does depend on the involution and therefore the invariant Δ_{2n} is not decomposable. Hence the class of Δ_{2n} in $\mathrm{Inv}^3(G,2)_{\mathrm{ind}}$ is nontrivial.

It follows from (7) that the class $\Delta_{2n}(A) \in H^3(L,\mathbb{Z}/2\mathbb{Z})/L^\times \cup [A]$ of $\Delta_{2n}(A,\sigma)$ depends only on the L-algebra A of degree $2n$ and exponent 2 but not on the involution. Since $\Delta_{2n}(A,\sigma)$ is not decomposable, it is not semidecomposable by our main theorem. The latter implies that $\Delta_{2n}(A)$ is *nontrivial generically*, i.e. that there is a central simple algebra A of degree $2n$ over a field extension of F with exponent 2 such that $\Delta_{2n}(A) \neq 0$. This answers a question raised in [GPT09]. (See [Bar14, Remark 4.10] for the $n = 4$ case.)

4b. Type A_{n-1}

Let $G = \mathbf{SL}_n/\boldsymbol{\mu}_m$, where n and m are positive integers such that n and m have the same prime divisors and $m \mid n$. Given a field extension L/F, the natural surjection $G \to \mathbf{PGL}_n$ yields a map

$$\alpha : H^1(L,G) \to H^1(L,\mathbf{PGL}_n) \subset \mathrm{Br}(L)$$

taking a G-torsor Y over L to the class of a central simple algebra $A(Y)$ of degree n and exponent dividing m. By definition, a decomposable invariant of G is of the form $Y \mapsto \phi \cup [A(Y)]$ for a fixed $\phi \in F^\times$.

The map $\mathbf{SL}_m \to \mathbf{SL}_n$ taking a matrix M to the tensor product $M \otimes I_{n/m}$ with the identity matrix gives rise to a group homomorphism $\mathbf{PGL}_m \to G$. The induced homomorphism

$$\varphi : \mathrm{Inv}^3(G,2)_{\mathrm{norm}} \to \mathrm{Inv}^3(\mathbf{PGL}_m,2)_{\mathrm{norm}} = F^\times/F^{\times m}$$

(see [Mer13a, Theorem 4.4]) is a splitting of the inclusion homomorphism

$$F^\times/F^{\times m} = \mathrm{Inv}^3(G,2)_{\mathrm{dec}} \hookrightarrow \mathrm{Inv}^3(G,2)_{\mathrm{norm}}.$$

Collecting descriptions of p-primary components of $\mathrm{Inv}^3(G,2)_{\mathrm{ind}}$ (see § 3b), we get

$$\mathrm{Inv}^3(G,2)_{\mathrm{ind}} \simeq \frac{m}{k}\mathbb{Z}q/m\mathbb{Z}q \quad \text{where } k = \begin{cases} \gcd\left(\dfrac{n}{m}, m\right) & \text{if } \dfrac{n}{m} \text{ is odd,} \\ \gcd\left(\dfrac{n}{2m}, m\right) & \text{if } \dfrac{n}{m} \text{ is even.} \end{cases} \tag{8}$$

Let $\Delta_{n,m}$ be a (unique) invariant in $\mathrm{Inv}^3(G,2)_{\mathrm{norm}}$ such that its class in $\mathrm{Inv}^3(G,2)_{\mathrm{ind}}$ corresponds to $(m/k)q+m\mathbb{Z}q$ and $\varphi(\Delta_{n,m})=0$. Note that the order of $\Delta_{n,m}$ in $\mathrm{Inv}^3(G,2)_{\mathrm{norm}}$ is equal to k. Therefore $\Delta_{n,m}$ takes values in $H^3(-,\mathbb{Z}/k\mathbb{Z}(2))\subset H^3(-,2)$.

Fix a G-torsor Y over F and consider the twists ${}^{Y}G$ and $\mathbf{SL}_1(A(Y))$ by Y of the groups G and $\mathbf{SL}_n$, respectively. The group $F^\times$ acts transitively on the fiber over $A(Y)$ of the map α. If $\phi\in F^\times$, we write ${}^{\phi}Y$ for the corresponding element in the fiber. By (8), the image of $\Delta_{n,m}$ under the natural composition

$$\mathrm{Inv}^3(G,2)_{\mathrm{norm}}\simeq \mathrm{Inv}^3({}^{Y}G,2)_{\mathrm{norm}}\longrightarrow \mathrm{Inv}^3(\mathbf{SL}_1(A(Y)),2)_{\mathrm{norm}}$$

is a m/k-multiple of the Rost invariant. Recall that the Rost invariant takes the class of ϕ in $F^\times/\mathrm{Nrd}(A(Y)^\times)=H^1(F,\mathbf{SL}_1(A(Y)))$ to the cup product $\phi\cup[A(Y)]\in H^3(F,2)$. So we get

$$\Delta_{n,m}({}^{\phi}Y)-\Delta_{n,m}(Y)\in F^\times\cup\frac{m}{k}[A(Y)]. \tag{9}$$

Given a central simple L-algebra A of degree n and exponent dividing m, we define an element

$$\Delta_{n,m}(A)\in\frac{H^3(L,\mathbb{Z}/k\mathbb{Z}(2))}{L^\times\cup(m/k)[A]}$$

as follows. Choose a G-torsor Y over L with $A(Y)\simeq A$ and set $\Delta_{n,m}(A)$ to be the class of $\Delta_{n,m}(Y)$ in the factor group. It follows from (9) that $\Delta_{n,m}(A)$ is independent of the choice of Y.

Proposition 4.1. *Let A be a central simple L-algebra of degree n and exponent dividing m. Then the order of $\Delta_{n,m}(A)$ divides k. If A is a generic algebra, then the order of $\Delta_{n,m}(A)$ is equal to k.*

Proof. If k' is a proper divisor of k, then the multiple $k'\Delta_{n,m}$ is not decomposable. By our theorem, $k'\Delta_{n,m}$ is not semidecomposable and hence $k'\Delta_{n,m}(A)\neq 0$. □

Example 4.2. Let A be a central simple F-algebra of degree $2n$ divisible by 8 and exponent 2. Choose a symplectic involution σ on A. The group $\mathbf{PGSp}_{2n}$ is a subgroup of $\mathbf{SL}_{2n}/\boldsymbol{\mu}_2$; hence, if $\mathrm{char}(F)\neq 2$, the restriction of the invariant $\Delta_{2n,2}$ on $\mathbf{PGSp}_{2n}$ is the invariant $\Delta_{2n}(A,\sigma)$ considered in § 4a. It follows that $\Delta_{2n,2}(A)=\Delta_{2n}(A)\in H^3(F,\mathbb{Z}/2\mathbb{Z})/(F^\times\cup[A])$.

The class $\Delta_{n,m}$ is trivial on decomposable algebras.

Proposition 4.3. *Let n_1,n_2 and m be positive integers such that m divides n_1 and n_2. Let A_1 and A_2 be two central simple algebras over F of degree n_1 and n_2, respectively, and of exponent dividing m. Then $\Delta_{n_1n_2,m}(A_1\otimes_F A_2)=0$.*

Proof. The tensor product homomorphism $\mathbf{SL}_{n_1}\times\mathbf{SL}_{n_2}\to\mathbf{SL}_{n_1n_2}$ yields a homomorphism

$$\mathrm{Sym}^2(T^*_{n_1n_2})\to\mathrm{Sym}^2(T^*_{n_1})\oplus\mathrm{Sym}^2(T^*_{n_2}),$$

where T_{n_1}, T_{n_2} and $T_{n_1n_2}$ are maximal tori of the respective groups. The image of the canonical Weyl-invariant generator $q_{n_1n_2}$ of $\mathrm{Sym}^2(T^*_{n_1n_2})$ is equal to $n_2q_{n_1}+n_1q_{n_2}$. Since n_1 and n_2 are divisible by m, the pullback of the invariant $\Delta_{n_1n_2,m}$ under the homomorphism $(\mathbf{SL}_{n_1}/\boldsymbol{\mu}_m)\times(\mathbf{SL}_{n_2}/\boldsymbol{\mu}_m)\to\mathbf{SL}_{n_1n_2}/\boldsymbol{\mu}_m$ is trivial. □

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Appendix A

Our aim in this appendix is to verify that the groups of decomposable and semidecomposable cohomological invariants of the group $\mathbf{PGO}_8$ coincide. Following the notation of [Mer13a], we have that:

- $\Lambda = \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_4 + \mathbb{Z}e$ where $e = \frac{1}{2}(e_1+e_2+e_3+e_4)$;
- T^* consists of all $\sum a_i e_i$ with $\sum a_i$ even;
- $S^2(\Lambda)^W = \mathbb{Z}q$ where $q = \frac{1}{2}(e_1^2+e_2^2+e_3^2+e_4^2)$;
- the fundamental weights are $\omega_1 = e_1$, $\omega_2 = e_1+e_2$, $\omega_3 = e - e_4$ and $\omega_4 = e$;
- the simple roots are $\lambda_1 = e_1 - e_2$, $\lambda_2 = e_2 - e_3$, $\lambda_3 = e_3 - e_4$ and $\lambda_4 = e_3 + e_4$;
- the Weyl group $W = S_4 \leftthreetimes (C_2)^3$ consists of permutations of e_i and sign changes of an even number of variables;
- the W-orbits of the fundamental weights are given by
  $W(\omega_1) = \{\pm e_1, \pm e_2, \pm e_3, \pm e_4\}$,
  $W(\omega_2) = \{\pm(e_1+e_2), \ldots, \pm(e_3+e_4), \pm(e_1-e_2), \ldots, \pm(e_3-e_4)\}$,
  $W(\omega_3) = \{\pm(e-e_1), \pm(e-e_2), \pm(e-e_3), \pm(e-e_4)\}$,
  $W(\omega_4) = \{e, -e, e-e_1-e_2, e-e_1-e_3, e-e_1-e_4, e-e_2-e_3, e-e_2-e_4, e-e_3-e_4\}$.

Let $\widehat{e^{\omega_i}}$ denote the sum $\sum_{\lambda\in W(\omega_i)} e^{\lambda} - 1$. Then the ideal $\widetilde{I}^W$ is generated by $\widehat{e^{\omega_i}}$ for $i = 1, \ldots, 4$. Note that $\mathbb{Z}[\Lambda]$ is the Laurent polynomial ring $\mathbb{Z}[e^{\pm e_1}, \ldots, e^{\pm e_4}, e^{\pm e}]$, so we represent it as a quotient of the polynomial ring

$$Z[\Lambda] = \mathbb{Z}[u_1, v_1, \ldots, u_4, v_4, u_5, v_5]/(u_i v_1 - 1, \ldots,\ u_5 v_5 - 1,\ u_5^2 = u_1u_2u_3u_4)$$

where $u_i = e^{e_i}$ and $v_i = e^{-e_i}$ for $i = 1, \ldots, 4$, $u_5 = e^e$ and $v_5 = e^{-e}$. Let $r_i = e^{\lambda_i}$ and $s_i = e^{-\lambda_i}$ for the simple roots λ_i. We have an exact sequence

$$0 \to J_1 \to \mathbb{Z}[u_1, \ldots, u_5, v_1, \ldots, v_5, r_1, \ldots, r_4, s_1, \ldots, s_4] \xrightarrow{\pi} \mathbb{Z}[\Lambda] \to 0,$$

where

$$\begin{aligned} J_1 = (&u_1v_1 - 1, \ldots, u_5v_5 - 1, r_1s_1 - 1, \ldots, r_4s_4 - 1,\\ &r_1 - u_1v_2, r_2 - u_2v_3, r_3 - u_3v_4, r_4 - u_3u_4, u_1u_2u_3u_4 - u_5^2), \end{aligned}$$

and the preimage

$$\begin{aligned} \pi^{-1}(\widetilde{I}^W) = J_1 + (&u_1 + \cdots + u_4 + v_1 + \cdots + v_4 - 8,\\ &u_1u_2 + \cdots + u_3u_4 + v_1v_2 + \cdots + v_3v_4 + \cdots\\ &+ u_1v_2 + \cdots + u_3v_4 + v_1u_2 + \cdots + v_3u_4 - 24,\\ &u_5v_1 + u_5v_2 + u_5v_3 + u_5v_4 + u_1v_5 + u_2v_5 + u_3v_5 + u_4v_5 - 8,\\ &u_5 + v_5 + u_5v_1v_2 + u_5v_1v_3 + u_5v_1v_4 + u_5v_2v_3 + u_5v_2v_4 + u_5v_3v_4 - 8). \end{aligned}$$

Note that $\mathbb{Z}[T^*] = \pi(\mathbb{Z}[r_1, \ldots, r_4, s_1, \ldots, s_4])$ and

$$\widetilde{I}^W \cap \mathbb{Z}[T^*] = \pi(\pi^{-1}(\widetilde{I}^W) \cap \mathbb{Z}[r_i, s_i]).$$

Now we use the software Maple to compute a basis of a bigger intersection,

$$\pi^{-1}(\widetilde{I}^W + \widetilde{I}^4) \cap \mathbb{Z}[r_i, s_i].$$

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We will prove that for any x in the basis of this intersection (see [Mer13a, § 4b, p. 19]),

$$c_2(\pi(x)) \in 4q\mathbb{Z} = \mathrm{Dec}(G).$$

Since $c_2(\widetilde{I}^3) = 0$, it is enough to consider the generators that are not contained in $\pi^{-1}(\widetilde{I}^3)$. We compute the basis of the intersection using the following code:

`with(PolynomialIdeals):`
`#relations ideal: J1 :=` $\langle u_1v_1-1, u_2v_2-1, u_3v_3-1, u_4v_4-1, u_5v_5-1, r_1s_1-1, r_2s_2-1, r_3s_3-1, r_4s_4-1, u_1u_2u_3u_4-u_5^2, r_1-u_1v_2, r_2-u_2v_3, r_3-u_3v_4, r_4-u_3u_4\rangle$
`#preimage of` $\widetilde{I}^W$`: J2 :=` $\langle u_1+u_2+u_3+u_4+v_1+v_2+v_3+v_4-8, u_5v_1+u_5v_2+u_5v_3+u_5v_4+u_1v_5+u_2v_5+u_3v_5+u_4v_5-8, u_5+v_5+u_5v_1v_2+u_5v_1v_3+u_5v_1v_4+u_5v_2v_3+u_5v_2v_4+u_5v_3v_4-8, -24+u_1v_2+u_2v_3+u_3v_4+u_3u_4+u_1u_2+u_1v_3+u_1v_4+u_2v_4+v_1u_2+u_1u_3+u_1u_4+u_2u_3+u_2u_4+v_1v_2+v_1v_3+v_1v_4+v_2v_3+v_2v_4+v_3v_4+v_1u_3+v_1u_4+v_2u_3+v_2u_4+v_3u_4\rangle$
`#preimage of the augmentation ideal:`
`augL :=` $\langle u[1]-1, v[1]-1, u[2]-1, v[2]-1, u[3]-1, v[3]-1, u[4]-1, v[4]-1, u[5]-1, v[5]-1\rangle$;
`#preimages of the square,cube and fourth power of the augmentation ideal: squarL := Add(Multiply(augL, augL), J1);`
`cubL := Add(Multiply(augL, Multiply(augL, augL)), J1);`
`quadL := Add(Multiply(augL, cubL), J1);`
`#preimage of` $\widetilde{I}^W + \widetilde{I}^4$`:`
`J := Add(Add(J1, J2), quadL)`
`#intersection with the subring` $\mathbb{Z}[r_i, s_i]$`:`
`K := EliminationIdeal(J, r[1], r[2], r[3], r[4], s[1], s[2], s[3], s[4]):`
`#basis of the intersection`
`Gen := IdealInfo[Generators](K):`
`#print out the elements of the basis that do not lie in` $\pi^{-1}(\widetilde{I}^3)$
`for x in Gen do`
`if not(IdealMembership(x, cubL)) then print(x) end if`
`end do`

This gives a list of 18 polynomials:

- $-34-r_1s_4-2r_2s_4+6r_1+6s_1+10r_2+10s_2+4r_3+4s_3+4r_4+4s_4-2s_2r_4-s_1r_4-2s_2r_3-s_1r_3+r_4r_3-2s_3r_2-2s_1r_2-s_3r_1-2s_2r_1+s_3s_4$;
- $38+r_1s_4+2r_2s_4+r_3s_4-6r_1-6s_1-10r_2-10s_2-6r_3-6s_3-6r_4-6s_4+s_3r_4+2s_2r_4+s_1r_4+2s_2r_3+s_1r_3+2s_3r_2+2s_1r_2+s_3r_1+2s_2r_1$;
- $-37-r_1s_4-2r_2s_4-3r_3s_4+6r_1+6s_1+10r_2+10s_2+7r_3+5s_3+4r_4+7s_4-2s_2r_4-s_1r_4-2s_2r_3-r_3^2-s_1r_3+r_4r_3-2s_3r_2-2s_1r_2-s_3r_1-2s_2r_1+r_3^2s_4$;
- $35+r_1s_4+2r_2s_4+r_3s_4-6r_1-6s_1-10r_2-10s_2-3r_3-5s_3-3r_4-6s_4+2s_2r_4+s_1r_4+2s_2r_3-r_3^2+s_1r_3-3r_4r_3+2s_3r_2+2s_1r_2+s_3r_1+2s_2r_1+r_3^2r_4$;
- $-118-6r_2s_4+14r_1+9s_1+74r_2+46s_2+14r_3+9s_3+14r_4+9s_4+r_4^2-10s_2r_4-10s_2r_3+r_3^2+r_4r_3-6s_3r_2-6s_1r_2-8r_4r_2-8r_3r_2+r_1^2-8r_2^2-10s_2r_1+r_4r_1+r_3r_1-8r_2r_1+3r_1r_3r_4$;
- $-92+28r_1+18s_1+52r_2+26s_2+34r_3+12s_3-8r_4+2r_4^2-2s_2r_4-8s_2r_3-r_3^2-9s_1r_3+2r_4r_3-6s_3r_2-12s_1r_2+2r_4r_2-16r_3r_2-r_1^2-4r_2^2-3s_3r_1-8s_2r_1+2r_4r_1-4r_3r_1-10r_2r_1+6r_2r_3s_1$;
- $-92+34r_1+12s_1+46r_2+32s_2+34r_3+12s_3-8r_4+2r_4^2-2s_2r_4-14s_2r_3-r_3^2-3s_1r_3+2r_4r_3-6s_3r_2-6s_1r_2+2r_4r_2-10r_3r_2-r_1^2-4r_2^2-3s_3r_1-14s_2r_1+2r_4r_1-10r_3r_1-10r_2r_1+6r_1r_3s_2$;

- $-92+34r_1+12s_1+52r_2+26s_2+28r_3+18s_3-8r_4+2r_4^2-2s_2r_4-8s_2r_3-r_3^2-3s_1r_3+2r_4r_3-12s_3r_2-6s_1r_2+2r_4r_2-10r_3r_2-r_1^2-4r_2^2-9s_3r_1-8s_2r_1+2r_4r_1-4r_3r_1-16r_2r_1+6r_1r_2s_3$;
- $-92-9r_1s_4-12r_2s_4+34r_1+12s_1+52r_2+26s_2-8r_3+28r_4+18s_4-r_4^2-8s_2r_4-3s_1r_4-2s_2r_3+2r_3^2+2r_4r_3-6s_1r_2-10r_4r_2+2r_3r_2-r_1^2-4r_2^2-8s_2r_1-4r_4r_1+2r_3r_1-16r_2r_1+6r_1r_2s_4$;
- $-92-3r_1s_4-6r_2s_4+28r_1+18s_1+52r_2+26s_2-8r_3+34r_4+12s_4-r_4^2-8s_2r_4-9s_1r_4-2s_2r_3+2r_3^2+2r_4r_3-12s_1r_2-16r_4r_2+2r_3r_2-r_1^2-4r_2^2-8s_2r_1-4r_4r_1+2r_3r_1-10r_2r_1+6r_2r_4s_1$;
- $-92-3r_1s_4-6r_2s_4+34r_1+12s_1+46r_2+32s_2-8r_3+34r_4+12s_4-r_4^2-14s_2r_4-3s_1r_4-2s_2r_3+2r_3^2+2r_4r_3-6s_1r_2-10r_4r_2+2r_3r_2-r_1^2-4r_2^2-14s_2r_1-10r_4r_1+2r_3r_1-10r_2r_1+6r_1r_4s_2$;
- $80-22r_1-12s_1-40r_2-26s_2-22r_3-12s_3+8r_4-2r_4^2+2s_2r_4+8s_2r_3+r_3^2+3s_1r_3-2r_4r_3+6s_3r_2+6s_1r_2-2r_4r_2+4r_3r_2+r_1^2+4r_2^2+3s_3r_1+8s_2r_1-2r_4r_1-2r_3r_1+4r_2r_1+6r_1r_2r_3$;
- $80+3r_1s_4+6r_2s_4-22r_1-12s_1-40r_2-26s_2+8r_3-22r_4-12s_4+r_4^2+8s_2r_4+3s_1r_4+2s_2r_3-2r_3^2-2r_4r_3+6s_1r_2+4r_4r_2-2r_3r_2+r_1^2+4r_2^2+8s_2r_1-2r_4r_1-2r_3r_1+4r_2r_1+6r_1r_2r_4$;
- $-34-3r_1s_4+26r_1+18s_1-10r_2+4s_2-4r_3+6s_3-4r_4+6s_4+r_4^2+2s_2r_4-3s_1r_4+2s_2r_3+r_3^2-3s_1r_3-2r_4r_3-6s_1r_2+4r_4r_2+4r_3r_2-2r_1^2+4r_2^2-3s_3r_1-4s_2r_1-2r_4r_1-2r_3r_1-2r_2r_1+6r_2r_3r_4$;
- $22+3r_1s_4-26r_1-18s_1+16r_2+2s_2+16r_3-6s_3+16r_4-6s_4-r_4^2-8s_2r_4+3s_1r_4-8s_2r_3-r_3^2+3s_1r_3-10r_4r_3+6s_1r_2-10r_4r_2-10r_3r_2+2r_1^2-4r_2^2+3s_3r_1+4s_2r_1+2r_4r_1+2r_3r_1+2r_2r_1+6r_3r_4s_2$;
- $112-3r_1s_4+6r_2s_4-3r_3s_4-8r_1-9s_1-74r_2-46s_2-8r_3-9s_3-11r_4-6s_4-r_4^2+10s_2r_4+10s_2r_3-r_3^2-4r_4r_3+6s_3r_2+6s_1r_2+8r_4r_2+8r_3r_2-r_1^2+8r_2^2+10s_2r_1-4r_4r_1-7r_3r_1+8r_2r_1+3r_1r_3s_4$;
- $112+6r_2s_4-11r_1-6s_1-74r_2-46s_2-8r_3-9s_3-8r_4-9s_4-r_4^2+10s_2r_4-3s_1r_4+10s_2r_3-r_3^2-3s_1r_3-7r_4r_3+6s_3r_2+6s_1r_2+8r_4r_2+8r_3r_2-r_1^2+8r_2^2+10s_2r_1-4r_4r_1-4r_3r_1+8r_2r_1+3r_3r_4s_1$;
- $22+3r_1s_4-6r_2s_4-6r_3s_4-26r_1-18s_1+22r_2-4s_2+16r_3-6s_3+10r_4-r_4^2-2s_2r_4+3s_1r_4-2s_2r_3-r_3^2+3s_1r_3-4r_4r_3+6s_1r_2-10r_4r_2-16r_3r_2+2r_1^2-4r_2^2+3s_3r_1+4s_2r_1+2r_4r_1+2r_3r_1+2r_2r_1+6r_2r_3s_4$.

Take the first element of the list,

$$\begin{aligned} y = {} & -34-r_1s_4-2r_2s_4+6r_1+6s_1+10r_2+10s_2+4r_3+4s_3+4r_4+4s_4 \\ & -2s_2r_4-s_1r_4-2s_2r_3-s_1r_3+r_4r_3-2s_3r_2-2s_1r_2-s_3r_1-2s_2r_1+s_3s_4. \end{aligned}$$

We compute $c_2(y)$ as the second term in the power series expansion of

$$\begin{aligned} & (1+(l_1-l_4)t)^{-1}(1+(l_2+l_4)t)^{-2}(1+l_1t)^6(1-l_1t)^6(1+l_2t)^{10} \\ & \times(1-l_2t)^{10}(1+l_3t)^4(1-l_3t)^4(1+l_4t)^4(1-l_4t)^4(1+(l_2+l_4)t)^{-2} \\ & \times(1+(-l_1+l_4)t)^{-1}(1+(-l_2+l_3)t)^{-2}(1+(-l_1+l_3)t)^{-1}(1+(l_4+l_3)t) \\ & \times(1+(-l_3+l_2)t)^{-2}(1+(-l_1+l_2)t)^{-2}(1+(-l_3+l_1)t)^{-1}(1+(-l_2+l_1)t)^{-2}(1+(-l_3+l_4)t) \end{aligned}$$

where $l_1=e_1-e_2$, $l_2=e_2-e_3$, $l_3=e_3-e_4$ and $l_4=e_3+e_4$, which gives

$$c_2(y)=-2(e_1^2+e_2^2+e_3^2+e_4^2)=-4q.$$

As the last step, we show that for every generator x that does not lie in $\pi^{-1}(\widetilde{I}^3)$, we have that either $x-y$, $x+y$, $x-2y$ or $x+2y$ lies in $\pi^{-1}(\widetilde{I}^3)$. To do this, we use the following Maple code:

```
for x in Gen do
if not IdealMembership(x, cubL) and IdealMembership(x, squarL) and
(IdealMembership(x-y, cubL) or IdealMembership(x+y, cubL) or
IdealMembership(x+2y, cubL) or IdealMembership(x-2y, cubL) )
then print(x) end if
end do
```

This returns the same list of 18 polynomials, so we see that for every generator x we have $c_2(x) \in 4q\mathbb{Z}$, and thus $\mathrm{SDec}(G) \subseteq \mathrm{Dec}(G)$.

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