ON CONVEX FUNDAMENTAL REGIONS FOR A LATTICE

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Let Λ be a lattice in Euclidean n-space, that is, Λ is a set of points $\xi_1 a_1 + \ldots + \xi_n a_n$ where $a_1, \ldots, a_n$ are linearly independent vectors and the ξ run over all integers. Let μ denote the Lebesgue measure. A closed convex set F is called a *fundamental region* for Λ if the sets $F + x$ $(x \in \Lambda)$ cover the whole space without overlapping; that is, if F^0 is the interior of F, and $0 \neq x \in \Lambda$, then $F^0 \cap (F^0 + x) = \phi$.

Let $f(x)$ be a positive definite quadratic form. The set F consisting of all x which satisfy the inequalities $f(x) \leqslant f(x + a)$, $0 \neq a \in \Lambda$, is clearly a fundamental region which (following Coxeter) we shall call the *Dirichlet region* of f and Λ. In his beautiful classical paper **(4)**, Voronoi showed that if F is a *primitive* fundamental region (that is, if each of its vertices is a vertex of exactly n neighbours $F + a$, $a \in \Lambda$), then F is the Dirichlet region associated with some quadratic form. The classification of non-primitive F has still not been achieved and, in particular, there is an unsettled conjecture of Voronoi that every F is a limit of primitive ones.

It follows from Voronoi's theorem that every primitive F possesses a centre of symmetry. In this note I prove the same result for non-primitive F. For a quite different proof, given in full only for three-space, see Minkowski **(3)**.

Let F be a fundamental region which is closed and convex. A set A is called a Λ-*packing* if $A \cap (A + x) = \phi$ for $0 \neq x \in \Lambda$, and it is known **(**see, for instance, **2)** that then $\mu(A) \leqslant \mu(F)$. The set A is a packing if and only if the equation $a_1 = a_2 + x$ has no solution with $a_1, a_2 \in A$, $0 \neq x \in \Lambda$, that is, on rewriting the equation $x = a_1 - a_2$, if and only if $A - A$ contains no lattice point except the origin.

Now F^0 is a Λ-packing, and, since F^0 is convex,

$$\begin{aligned} F^0 - F^0 &= \tfrac{1}{2}(F^0 + F^0) - \tfrac{1}{2}(F^0 + F^0) \\ &= \tfrac{1}{2}(F^0 - F^0) - \tfrac{1}{2}(F^0 - F^0), \end{aligned}$$

it follows that $\frac{1}{2}(F^0 - F^0)$ is a packing, and therefore

$$\mu(\tfrac{1}{2}(F^0 - F^0)) \leqslant \mu(F) = \mu(F^0). \tag{1}$$

Now by the Brunn-Minkowski theorem **(1**, pp. 88-91**)**, we have

$$\mu(\tfrac{1}{2}F^0 + \tfrac{1}{2}(-F^0))^{1/n} \geqslant \mu(F^0)^{1/n}. \tag{2}$$

From (1), (2), equality must hold in the Brunn-Minkowski theorem, so F^0 is homothetic to $-F^0$, that is, F^0 has a centre of symmetry.

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