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2-ARC-TRANSITIVE REGULAR COVERS OF $K_{n,n} - nK_2$ HAVING THE COVERING TRANSFORMATION GROUP $\mathbb{Z}_p^3$

SHAOFEI DU✉ and WENQIN XU

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Abstract

This paper contributes to the regular covers of a complete bipartite graph minus a matching, denoted $K_{n,n} - nK_2$, whose fiber-preserving automorphism group acts 2-arc-transitively. All such covers, when the covering transformation group K is either cyclic or $\mathbb{Z}_p^2$ with p a prime, have been determined in Xu and Du ['2-arc-transitive cyclic covers of $K_{n,n} - nK_2$', *J. Algebraic Combin.* **39** (2014), 883–902] and Xu *et al.* ['2-arc-transitive regular covers of $K_{n,n} - nK_2$ with the covering transformation group $\mathbb{Z}_p^2$', *Ars. Math. Contemp.* **10** (2016), 269–280]. Finally, this paper gives a classification of all such covers for $K \cong \mathbb{Z}_p^3$ with p a prime.

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1. Introduction

Throughout this paper graphs are finite, simple and undirected. For the group- and graph-theoretic terminology we refer the reader to [18, 20]. For a graph X, let $V(X)$, $E(X)$, $A(X)$ and $\mathrm{Aut}\,X$ denote the vertex set, edge set, arc set and the full automorphism group of X, respectively. An edge and an arc of X are denoted by $\{u, v\}$ and (u, v), respectively. An *s-arc* of X is a sequence $(v_0, v_1, \ldots, v_s)$ of $s + 1$ vertices such that $(v_i, v_{i+1}) \in A(Y)$ and $v_i \neq v_{i+2}$, and X is said to be 2-*arc-transitive* if $\mathrm{Aut}\,X$ acts transitively on the set of 2-arcs of X.

Let X be a graph and let $\mathcal{P}$ be a partition of $V(X)$ into disjoint sets of equal cardinality m. The quotient graph $Y := X/\mathcal{P}$ is the graph with vertex set $\mathcal{P}$ and two vertices P_1 and P_2 of Y are adjacent if there is at least one edge between a vertex of P_1 and a vertex of P_2 in X. We say that X is an *m-fold cover* of Y if the edge set between P_1 and P_2 in X is a matching whenever $P_1P_2 \in E(Y)$. In this case Y is called

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the *base graph* of X and the sets P_i are called the *fibers* of X. An automorphism of X which maps a fiber to a fiber is said to be *fiber-preserving*. The subgroup K of all those automorphisms of X which fix each of the fibers setwise is called the *covering transformation group*. It is easy to see that if X is connected, then the action of K on the fibers of X is necessarily semiregular, that is, $K_v = 1$ for each $v \in V(X)$. In particular, if this action is regular, we say that X is a *regular cover* of Y.

By [28, Theorem 4.1], the class of finite 2-arc-transitive graphs X can be divided into the following three subclasses:

(1) quasiprimitive type: every nontrivial normal subgroup of $\mathrm{Aut}\,X$ acts transitively on vertices;
(2) bipartite type: every nontrivial normal subgroup of $\mathrm{Aut}\,X$ has at most two orbits on vertices and at least one of them has two orbits on vertices;
(3) covering type: there exists a normal subgroup of $\mathrm{Aut}\,X$ having at least three orbits on vertices and thus X is a regular cover of some graph in cases (1) or (2).

During the past twenty years, a lot of results regarding the primitive, quasiprimitive and bipartite 2-arc-transitive graphs have appeared; see [12, 13, 21–23, 28, 29]. However, very few results concerning the 2-arc-transitive covers are known, except for some covers of graphs with small valency and small order. The first worthy class of graphs to be studied might be complete graphs. In [10], a classification of covers of complete graphs is given, whose fiber-preserving automorphism group acts 2-arc-transitively and whose covering transformation group is either cyclic or $\mathbb{Z}_p^2$ with p a prime, and it is generalized in [8] to the covering transformation group $\mathbb{Z}_p^3$ with p a prime. In [32], the same problem as in [10] and [8] is considered, where the covering transformation group is a metacyclic group, which is by definition an extension of one cyclic group by another.

As for covers of bipartite type, in [31] and [33], all regular covers of a complete bipartite graph minus a matching $K_{n,n} - nK_2$ were classified, whose covering transformation group is cyclic or $\mathbb{Z}_p^2$ with p a prime, and whose fiber-preserving automorphism group acts 2-arc-transitively. In this paper, we shall extend the covering transformation group to $\mathbb{Z}_p^3$ with p a prime. Interestingly, we find several new covers of $K_{n,n} - nK_2$. For further reading on the topic of covers, see [5, 6, 9, 14–16, 26].

A combinatorial description of a covering is introduced through a voltage graph, in the next section. Before stating the main theorem, we first introduce several families of covers $Y \times_f K$ of $Y := K_{n,n} - nK_2$ with the covering transformation group $K \cong \mathbb{Z}_p^3$ for a prime p and a voltage assignment f, where

$$V(Y) = \{i, i' \mid 1 \le i \le n\}, \quad E(Y) = \{\{i, j'\} \mid i \ne j, i, j' \in V(Y)\}$$

and K is identified with the additive group of the three-dimensional vector space $V(3, p)$ over $\mathbb{F}_p$.

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(1) $n = 4$ and $X_1(4, p) = Y \times_f K$, where

$$f_{12'} = f_{13'} = f_{14'} = f_{24'} = f_{21'} = f_{31'} = f_{41'} = (0,0,0),$$
$$f_{23'} = (1,0,0), \quad f_{42'} = (0,1,0), \quad f_{34'} = (0,0,1),$$
$$f_{43'} = (0,1,-1), \quad f_{32'} = (-1,1,0).$$

(2) $n = 5$, $p = \pm 1 \bmod 10$ and $X_{21}(5, p) = Y \times_f K$, where

$$f_{1,2'} = (0,2t,1-2t), \quad f_{1,3'} = (2t,1-2t,0), \quad f_{1,4'} = (1-2t,0,2t),$$
$$f_{1,5'} = (-1,-1,-1), \quad f_{2,3'} = (1-2t,0,-2t), \quad f_{2,4'} = (2t,2t-1,0),$$
$$f_{2,5'} = (-1,1,1), \quad f_{3,4'} = (0,-2t,1-2t), \quad f_{3,5'} = (1,1,-1),$$
$$f_{4,5'} = (1,-1,1), \quad f_{i,j'} = f_{i',j} \quad \text{for } i,j \in \{1,2,3,4,5\}, \quad \text{where } t = \frac{1+\sqrt{5}}{4} \in \mathbb{F}_p^*.$$

$n = p = 5$ and $X_{22}(5, 5) = Y \times_f K$, where

$$f_{1,2'} = (0,-1,0), \quad f_{1,3'} = (3,-1,2), \quad f_{1,4'} = (2,3,-1), \quad f_{1,5'} = (0,1,2),$$
$$f_{2,3'} = (0,-1,3), \quad f_{2,4'} = (3,0,1), \quad f_{2,5'} = (2,2,-1), \quad f_{3,4'} = (0,-1,1),$$
$$f_{3,5'} = (3,1,2), \quad f_{4,5'} = (0,-1,-1), \quad f_{i,j'} = f_{i',j} \quad \text{for } i,j \in \{1,2,3,4,5\}.$$

(3) Label $V(Y) = \{i, j' \mid i, j \in \mathrm{PG}(1, p)\}$ and $E(Y) = \{\{i, j'\} \mid i, j' \in V(Y), i \neq j\}$. $n = 1 + p$, $p \geq 5$ and $X_{31}(p + 1, p) = Y \times_f K$, where

$$f_{\infty,i'} = f_{\infty',i} = (0,1,2i) \quad \text{and}$$
$$f_{i,j'} = f_{i',j} = \left(\frac{1}{i-j}, \frac{i+j}{i-j}, \frac{2ij}{i-j}\right) \quad \text{for all } i \neq j \text{ in } \mathbb{F}_p.$$

$n = 6$, $p = 5$ and $X_{32}(6, 5) = Y \times_f K$, where

$$f_{\infty,i'} = f_{\infty',i} = (-i,-i^2,i^3),$$
$$f_{i,j'} = (0,\pm 2,\pm 2(i+j)) \quad \text{for } (i-j)^2 = \mp 1, \text{ where } i,j \in \mathbb{F}_5.$$

(4) Let $\Omega = \mathrm{PG}(2, 2)$ be the two-dimensional projective space over the field $\mathbb{F}_2$, while we identify Ω with $V(3, 2) \setminus \{0\}$. Let χ_Δ denote the characteristic function of Δ, that is, if $\chi_\Delta(i) = 1$ for $i \in \Delta$ and $\chi_\Delta(i) = 0$ for $i \notin \Delta$, then the set $V = V(\Omega)$ of all characteristic functions χ_Δ, where $\Delta \in P(\Omega)$, forms a seven-dimensional vector space over $\mathbb{F}_2$ with the rule: $(a\chi_\Delta + b\chi_\Gamma)(i) = a\chi_\Delta(i) + b\chi_\Gamma(i)$ for any $a, b \in \mathbb{F}_2$ and $\chi_\Delta, \chi_\Gamma \in V(\Omega)$. Clearly, a natural basis for $V(\Omega)$ is the set of characteristic functions $\chi_{\{i\}}$ for all $i \in \Omega$. Note that a one-dimensional subspace of $\mathrm{PG}(2, 2)$ can be written as $\{i, j, i + j\}$ for all $i \neq j$ in Ω, while a two-dimensional subspace of $\mathrm{PG}(2, 2)$ can be written as $\{i, j, k, i + j, j + k, k + i, i + j + k\}$ for any three distinct elements i, j, k in Ω. Let V_1 and V_2 be the subspaces of V generated by the characteristic functions of all one-dimensional subspaces and of all two-dimensional subspaces of $\mathrm{PG}(2, 2)$, respectively.
Let $Y = K_{8,8} - 8K_2$, where $V(Y) = \{i, j' \mid i, j \in V(3, 2)\}$, $E(Y) = \{\{i, j'\} \mid i, j' \in V(Y), i \neq j\}$, and let K be the corresponding additive group of V_1/V_2.

We have $n = 8$, $p = 2$ and $X_4(8, 2) = Y \times_f K$, where

$$f_{0,j'} = \overline{0} := V_2 \quad \text{and} \quad f_{i,j'} = \overline{\chi}_{\{i,j,i+j\}} := \chi_{\{i,j,i+j\}} + V_2 \text{ for all } i \neq j \text{ in } \Omega.$$

Now we are ready to state the main result of this paper, which will be proved in Section 3.

THEOREM 1.1. *Let X be a connected regular cover of $K_{n,n} - nK_2$ $(n \geq 3)$, whose covering transformation group K is isomorphic to $\mathbb{Z}_p^3$ with p a prime and whose fiber-preserving automorphism group acts 2-arc-transitively. Then one of the following holds:*

(1) $n = 4$ *and* $X \cong X_1(4, p)$*;*
(2) $n = 5$ *and* $X \cong X_{21}(5, p)$ *for* $p \equiv \pm 1 \bmod 10$*, or* $X_{22}(5, 5)$ *for* $p = 5$*;*
(3) $n = p + 1 \geq 6$ *and* $X \cong X_{31}(p + 1, p)$ *for* $p \geq 5$*, or* $X_{32}(6, 5)$ *for* $p = 5$*;*
(4) $n = 8$ *and* $X \cong X_4(8, 2)$ *for* $p = 2$.

2. Preliminaries

In this section we introduce some preliminary results needed in Section 3.

To describe a covering graph, we need the following definition. A combinatorial description of a covering was introduced through a voltage graph by Gross and Tucker [17, 18]. Let Y be a graph and K a finite group. A *voltage assignment* (or *K-voltage assignment*) of the graph Y is a function $f : A(Y) \to K$ with the property that $f(u, v) = f(v, u)^{-1}$ for each $(u, v) \in A(Y)$. For convenience, we denote $f(u, v)$ by $f_{u,v}$. The values of f are called *voltages* and K is called the *voltage group*. The *derived graph $Y \times_f K$ from a voltage assignment f* has its vertex set $V(Y) \times K$ and its edge set $E(Y) \times K$, so that an edge (e, g) of $Y \times_f K$ joins a vertex (u, g) to $(v, f_{v,u}g)$ for $(u, v) \in A(Y)$ and $g \in K$, where $e = \{u, v\}$. Clearly, the graph $Y \times_f K$ is a covering of the graph Y with the first coordinate projection $p : Y \times_f K \to Y$, which is called the *natural projection*. For each $u \in V(Y)$, $\{(u, g) \mid g \in K\}$ is a fiber of u. Moreover, by defining $(u, g')^g := (u, g'g)$ for any $g \in K$ and $(u, g') \in V(Y \times_f K)$, K can be identified with a subgroup of $\mathrm{Aut}(Y \times_f K)$ fixing each fiber setwise and acting regularly on each fiber. Therefore, p can be viewed as a *K-covering*. Conversely, each connected regular cover X of Y with the covering transformation group K can be described by a derived graph $Y \times_f K$ from some voltage assignment f. Given a spanning tree T of the graph Y, a voltage assignment f is said to be *T-reduced* if the voltages on the tree arcs are the identity. Gross and Tucker [17] showed that every regular cover X of a graph Y can be derived from a T-reduced voltage assignment f with respect to an arbitrary fixed spanning tree T of Y. Moreover, the voltage assignment f naturally extends to walks in Y. For any walk W of Y, let f_W denote the voltage of W. Finally, we say that an automorphism α of Y *lifts* to an automorphism $\overline{\alpha}$ of X if $\alpha p = p\overline{\alpha}$, where p is the covering projection from X to Y.

The first proposition is related to a lifting criterion of an automorphism of a base graph with respect to a voltage assignment.

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Proposition 2.1 [25, Corollary 4.3]. *Let Y be a connected graph and let X be a cover of Y derived from a voltage assignment f. Then an automorphism α of Y can be lifted to an automorphism of X if and only if, for each closed walk W in Y, we have that $f_{W^\alpha} = 1$ implies $f_W = 1$.*

Let G be a finite group and H a proper subgroup of G, and let $D = D^{-1}$ be an inverse-closed union of some double cosets of H in $G - H$. Then the *coset graph* $X = X(G; H, D)$ is defined by taking $V(X) = \{Hg \mid g \in G\}$ as the vertex set and $E(X) = \{\{Hg_1, Hg_2\} \mid g_2g_1^{-1} \in D\}$ as the edge set. By the definition, the order of $V(X)$ is the number of left cosets of H in G and its valency is the number of left cosets of H in D. It follows that the group G in its coset action by right multiplication on $V(X)$ is transitive, and the kernel of this representation of G is the intersection of all the conjugates of H in G. If this kernel is trivial, then we say that the subgroup H is *core-free*. In particular, if $H = 1$, then we get a *Cayley graph*. Conversely, each vertex-transitive graph is isomorphic to a coset graph (see [24]).

Let G be a group, let L and R be subgroups of G and let D be a union of double cosets of R and L in G, namely, $D = \bigcup_i Rd_iL$. By $[G : L]$ and $[G : R]$, we denote the sets of cosets G relative to L and R, respectively. Define a bipartite graph $X = \mathbf{B}(G, L, R; D)$ with bipartition $V(X) = [G : L] \cup [G : R]$ and edge set $E(X) = \{\{Lg, Rdg\} \mid g \in G, d \in D\}$. This graph is called the *bicoset graph* of G with respect to L, R and D (see [11]).

Proposition 2.2 [11, Lemmas 2.3 and 2.4].

(i) *The bicoset graph $X = \mathbf{B}(G, L, R; D)$ is connected if and only if G is generated by elements of $D^{-1}D$.*
(ii) *Let Y be a bipartite graph with bipartition $V(Y) = U(Y) \cup W(Y)$, let G be a subgroup of* $\mathrm{Aut}(Y)$ *acting transitively on both U and W, let $u \in U(Y)$ and $w \in W(Y)$ and set $D = \{g \in G \mid w^g \in Y_1(u)\}$, where $Y_1(u)$ is the neighborhood of u. Then D is a union of double cosets of G_w and G_u in G, and $Y \cong \mathbf{B}(G, G_u, G_w; D)$. In particular, if $\{u, w\} \in E(Y)$ and G_u acts transitively on its neighbor, then $D = G_wG_u$.*

The following result may be deduced from the classification of doubly transitive groups (see [3] and [4, Corollary 8.3]).

Proposition 2.3. *Let G be a 3-transitive permutation group of degree at least four. Then one of the following occurs:*

(i) $G \cong S_4$;
(ii) $\mathrm{soc}(G)$ *is* 4-*transitive;*
(iii) $\mathrm{soc}(G) \cong M_{22}$ *or* A_5, *which are* 3-*transitive but not* 4-*transitive;*
(iv) $\mathrm{PSL}(2, q) \le G \le \mathrm{P\Gamma L}(2, q)$, *where the projective special linear group* $\mathrm{PSL}(2, q)$ *is the socle of G which does not act 3-transitively, and G acts on the projective geometry* $\mathrm{PG}(1, q)$ *in a natural way, having degree $q + 1$ with $q \ge 5$ an odd prime power;*

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(v) $G \cong \mathrm{AGL}(m, 2)$ *with* $m \geq 3$*; or*
(vi) $G \cong \mathbb{Z}_2^4 \rtimes A_7 < \mathrm{AGL}(4, 2)$.

The next two propositions deal with two basic group-theoretic results.

PROPOSITION 2.4 [20, Satz 4.5]. *Let* H *be a subgroup of a group* G. *Then* $C_G(H)$ *is a normal subgroup of* $N_G(H)$ *and the quotient* $N_G(H)/C_G(H)$ *is isomorphic with a subgroup of* $\mathrm{Aut}\, H$.

PROPOSITION 2.5 [20, Satz 17.4]. *Let* G *be a finite group. Let* A *and* B *be two subgroups of* G *such that* A *is abelian normal in* G, $A \leq B \leq G$ *and* $(|A|, |G : B|) = 1$. *If* A *has a complement in* B, *then* A *has a complement in* G.

The following result may be deduced from Bloom's determination of the subgroups of $\mathrm{PSL}(3, q)$ in [1].

PROPOSITION 2.6. *Let* $G = \mathrm{GL}(3, p)$ *for an odd prime* p. *Then:*

(1) *any nontrivial subgroup* H *of* G *which does not contain an elementary abelian normal subgroup of order* ≥ 2 *is isomorphic to one of the following groups:*

   (i) $\mathrm{PSL}(2, 5)$ *with* $p \equiv \pm 1 \bmod 10$*;*
   (ii) $\mathrm{PSL}(2, 7)$ *with* $p^3 \equiv 1 \bmod 7$*;*
   (iii) $\mathrm{PSL}(2, p)$ *for* $p \geq 5$*; or*
   (iv) $\mathrm{PGL}(2, p)$ *for* $p \geq 5$.

   Moreover, G *has exactly one conjugacy class of subgroups isomorphic to each subgroup* H *listed in* (i)–(iii)*;*

(2) G *contains neither the affine group* $\mathrm{AGL}(m, 2)$ *for* $m \geq 3$ *nor* $\mathbb{Z}_2^4 \rtimes A_7$.

The next proposition shows a property of $\mathrm{PSL}(2, 7)$ acting on the vector space $V(3, p)$.

PROPOSITION 2.7 [8, Lemmas 2.7 and 2.8]. *Let* p *be an odd prime and* $p^3 \equiv 1 \bmod 7$ *or* $p = 7$. *Then, as a subgroup of* $\mathrm{GL}(3, p)$, $\mathrm{PSL}(2, 7)$ *has no orbits of length seven in its action on the space* $V(3, p)$.

For a group G, we let G' denote the commutator subgroup of G. Recall that a group G is an *extension* of N by H if G has a normal subgroup N such that the quotient group G/N is isomorphic to H. In particular, G is a *proper central extension* of N by H if $N \leq Z(G) \cap G'$ is a central subgroup. Such central subgroups are all quotients of a largest group, called the *Schur multiplier* $\mathrm{Mult}(G)$ of G.

PROPOSITION 2.8 [7, page xv]. *The Schur multiplier of the simple group* $\mathrm{PSL}(2, q)$ *is* $\mathbb{Z}_2$ *for* $q \neq 9$, *and* $\mathbb{Z}_6$ *for* $q = 9$.

The next result is a simple observation and it was first mentioned in [9].

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Proposition 2.9 [9, Lemma 2.5]. *Let Y be a graph and let $\mathcal{B}$ be a set of cycles of Y spanning the cycle space C_Y of Y. If X is a cover of Y given by a voltage assignment f for which each $C \in \mathcal{B}$ vanishes, then X is disconnected.*

The following proposition may be extracted from [33].

Proposition 2.10. *Let X be a connected regular cover of $K_{n,n} - nK_2$ $(n \geq 3)$, whose covering transformation group K is isomorphic to $\mathbb{Z}_p^2$ with p a prime and whose fiber-preserving automorphism group acts 2-arc-transitively. Then X exists if and only if $n = 4$.*

3. Proof of Theorem 1.1

To prove Theorem 1.1, let $U = \{1, 2, \ldots, n\}$ and $W = \{1', 2', \ldots, n'\}$. Set $Y = K_{n,n} - nK_2$ $(n \geq 3)$ with the vertex set $V(Y) = U \cup W$ and the edge set $E(Y) = \{\{i, j'\} \mid i \neq j, i, j = 1, 2, \ldots, n\}$. Let X be a cover of Y with covering projection $f : X \to Y$ and covering transformation group $K = V^+(3, p)$, the additive group of $V(3, p)$.

Suppose that $n = 3$. Then Y is a circle and there is only one cotree arc. Since X is assumed to be connected, all voltages assigned to the cotree arcs in Y should generate K. It means that K is a cyclic group, which is a contradiction. Therefore, we assume that $n \geq 4$.

Let A be a 2-arc-transitive group of automorphisms of the base graph Y and let $G = A_U = A_W$. Let $\widetilde{A}$ and $\widetilde{G}$ be the respective lifts of A and G. Clearly, $\mathrm{Aut}(Y) = S_n \times \langle\sigma\rangle$, where σ is the involution exchanging every pair i and i'.

Since A acts 2-arc-transitively on Y, G has a faithful 3-transitive representation on both U and W, so that G should be one of the 3-transitive groups listed in Proposition 2.3. Moreover, for the case $n = 4$, it has been proved in [30] that $X \cong X_1(4, p)$. So, we need to consider the following remaining cases in four separate subsections:

(1) either $\mathrm{soc}(G)$ is 4-transitive or $\mathrm{soc}(G) \cong M_{22}$, and it will be proved in Section 3.1 that the covering graph X does not exist;
(2) $n = 5$ and $\mathrm{soc}(G) = A_5$, and it will be proved in Section 3.2 that $X \cong X_{21}(5, p)$ or $X_{22}(5, 5)$;
(3) $n \geq 6$ and $\mathrm{soc}(G) = \mathrm{PSL}(2, q)$ with $q \geq 5$, and it will be proved in Section 3.3 that $X \cong X_{31}(p + 1, p)$ or $X_{32}(6, 5)$;
(4) G is of affine type and it will be proved in Section 3.4 that $X \cong X_4(8, 2)$.

3.1. Either soc(G) is 4-transitive or soc(G) $\cong M_{22}$.

Lemma 3.1. *There exist no regular covers X of $K_{n,n} - nK_2$, whose fiber-preserving automorphism group acts 2-arc-transitively and whose covering transformation group is isomorphic to $\mathbb{Z}_p^3$ with p a prime, provided either* $\mathrm{soc}(G)$ *acts 4-transitively on two biparts or* $\mathrm{soc}(G) \cong M_{22}$.

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Proof. Suppose that G has a nonabelian simple socle $T := \mathrm{soc}(G)$ which is either 4-transitive or isomorphic to M_{22}. Let $\widetilde{T}$ be the lift of T, so that $\widetilde{T}/K = T$. In view of Proposition 2.4,

$$(\widetilde{T}/K)/(C_{\widetilde{T}}(K)/K) \cong \widetilde{T}/C_{\widetilde{T}}(K) \leq \mathrm{Aut}(K) \cong \mathrm{GL}(3, p). \tag{3.1}$$

Since $C_{\widetilde{T}}(K)/K \trianglerighteq \widetilde{T}/K$ and $\widetilde{T}/K$ is simple, we get $C_{\widetilde{T}}(K)/K = 1$ or $\widetilde{T}/K$. If the first case happens, then (3.1) implies that $\mathrm{GL}(3, p)$ contains a nonabelian simple subgroup which is either 4-transitive or isomorphic to M_{22}. This contradicts Proposition 2.6. Thus, $C_{\widetilde{T}}(K) = \widetilde{T}$, that is, $K \leq Z(\widetilde{T})$. Let $\mathbb{Z}_p \cong K_1 \leq K$. Since $K \leq Z(\widetilde{T})$, it follows that $K_1 \trianglerighteq \widetilde{T}$. Consider the quotient graph Z induced by the normal subgroup K_1. Then Z is a $\mathbb{Z}_p^2$-cover of the base graph Y. However, by Proposition 2.10, there exists no such cover. This completes our proof of this lemma. □

3.2. $n = 5$ and $\mathrm{soc}(G) = A_5$. Suppose that $n = 5$ and $\mathrm{soc}(G) = A_5$, so that $Y = K_{5,5} - 5K_2$. Since G is isomorphic to either A_5 or S_5 and since A_5 is a 3-transitive group of degree five, it suffices to find all the covers for which A_5 lifts. Suppose that $G \cong A_5$ and let $\widetilde{G}$ be the lift of G, that is, $\widetilde{G}/K = G$. As A_5 is simple, we have $C_{\widetilde{G}}(K)/K \cong 1$ or A_5. For the case $C_{\widetilde{G}}(K)/K \cong A_5$, which means that $K \leq Z(\widetilde{G})$, with the same arguments as Lemma 3.1, one may get that there exist no connected covers occurring. Therefore, $C_{\widetilde{G}}(K) = K$. Moreover, it follows from Proposition 2.4 that

$$A_5 \cong \widetilde{G}/K = \widetilde{G}/C_{\widetilde{G}}(K) \leq \mathrm{Aut}(K) \cong \mathrm{GL}(3, p).$$

So, by Proposition 2.6, we have either $p \equiv \pm 1 \bmod 10$ or $p = 5$. In what follows, we deal with these two cases in Lemmas 3.2 and 3.3 separately.

Lemma 3.2. *If $p \equiv \pm 1 \bmod 10$, then $X \cong X_{21}(5, p)$.*

Proof. Let F be a fiber and take a vertex $\widetilde{v} \in F$. Then $\widetilde{G}_F = K \rtimes \widetilde{G}_{\widetilde{v}}$. Since $(|\widetilde{G} : \widetilde{G}_F|, |K|) = (5, p^3) = 1$ and K is an abelian normal subgroup of $\widetilde{G}$, by Proposition 2.5, K has a complement in $\widetilde{G}$, say T. Thus, $\widetilde{G} = K \rtimes T$, where $T \cong A_5$.

Let $K = V^+(3, p)$. By [1, Lemma 6.4], $\mathrm{GL}(3, p)$ has only one conjugacy class of subgroups isomorphic to A_5, for $p \equiv \pm 1 \bmod 10$, given as follows:

$$a_1 = (12)(34) \longmapsto a = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad c_1 = (234) \longmapsto c = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

$$x_1 = (345) \longmapsto x = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} - t & -t \\ t - \frac{1}{2} & t & -\frac{1}{2} \\ t & -\frac{1}{2} & \frac{1}{2} - t \end{pmatrix},$$

where $t = ((1 + \sqrt{5})/4) \in \mathbb{F}_p^*$ and multiplication in A_5 is chosen from right to left (for example, $(123)(234) = (12)(34)$ but not $(13)(24)$). For any $k = (x, y, z) \in K$ and any

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matrix $g \in T$, we may write $k^g := (x, y, z)g$. Moreover, under this isomorphism,

$$d_1 = (15)(24) = (345)(14)(23)(354) \longmapsto d = \begin{pmatrix} -t & -\frac{1}{2} & t - \frac{1}{2} \\ -\frac{1}{2} & t - \frac{1}{2} & -t \\ t - \frac{1}{2} & -t & -\frac{1}{2} \end{pmatrix},$$

$$b_1 = (13)(24) = (234)(12)(34)(243) \longmapsto b = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Acting on $V(Y) = U \cup W$, where $U = \{1, 2, 3, 4, 5\}$ and $W = \{1', 2', 3', 4', 5'\}$, let $H := \langle a, b \rangle \rtimes \langle c \rangle \cong A_4$ be the point stabilizer for the vertex $5 \in U$ and so the other vertices in $U \setminus \{5\}$ correspond to the cosets $\{Hd,\ Hda,\ Hdb,\ Hdab\}$. Then we carry out the proof by the following four steps.

Step 1. Determination of the point stabilizers $\widetilde{G}_{\widetilde{u}}$.

Taking $\widetilde{u} \in f^{-1}(5)$, the fiber over 5, we have $A_4 \cong \widetilde{G}_{\widetilde{u}} \leq K \rtimes H \cong \mathbb{Z}_p^3 \rtimes A_4$. Since $p \equiv \pm 1 \bmod 10$, p cannot be 2 or 3. Thus, K is a normal π-Hall subgroup of $K \rtimes H$. So, by the Schur–Zassenhaus theorem, we get that the subgroups of $K \rtimes H$ which are isomorphic to H are all conjugate. Therefore, one may set $L := \widetilde{G}_{\widetilde{u}} = H$ and $R := \widetilde{G}_{\widetilde{u}'} = H$, where $\widetilde{u}' \in f^{-1}(5')$.

Step 2. Determination of the bicoset graphs of $\widetilde{G}$ relative to L and R.

Now, by Proposition 2.2, our graph X is isomorphic to a bicoset graph $X' = \mathbf{B}(\widetilde{G}, L, R; D)$, where $D = Rdk_1L$ for $k_1 \in K$, with two biparts:

$$[\widetilde{G} : L] = \{Lk, Ldk, Ldak, Ldbk, Ldabk \mid k \in K\},$$
$$[\widetilde{G} : R] = \{Rk, Rdk, Rdak, Rdbk, Rdabk \mid k \in K\}.$$

Since the length of the orbit of L containing the vertex Rdk_1 is four, the element c should fix the vertex Rdk_1, that is,

$$\begin{aligned} Rdk_1 &= Rdk_1c = Rdk_1c(dk_1)^{-1}dk_1 = Rc^d(k_1^ck_1^{-1})^d dk_1 \\ &= Rc^{-1}(k_1^ck_1^{-1})^d dk_1 = R(k_1^ck_1^{-1})^d dk_1, \end{aligned}$$

which forces $k_1^c = k_1$. This in turn gives $k_1 = (x, x, x)$ for some $x \in \mathbb{F}_p^*$.

Step 3. Show that $X' \cong X_{21}(5, p)$.

Since the neighbor of L corresponds to the bicoset $D = Rdk_1L$, the vertex L is adjacent to

$$\{Rd(x, x, x), Rda(x, -x, -x), Rdb(-x, -x, x), Rdab(-x, x, -x)\}.$$

Therefore, the neighbors of Ld, Lda, Ldb and $Ldab$ are, respectively,

$$\{R(-x, -x, -x), Rda(0, 2tx, (1 - 2t)x), Rdb(2tx, (1 - 2t)x, 0), Rdab((1 - 2t)x, 0, 2tx)\},$$
$$\{R(-x, x, x), Rdb((1 - 2t)x, 0, -2tx), Rdab(2tx, (2t - 1)x, 0), Rd(0, -2tx, (2t - 1)x)\},$$
$$\{R(x, x, -x), Rda((2t - 1)x, 0, 2tx), Rdab(0, -2tx, (1 - 2t)x), Rd(-2tx, (2t - 1)x, 0)\}$$

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and

$$\{R(x,-x,x), Rda(-2tx,(1-2t)x,0), Rdb(0,2tx,(2t-1)x), Rd((2t-1)x,0,-2tx)\}.$$

Define a map $\eta\colon V(X') \to V(X_{21}(5,p))$ by the rule

$$\begin{aligned} \eta(Lk) &= (5, x^{-1}k), & \eta(Rk) &= (5', x^{-1}k),\\ \eta(Ldk) &= (1, x^{-1}k), & \eta(Rdk) &= (1', x^{-1}k),\\ \eta(Ldak) &= (2, x^{-1}k), & \eta(Rdak) &= (2', x^{-1}k),\\ \eta(Ldbk) &= (3, x^{-1}k), & \eta(Rdbk) &= (3', x^{-1}k) \quad \text{and}\\ \eta(Ldabk) &= (4, x^{-1}k), & \eta(Rdabk) &= (4', x^{-1}k), \end{aligned}$$

where $k \in K$. It can be checked that $X' \cong X_{21}(5,p)$ via η.

Step 4. The connectedness of $X_{21}(5,p)$. Take three closed walks:

$$W_1 = 1,2',3,4',1, \quad W_2 = 1,3',4,2',1, \quad W_3 = 1,4',2,3',1.$$

Then it is easy to get $f_{W_1} = (0,0,2(1-4t))$, $f_{W_2} = (0,2(1-4t),0)$ and $f_{W_3} = (2(1-4t),0,0)$, where t is given as above. Then f_{W_1}, f_{W_2} and f_{W_3} can generate K. Hence, $X_{21}(5,p)$ is connected.

Finally, in view of the voltage assignment f of $X_{21}(5,p)$, for σ which exchanges every pair in Y and for any i,j, we have $f_{i^\sigma,j'^\sigma} = f_{i',j} = f_{i,j'}$. Thus, $f_{W^\sigma} = f_W$ for any closed walk W. So, by Proposition 2.1, σ lifts. □

Lemma 3.3. *If $p = 5$, then $X \cong X_{22}(5,5)$.*

Proof. Suppose that $n = p = 5$. By [1, Lemma 6.3], GL(3, 5) has only one conjugacy class of subgroups isomorphic to PSL(2, 5) given as follows:

$$\varphi : \overline{\begin{pmatrix} r & s\\ t & v\end{pmatrix}} \mapsto (rv-st)^{-1}\begin{pmatrix} r^2 & 2rs & 2s^2\\ rt & rv+st & 2sv\\ t^2/2 & tv & v^2\end{pmatrix}.$$

In particular,

$$\bar a = \overline{\begin{pmatrix} 1 & 1\\ 0 & 1\end{pmatrix}} \longmapsto a = \begin{pmatrix} 1 & 2 & 2\\ 0 & 1 & 2\\ 0 & 0 & 1\end{pmatrix}, \quad \bar b = \overline{\begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix}} \longmapsto b = \begin{pmatrix} 0 & 0 & 2\\ 0 & -1 & 0\\ 3 & 0 & 0\end{pmatrix},$$
$$\bar c = \overline{\begin{pmatrix} 3 & 0\\ 0 & 2\end{pmatrix}} \longmapsto c = \begin{pmatrix} -1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{pmatrix}, \quad \bar d = \overline{\begin{pmatrix} 1 & 1\\ 2 & 3\end{pmatrix}} \longmapsto d = \begin{pmatrix} 1 & 2 & 2\\ 2 & 0 & 1\\ 2 & 1 & -1\end{pmatrix}.$$

Acting on $V(Y) = U \cup W$, where $U = \{1,2,3,4,5\}$ and $W = \{1',2',3',4',5'\}$, let $H := \langle b,c\rangle \rtimes \langle d\rangle \cong A_4$ correspond to the vertex $1 \in U$ and the other vertices in $U \setminus \{1\}$ correspond to the cosets $\{Ha,\ Ha^2,\ Ha^3,\ Ha^4\}$. Take $\widetilde{u} \in F := f^{-1}(1)$ and $\widetilde{u}' \in F' := f^{-1}(1')$. Let $L := \widetilde{G}_{\widetilde{u}}$ and $R := \widetilde{G}_{\widetilde{u}'}$, and the two biparts in the bicoset graph are

$$[\widetilde{G} : L] = \{La^ik \mid i \in \mathbb{Z}_5, k \in K\} \quad \text{and} \quad [\widetilde{G} : R] = \{Ra^ik \mid i \in \mathbb{Z}_5, k \in K\}.$$

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Then we carry out the proof by the following six steps.

Step 1. Show that K has no complement in $\widetilde{G}$.

On the contrary, assume that $\widetilde{G} = K \rtimes T$, where $T \cong \mathrm{PSL}(2,5)$. Since $\widetilde{G}_F = \widetilde{G}_{F'} = K \rtimes H \cong \mathbb{Z}_5^3 \rtimes A_4$ and there is only one conjugacy class of A_4 in $K \rtimes H$, we may set $L := \widetilde{G}_{\widetilde{u}} = H$ and $R := \widetilde{G}_{\widetilde{u}'} = H$. Then $X \cong X' = \mathbf{B}(\widetilde{G}, L, R; D)$, where $D = Rak_1L$ for some $k_1 \in K \setminus \{0\}$, noting that $R = L = H$.

As the length of the orbit of L containing the vertex Rak_1 is four, the element d should fix the vertex Rak_1, that is,

$$Rak_1 = Rak_1d = Rak_1d(ak_1)^{-1}ak_1 = Rd^{a^{-1}}(k_1^dk_1^{-1})^{a^{-1}}ak_1 = R(k_1^dk_1^{-1})^{a^{-1}}ak_1,$$

forcing $k_1^d = k_1$. This gives $k_1 = (x, x, -x)$ for some $x \in \mathbb{F}_5^*$.

Now

$$\langle D^{-1}D\rangle = \langle L(ak_1)^{-1}Rak_1L\rangle = \langle b, c, d, b^{ak_1}, c^{ak_1}, d^{ak_1}\rangle = \widetilde{G}.$$

By computation, one may get $c^{ak_1}d^{ak_1}b = ak_1$ and so $ak_1 \in \langle D^{-1}D\rangle$. Thus, $\langle D^{-1}D\rangle = \langle b, d, ak_1\rangle$. Moreover, we have $d = (ak_1)b(ak_1)^2b(ak_1)^2$, which means that $\langle D^{-1}D\rangle = \langle b, ak_1\rangle$. Since $(ak_1)^5 = b^2 = (ak_1b)^3 = 1$, it follows that $\langle D^{-1}D\rangle \cong A_5 < \widetilde{G}$ and thus, by Proposition 2.2, X' is disconnected.

Step 2. Determination of the defining relations of $\widetilde{G}$.

Now assume that K has no complement in $\widetilde{G}$. Then our group $\widetilde{G} = \langle a_1, b_1, x_1, y_1, z_1\rangle$ has the following defining relations:

$$a_1^5 = (0,0,i), \quad b_1^2 = (3j, 0, j), \quad (a_1b_1)^3 = (l, -l, 2l), \quad x_1^{a_1} = x_1y_1^2z_1^2,$$
$$y_1^a = y_1z_1^2, \quad z_1^{a_1} = z_1, \quad {x_1}^{b_1} = z_1^2, \quad y_1^{b_1} = y_1^4, z_1^{b_1} = x_1^3,$$

where $i,\ j,\ l \in \mathbb{F}_5$ and $x_1 = (1,0,0)$, $y_1 = (0,1,0)$, $z_1 = (0,0,1) \in K$. If $i = 0$, then $(|\widetilde{G} : K \rtimes \langle a_1\rangle|, |K|) = (12, 5^3) = 1$; Proposition 2.5 implies that K has a complement in $\widetilde{G}$, which contradicts our assumption. Hence, $i \neq 0$.

Set $H := \langle a, b, x, y, z\rangle$, which has the following defining relations:

$$a^5 = (0,0,1), \quad b^2 = 1, \quad (ab)^3 = (l, -l, 2l), x^a = xy^2z^2,$$
$$y^a = yz^2, z^a = z, x^b = z^2, y^b = y^4, z^b = x^3,$$

where $l \in \mathbb{F}_5$ and $x = (1,0,0)$, $y = (0,1,0)$, $z = (0,0,1) \in K$.

Define a map from H to $\widetilde{G}$:

$$\varphi : a \mapsto a_1(0, l, j), b \mapsto b_1(j, 0, 2j), x \mapsto x_1^i, y \mapsto y_1^i, z \mapsto z_1^i.$$

Then φ can be extended to an isomorphism from H to $\widetilde{G}$. Therefore, let $\widetilde{G} = H$.

Step 3. Determination of the point stabilizers $\widetilde{G}_{\widetilde{u}}$.

Since $\widetilde{G}_{\widetilde{u}}$ is the lift of $H = \langle \overline{b}, \overline{d}\rangle$, where $\overline{d} = \overline{a}\overline{b}\overline{a}^2\overline{b}\overline{a}^2$, we may set $\widetilde{G}_{\widetilde{u}} := \langle bk_1,\ aba^2ba^2k_2\rangle$ for some $k_1,\ k_2 \in K$. As $(bk_1)^2 = 1$, we get $k_1 = (r_1, s_1, 3r_1)$ for some $r_1, s_1 \in \mathbb{F}_5$.

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For the generators a and b, one may get the following relations:

$$\begin{aligned} &bab = a^{-1}ba^{-1}, \quad ba^{-1}b = aba, \quad ba^{-2}b = a(ba^2b)a, \\ &ba^2b = a^{-1}(ba^{-2}b)a^{-1}, \quad ba^2ba^2b = a^{-1}(ba^{-3}b)a^{-1}, \\ &ba^{-2}ba^{-2}b = a(ba^3b)a, \quad ba^2ba^3b = a^{-1}ba^{-2}ba^2b, \\ &ba^{-2}ba^2b = a(ba^2ba^3b) = (ba^{-3}ba^{-2}b)a^{-1}. \end{aligned} \tag{3.2}$$

Since $\widetilde{G}_{\widetilde{u}} \cong A_4$ and $aba^2ba^2k_2$ is the lift of $\overline{d}$, it follows that

$$(aba^2ba^2k_2)^3 = 1. \tag{3.3}$$

According to (3.2) and $a^5 = (0, 0, 1)$,

$$\begin{aligned} (aba^2ba^2)^3 &= a(ba^2ba^2ab)a^2ba^2aba^2ba^2 = ba^{-2}(ba^2ba^2b)a^3ba^2ba^2 \\ &= ba^{-3}ba^{-3}(ba^2ba^2b)a^2 = ba^{-3}ba^{-4}ba^{-3}ba \\ &= (a^{-5})^bba^2ba^{-5}aba^{-3}ba = (a^{-5})^bba^2(bab)a^{-3}ba(a^{-5})^{ba^{-3}ba} \\ &= (a^{-5})^b(bab)a^{-4}ba(a^{-5})^{ba^{-3}ba} = (a^{-5})^ba^{-1}ba^{-5}ba(a^{-5})^{ba^{-3}ba} \\ &= (a^{-5})^b(a^{-5})^{ba}(a^{-5})^{ba^{-3}ba} = (2, 2, 3). \end{aligned}$$

Set $k_2 := (r_2, s_2, t_2)$. It follows from (3.3) that

$$k_2^{I+aba^2ba^2+(aba^2ba^2)^2} = (3, 3, 2),$$

that is, $r_2 + s_2 - t_2 = 3$.

Letting $k = (2s_1 + 2s_2 + t_2, 2s_1, 3s_1 + 2s_2 + 3t_2) \in K$,

$$\widetilde{G}_{\widetilde{u}^k} = k^{-1}\widetilde{G}_{\widetilde{u}}k = \langle b(k^{-1})^bk_1k,\ aba^2ba^2(k^{-1})^{aba^2ba^2}k_2k\rangle = \langle bk_1',\ aba^2ba^2(3, 0, 0)\rangle,$$

where $k_1' = (s_1 + 3s_1 + s_2 + 2t_2, 0, 3(r_1 + 3s_1 + s_2 + 2t_2))$. Moreover,

$$(bk_1'aba^2ba^2(3, 0, 0))^3 = (baba^2ba^2(k_1')^{aba^2ba^2}(3, 0, 0))^3 = 1. \tag{3.4}$$

By (3.2), one may get $(baba^2ba^2)^3 = 1$. Then, from (3.4), we have $k_1' = (2, 0, 1)$. Hence, we may assume that

$$L := \widetilde{G}_{\widetilde{v}} = \langle b(2, 0, 1), aba^2ba^2(3, 0, 0)\rangle \quad \text{and} \quad R := \widetilde{G}_{\widetilde{v}'} = \langle b(2, 0, 1), aba^2ba^2(3, 0, 0)\rangle,$$

where $\widetilde{v} \in f^{-1}(1)$ and $\widetilde{v}' \in f^{-1}(1')$.

Step 4. Determination of the bicoset graphs $\mathbf{B}(\widetilde{G}, L, R; D)$ of $\widetilde{G}$.

Set $D = Rak_3L$ for some $k_3 \in K$ and $X' := \mathbf{B}(\widetilde{G}, L, R; D)$.

As the length of the orbit of L containing the vertex Rak_3 is four, the element $aba^2ba^2(3, 0, 0)$ should fix the vertex Rak_3, that is,

$$\begin{aligned} Rak_3 &= Rak_3(aba^2ba^2(3, 0, 0)) = Rak_3aba^2ba^2(3, 0, 0)(ak_3)^{-1}ak_3 \\ &= Ra^2ba^2ba(k_3^{aba^2ba^2}(3, 0, 0)k_3^{-1})^{a^{-1}}ak_3, \\ &= R[(-1, 1, 1) + (k_3^{aba^2ba^2}(3, 0, 0)k_3^{-1})^{a^{-1}}]ak_3, \end{aligned}$$

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forcing

$$(-1,1,1) + (k_3^{aba^2ba^2}(3,0,0)k_3^{-1})^{a^{-1}} = 0. \tag{3.5}$$

By (3.5), we get $k_3 = (x, x-1, -x)$ for some $x \in \mathbb{F}_5$.

Let $D' = Ra(0,-1,0)L$. Define the map

$$\delta:\ a \mapsto a(-x,-x,x), \quad b \mapsto b.$$

It is easy to check that δ gives an automorphism of $\widetilde{G}$ fixing R and L and maps D to D'. Then δ induces an isomorphism from $\mathbf{B}(\widetilde{G}, L, R; D)$ to $\mathbf{B}(\widetilde{G}, L, R; D')$. Therefore, we let $D = Ra(0,-1,0)L$.

Step 5. Show that $X' \cong X_{22}(5,5)$.

Since the neighbor of L corresponds to the bicoset $D = Ra(0,-1,0)L$, the vertex L is adjacent to

$$\{Ra(0,-1,0), Ra^2(3,-1,2), Ra^3(2,3,-1), Ra^4(0,1,2)\}.$$

Therefore, the neighbors of La, La^2, La^3 and La^4 are respectively

$$\begin{gathered}\{R(0,1,0), Ra^2(0,-1,3), Ra^3(3,0,1), Ra^4(2,2,-1)\},\\ \{R(2,1,3), Ra(0,1,2), Ra^3(0,-1,1), Ra^4(3,1,2)\},\\ \{R(3,2,1), Ra(2,0,-1), Ra^2(0,1,-1), Ra^4(0,-1,-1)\} \quad \text{and}\\ \{R(0,-1,3), Ra(3,3,1), Ra^2(2,-1,3), Ra^3(0,1,1)\}.\end{gathered}$$

Define a map $\eta\colon V(X') \to V(X_{22}(5,5))$ by the rule

$$\begin{aligned} \eta(Lk) &= (1,k), & \eta(Rk) &= (1',k),\\ \eta(Lak) &= (2,k), & \eta(Rak) &= (2',k),\\ \eta(La^2k) &= (3,k), & \eta(Ra^2k) &= (3',k),\\ \eta(La^3k) &= (4,k), & \eta(Ra^3k) &= (4',k) \quad \text{and}\\ \eta(La^4k) &= (5,k), & \eta(Ra^4k) &= (5',k), \end{aligned}$$

where $k \in K$. Then $X' \cong X_{22}(5,5)$ via η.

Step 6. The connectedness of $X_{22}(5,5)$.

Take three closed walks in Y:

$$W_1 = 1,2',3,4',1, \quad W_2 = 1,3',4,2',1, \quad W_3 = 1,4',2,3',1.$$

Then $f_{W_1} = (3,-1,0)$, $f_{W_2} = (0,-1,2)$ and $f_{W_3} = (1,3,-1)$. Thus, f_{W_1}, f_{W_2} and f_{W_3} can generate K, showing the connectedness of $X_{22}(5,5)$.

Finally, similarly to Lemma 3.2, σ exchanging every pair in Y lifts. □

3.3. soc(G) = PSL(2, q) with $q \geq 5$ and $n = 1 + q \geq 6$.

Lemma 3.4. *Suppose that* $\mathrm{PSL}(2, q) \leq G \leq \mathrm{P\Gamma L}(2, q)$, *where q is an odd prime power. Then the following hold.*

(1) $G = \mathrm{PGL}(2, p)$ *and* $A = G \times \langle \sigma \rangle$, *where σ is an involution exchanging two biparts.*

(2) $\widetilde{G} = K \rtimes T$, $\widetilde{A} = K \rtimes (T \times \langle \tau \rangle)$, *where τ is an involution which is a lift of σ and T is the image of one of the following faithful irreducible p-modular representations φ of degree three of* $\mathrm{PGL}(2, p)$, *up to equivalence, either:*

(i) $p \geq 5 : \varphi_1 : \overline{\begin{pmatrix} r & s \\ u & v \end{pmatrix}} \mapsto (rv - su)^{-1} \begin{pmatrix} r^2 & 2rs & 2s^2 \\ ru & rv + su & 2sv \\ u^2/2 & uv & v^2 \end{pmatrix}$; *or*

(ii) $p \geq 5 : \varphi_2 : \overline{g} \mapsto \det(g)^{(p-1)/2} \varphi_1(\overline{g}), \overline{g} \in \mathrm{PGL}(2, p)$.

Proof. (1) Let $\widetilde{G}$ be the lift of G, so that $\widetilde{G}/K = G$, where $\mathrm{PSL}(2, q) \leq G \leq \mathrm{P\Gamma L}(2, q)$. Since $C_{\widetilde{G}}(K)/K$ is normal in $\widetilde{G}/K$ and $\mathrm{soc}(\widetilde{G}/K) \cong \mathrm{PSL}(2, q)$, we deduce that $C_{\widetilde{G}}(K)/K = 1$ or $\mathrm{PSL}(2, q) \leq C_{\widetilde{G}}(K)/K$. If the latter case happens, then, with the same arguments as Lemma 3.1, one may get that there exist no covers occurring. So, $C_{\widetilde{G}}(K) = K$.

Since $\mathrm{PSL}(2, q) \leq \widetilde{G}/K = \widetilde{G}/C_{\widetilde{G}}(K) \leq \mathrm{Aut}(K) \cong \mathrm{GL}(3, p)$, it follows from Proposition 2.6 that $q = p$ and $G \cong \mathrm{PGL}(2, p)$. Moreover, $A = G \times \langle \sigma \rangle$, where σ is an involution exchanging two biparts.

(2) In what follows, we identify $V(Y)$ with two copies of the projective line $\mathrm{PG}(1, p)$. Let F be a fiber over ∞ and pick $\widetilde{u} \in F$. Then $\widetilde{A}_F = K \rtimes \widetilde{A}_{\widetilde{u}}$. Since K is an abelian normal subgroup of $\widetilde{A}$ and $(|K|, |\widetilde{A} : \widetilde{A}_F|) = (p^3, 2(1 + p)) = 1$, it follows from Proposition 2.5 that K has a complement in $\widetilde{A}$, which is of course isomorphic to $\mathrm{PGL}(2, p) \times \mathbb{Z}_2$. Therefore, we may set $\widetilde{G} = K \rtimes T$, where T is the image of one of faithful irreducible p-modular representations φ of degree three of $\mathrm{PGL}(2, p)$. Consequently, $\widetilde{A} = K \rtimes (T \times \langle \tau \rangle)$ for an involution τ which is a lift of σ.

By [1, Lemma 6.3], the map φ_1 in (2)(i) of the present lemma gives an irreducible p-modular representation of degree three of $\mathrm{PGL}(2, p)$. Clearly, φ_2 is another such representation, which is inequivalent to φ_1.

In view of Proposition 2.6, all the subgroups isomorphic to $\mathrm{PSL}(2, p)$ (respectively $\mathrm{PGL}(2, p)$) contained in $\mathrm{SL}(3, p)$ form a conjugacy class of $\mathrm{GL}(3, p)$, given by φ_1, noting that $\varphi_1(g) = \varphi_2(g)$ for any $g \in \mathrm{PSL}(2, p)$.

Let φ be any irreducible p-modular representation of degree three of T, where $\varphi(T)$ is not contained in $\mathrm{SL}(3, p)$. Then we show that φ is equivalent to φ_2.

Take an involution $\overline{b} \in \mathrm{PGL}(2, p) \setminus \mathrm{PSL}(2, p)$. Then $\varphi(\mathrm{PSL}(2, p)) \leq \mathrm{SL}(3, p)$ and $\varphi(\overline{b}) \in \mathrm{GL}(3, p) \setminus \mathrm{SL}(3, p)$. Now $\det(\varphi(\overline{b})) = -1$. Let $e = \| -1, -1, -1 \|$ be the central involution of $\mathrm{GL}(3, p)$. Then $e\varphi(\overline{b}) \leq \mathrm{SL}(3, p)$, so that $\langle \varphi(\mathrm{PSL}(2, p)), e\varphi(\overline{b}) \rangle \leq \mathrm{SL}(3, p)$. Since all the subgroups isomorphic to $\mathrm{PGL}(2, p)$ contained in $\mathrm{SL}(3, p)$ are conjugate in GL(3, p), there exists a $g \in \mathrm{GL}(3, p)$ such that $\langle \varphi(\mathrm{PSL}(2, p)), e\varphi(\overline{b})\rangle = \varphi_1(\mathrm{PGL}(2, p))^g$. Then

$$\varphi(\mathrm{PSL}(2, p) = \varphi_1(\mathrm{PSL}(2, p))^g \quad \text{and} \quad e\varphi(\overline{b}) = \varphi_1(\overline{x})^g,$$

for some involution $\overline{x} \in \mathrm{PGL}(2, p) \setminus \mathrm{PSL}(2, p)$, that is, $\varphi(\overline{b}) = e(\varphi_1(\overline{x}))^g = (-\varphi_1(\overline{x}))^g = \varphi_2(\overline{x})^g$. Now

$$\begin{aligned}\varphi(\mathrm{PGL}(2, p)) &= \langle \varphi(\mathrm{PSL}(2, p)), \varphi(b)\rangle \\ &= \langle \varphi_1(\mathrm{PSL}(2, p))^g, \varphi_2(\overline{x})^g\rangle \\ &= \langle \varphi_2(\mathrm{PSL}(2, p)), \varphi_2(\overline{x})\rangle^g \\ &= \varphi_2(\mathrm{PGL}(2, p))^g.\end{aligned}$$

Therefore, up to equivalence, φ_1 and φ_2 are all irreducible p-modular representations of degree three of PGL(2, p). □

For $m = 1, 2$, let

$$S = \varphi_m(\mathrm{PSL}(2, p)), \quad T_m = \varphi_m(\mathrm{PGL}(2, p)), \quad \widetilde{G}_m = K \rtimes T_m \quad \text{and} \quad \widetilde{A}_m = \widetilde{G}_m \rtimes \langle \tau \rangle,$$

where the operation between K and τ is yet to be determined. Then both $\widetilde{G}_1$ and $\widetilde{G}_2$ are subgroups of AGL(3, p) and $\widetilde{G}_1 \cap \widetilde{G}_2 = K \rtimes S$. Again, $K = V^+(3, p)$. However, we adopt a multiplication notation for K when considering K as a subgroup of $\widetilde{G}_i$.

In PGL(2, p), set

$$t_1 = \overline{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}}, \quad a_1 = \overline{\begin{pmatrix} \theta & 0 \\ 0 & 1 \end{pmatrix}}, \quad g_1 = \overline{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}},$$

where $\mathbb{F}_p^* = \langle \theta \rangle$, and set $H_1 = \langle t_1, a_1\rangle$.

Let $\mathrm{PG}(1, p) = \{\infty, 0, 1, \ldots, p-1\}$ be the projective line over $\mathbb{F}_p$, where we identify $\langle (0, 1)\rangle$ and $\langle (1, \ell)\rangle$ with ∞ and ℓ, respectively. Then H_1 fixes $\infty \in \mathrm{PG}(1, p)$ and t_1^i maps ℓ into $\ell + i$. Furthermore, let $\varphi = \varphi_m$, where $m = 1, 2$, and set $t = \varphi(t_1)$, $a = \varphi(a_1)$ and $g = \varphi(g_1)$. Then, for any i,

$$t^i = \varphi(t_1^i) = \begin{pmatrix} 1 & 2i & 2i^2 \\ 0 & 1 & 2i \\ 0 & 0 & 1 \end{pmatrix}, \quad a^i = \varphi(a_1^i) = (-1)^{(m-1)i} \begin{pmatrix} \theta^i & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \theta^{-i} \end{pmatrix},$$

$$g = \phi(g_1) = \begin{pmatrix} 0 & 0 & 2 \\ 0 & -1 & 0 \\ 1/2 & 0 & 0 \end{pmatrix}.$$

Then we have the following lemma.

LEMMA 3.5. *With the above notation, X is isomorphic to either:*

(i) $p \geq 5$, *$Cos(\widetilde{A}_1; \langle t, a\rangle, \langle t, a\rangle gk\tau\langle t, a\rangle)$, where $k = (0, 1, 0)$ and $[\tau, K] = 1$; or*

(ii) $p = 5$, *$Cos(\widetilde{A}_2; \langle k't, a\rangle, \langle t, a\rangle g\tau\langle k't, a\rangle)$, where $k' = (1, -1, -1)$ and $k^\tau = k^{-1}$ for any $k \in K$.*

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PROOF. First note that our graph X is isomorphic to a coset graph arising from $\widetilde{A} = \widetilde{A}_m$, where $m = 1, 2$. Set $T = T_m$ and $H = \varphi(H_1) = \langle t, a\rangle$. Set $M := \widetilde{G}_F = K \rtimes H$, where $F = f^{-1}(\infty)$. Take $\widetilde{u} \in F$.

Step 1. Determination of $\widetilde{A}_{\widetilde{u}} = \widetilde{G}_{\widetilde{u}}$.

Clearly, $\widetilde{G}_{\widetilde{u}} \leq M$. Note that $|M| = |K \rtimes H| = |(K \rtimes \langle t\rangle) \rtimes \langle a\rangle| = p^4(p-1)$. Let $P = K \rtimes \langle t\rangle$. Then P is a p-group of order p^4. Since $p \geq 5$ by assumption, P is a regular p-group (for the definition of regular p-groups, see [20, Kapitel III, Definitionen 10.2]). Since $\Phi(P) \leq K$ and the order of t is p, P has exponent p. Clearly, M has only one conjugacy class of subgroups isomorphic to $\langle a\rangle$. Assume that L is a subgroup of M such that $\langle a\rangle \leq L \cong H$ and $L \cap K = 1$. Then we may assume that $L = \langle kt\rangle \rtimes \langle a\rangle$ for some $k = (x, y, z) \in K$. Suppose that $(kt)^a = (kt)^i$. Then

$$(kt)^a = k^a t^a = (-1)^{m-1}(\theta x, y, \theta^{-1}z)t^{\theta^{-1}}$$

and

$$\begin{aligned}(kt)^i &= (kk^{t^{-1}}k^{t^{-2}}\cdots k^{t^{-i+1}})t^i \\ &= ((x, y, z) + (x, -2x + y, 2x - 2y + z) + \cdots \\ &\quad + (x, -2(i-1)x + y, 2(i-1)^2x - 2(i-1)y + z))t^i \\ &= \Big(ix, -(i-1)ix + iy, \frac{(i-1)i(2i-1)}{3}x - (i-1)iy + iz\Big)t^i.\end{aligned}$$

Thus, we get $i = \theta^{-1}$ and

$$(-1)^{m-1}(\theta x, y, \theta^{-1}z) = \Big(ix, -(i-1)ix + iy, \frac{(i-1)i(2i-1)}{3}x - (i-1)iy + iz\Big). \quad (3.6)$$

(1) First, suppose that $m = 1$. From (3.6), we have $\theta x = ix = \theta^{-1}x$ and so $\theta^2 x = x$. Since $p \geq 5$, we get $\theta^2 \neq 1$ and so $x = 0$ and $y = 0$ by the second equation again. Hence, $k = (0, 0, z)$ for any $z \in \mathbb{F}_p$, which means that k has p possibilities. For each k, we get an $L = \langle kt\rangle \rtimes \langle a\rangle$; in particular, $L = H$ when $z = 0$. Furthermore, these p subgroups are conjugate in M. In fact, for any $k = (0, 0, z)$, by taking $k' = (0, z/2, 0)$,

$$\begin{aligned}(kt)^{k'} &= k(k')^{-1}tk' = k(k')^{-1}(k')^{t^{-1}}t \\ &= \Big((0,0,z) - \Big(0, \frac{z}{2}, 0\Big) + \Big(0, \frac{z}{2}, -z\Big)\Big)t = (0,0,0)t = t\end{aligned}$$

and

$$a^{k'} = k'^{-1}ak' = k'^{-1}(k')^{a^{-1}}a = \Big(\Big(0, -\frac{z}{2}, 0\Big) + \Big(0, \frac{z}{2}0\Big)\Big)a = a,$$

which forces $L^{k'} = H$. Therefore, we choose $\widetilde{G}_{\widetilde{u}} = H$.

(2) Now suppose that $m = 2$. From (3.6), we get $-\theta x = ix = \theta^{-1}x$ and so $x(\theta^2 + 1) = 0$.

If $x = 0$, then from (3.6) it can be easily deduced that $y = z = 0$.

Suppose that $x \neq 0$. Then $\theta^2 = -1$, that is, $p = 5$. Solving (3.6) again, we get $k' = (y, -y, -y)$. Therefore, we choose $\widetilde{G}_{\widetilde{u}} = \langle k't, a\rangle$.

Step 2. Determination of the coset graphs.

Set $L := \widetilde{A}_{\widetilde{u}}$. Assume that $X' \cong Cos(\widetilde{G}; L, D)$, where the neighbor of L corresponds to a bicoset $D = Lgk_1\tau L$ for some $k_1 = (x_1, y_1, z_1) \in K$. Then the following conditions should be satisfied:

(1) $d(X') = p$.

As the length of the orbit of L containing the vertex $Lgk_1\tau$ is p, the element a should fix the vertex $Lgk_1\tau$, that is, $Lgk_1\tau a = Lgk_1\tau$. Then, noting that $g^2 = 1$ and $[\tau, T] = 1$,

$$L = Lgk_1ak_1^{-1}g = La^g(k_1^ak_1^{-1})^g = La^{-1}(k_1^ak_1^{-1})^g = L(k_1^ak_1^{-1})^g.$$

Therefore, $(k_1^ak_1^{-1})^g \in K \cap L = 1$, that is, $k_1^ak_1^{-1} = 0$. Then

$$k_1^ak_1^{-1} = (((-1)^{m-1}\theta - 1)x_1, ((-1)^{m-1} - 1)y_1, ((-1)^{m-1}\theta^{-1} - 1)z_1) = 0.$$

Therefore, if $m = 1$, then $k_1 = (0, y, 0)$ for some $y \in \mathbb{F}_p^*$; and, if $m = 2$, then $k_1 = 0$.

In summary, we get $X' = Cos(\widetilde{A}; L, D)$, where

$$m = 1, \quad L = \langle t, a\rangle \quad \text{and} \quad D = Lg(0, y, 0)\tau L, \quad \text{where } y \in \mathbb{F}_p^*, \text{ and}$$
$$m = 2, \quad L = \langle (y, -y, -y)t, a\rangle \quad \text{and} \quad D = Lg\tau L, \quad \text{where } y \in \mathbb{F}_p^*.$$

Suppose that $[\tau, K] \neq 1$. Then τ can be viewed as an involution of $\mathrm{GL}(3, p)$. By Lemma 3.4, we have $C_{\mathrm{GL}(3,p)}(\mathrm{PGL}(2, p)) = Z(\mathrm{GL}(3, p))$. In view of $[\tau, T] = 1$, we get that τ is the central involution of $\mathrm{GL}(3, p)$ and, in particular, $k^\tau = k^{-1}$ for any $k \in K$. For any $y \in \mathbb{F}_p^*$, define a map $\lambda(y)$ on $\widetilde{A}$ by

$$\lambda(y)(k) = yk, \quad \lambda(y)(d) = d, \quad \lambda(y)(\tau) = \tau,$$

where $k \in K$ and $d \in T$. Clearly, $\lambda(y)$ can be extended to an automorphism of $\widetilde{A}$ and, moreover, $\lambda(y^{-1})$ fixes L and moves $L(0, y, 0)L$ to $L(0, 1, 0)L$ for $m = 1$ and moves $L = \langle (y, -y, -y)t, a\rangle$ to $L = \langle (1, -1, -1)t, a\rangle$. Therefore, L and D can be chosen as follows:

$$m = 1, \quad L = \langle t, a\rangle, \quad D = Lg(0, 1, 0)\tau L; \quad m = 2, L = \langle (1, -1, -1)t, a\rangle, \quad D = Lg\tau L.$$

(2) *Undirected property.*

For $m = 2$, we have $D = Lg\tau L$, where $g\tau$ is an involution and so $D = D^{-1}$.

Let $m = 1$. First, suppose that $[\tau, K] = 1$. Note that $D = Lgk\tau L$, where $L = \langle t, a\rangle$ and $k = (0, 1, 0)$. Then $(gk\tau)^2 = gkgk = k^{-1}k = 1$ and so $D^{-1} = D$.

Next, suppose that $[\tau, K] \neq 1$. Then $\tau = e$, as stated before. Assume that $D^{-1} = D$. Then there exist $h_1, h_2 \in H$ such that $(gk\tau)^{-1} = h_1gk\tau h_2$, that is,

$$kg = h_1gkh_2 = h_1k^{-1}gh_2 = (k^{h_1^{-1}})^{-1}h_1gh_2,$$

which forces $k = (k^{h_1^{-1}})^{-1}$. However, for any $h_1^{-1} = t^ia^j$, we have $(k^{h_1^{-1}})^{-1} = (0, -1, -2i\theta^j) \neq k$. Therefore, $[\tau, K] = 1$ and so τ is a central involution of $\widetilde{A}$.

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(3) *Connectedness property.*

(i) $m = 1$:

It has been shown in (2) that $[K, \tau] = 1$. Now X is connected if and only if

$$\langle D\rangle = \langle L, gk\tau\rangle = \langle t, a, gk\tau\rangle = \widetilde{A}.$$

By computation, we get the following equations:

$$\begin{aligned} t^{gk\tau} &= t^g(1,0,0), \quad t^g(1,0,0)tt^g(1,0,0) = g(1,0,2),\\ t^{g(1,0,2)} &= t^g(-1,2,0), \quad (t^g(-1,2,0))^{-1}t^g(1,0,0) = (2,-2,0).\end{aligned}$$

Thus, $(2,-2,0) \in \langle D\rangle$. Furthermore, we have $(2,-2,0)^t = (2,2,0) \in \langle D\rangle$ and $(2,-2,0)^{gk\tau} = (0,2,4) \in \langle D\rangle$. Hence, $K \le \langle D\rangle$, so that $\langle D\rangle = \widetilde{A}$, as desired.

(ii) $m = 2$:

Note that in this case $p = 5$ and

$$\langle D\rangle = \langle L, g\tau\rangle = \langle (1,-1,-1)t, a, g\tau\rangle.$$

First, suppose that $[K, \tau] = 1$. By computation, we get the following equations:

$$((1,-1,-1)t)^{g\tau} = t^g(2,-1,2), \quad t^g(2,-1,2)(1\ ,-1,-1)tt^g(2,-1,2) = g,$$
$$((1,-1,-1)tg)^3 = (0,0,0).$$

Thus, $\langle (1,-1,-1)t, g\rangle \cong \mathrm{PSL}(2,5)$. Moreover,

$$((1,-1,-1)t)^3 g((1,-1,-1)t)^2 g((1,-1,-1)t)^3 g = a^2,$$

which means that $\langle (1,-1,-1)t, g, a\rangle \cong \mathrm{PGL}(2,5)$. Therefore,

$$\langle D\rangle = \langle (1,-1,-1)t, g, a\rangle \times \langle \tau\rangle \cong \mathrm{PGL}(2,5) \times \mathbb{Z}_2 < \widetilde{A},$$

so that X' is disconnected in this case.

Next, suppose that $k^\tau = k^{-1}$ for any $k \in K$. Then

$$\begin{aligned} ((1,-1,-1)t)^{g\tau} &= t^g(3,1,-2), \quad t^g(3,1,-2)(1,-1,-1)tt^g(3,1,-2) = g(-1,-1,-2),\\ ((1,-1,-1)t)^{g(-1,-1,-2)} &= t^g(2,2,2), \quad (t^g(2,2,2))^{-1}t^g(3,1,-2) = (1,-1,1).\end{aligned}$$

Therefore, $K \le \langle D\rangle$, so that $\langle D\rangle = \widetilde{A}$, proving the connectedness. □

Lemma 3.6. *The following hold:*

(i) $p \ge 5$, $Cos(\widetilde{A}_1; \langle t, a\rangle, \langle t, a\rangle gk\tau\langle t, a\rangle) \cong X_{31}(p+1, p)$, *where* $k = (0,1,0)$ *and* $[\tau, K] = 1$;

(ii) $p = 5$, $Cos(\widetilde{A}_2; \langle k't, a\rangle, \langle t, a\rangle g\tau\langle k't, a\rangle) \cong X_{32}(6,5)$, *where* $k' = (1,-1,-1)$ *and* $k^\tau = k^{-1}$ *for any* $k \in K$.

Proof. We discuss the two covers separately.

Step 1. Show that $Cos(\widetilde{A}_1; \langle t, a\rangle, \langle t, a\rangle gk\tau\langle t, a\rangle) \cong X_{31}(p+1, p)$.

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Note that L is adjacent to $Lgt^j(0, 1, 2j)\tau$ with $j \in \mathbb{F}_p$. Moreover, for any $k \in K$,

$$\{Lgt^i, Lgt^j k\tau\} \in E(X') \quad \text{if and only if } \{L, Lgt^j kt^{-i} g\tau\} \in E(X').$$

By computation,

$$Lgt^j kt^{-i} g\tau = Lgt^{j-i} gk^{t^{-i}g}\tau = Lgt^{(i-j)^{-1}} k^{t^{-i}g}\tau.$$

Therefore, $k^{t^{-i}g} = (0, 1, 2/(i-j))$, that is,

$$k = \left(0, 1, \frac{2}{i-j}\right)^{gt^i} = \left(\frac{1}{i-j}, \frac{i+j}{i-j}, \frac{2ij}{i-j}\right).$$

Hence, Lgt^i is adjacent to $Lgt^j(1/(i-j), (i+j)/(i-j), 2ij/(i-j))\tau$.

Set $X' := Cos(\widetilde{A}_1; \langle t, a\rangle, \langle t, a\rangle gk\tau\langle t, a\rangle)$. Define a map $\phi_1\colon V(X') \to V(X_{31}(p+1, p))$ by the rule

$$\phi_1(Lk) = (\infty, k), \quad \phi_1(Lgt^i k) = (i, k),$$
$$\phi_1(Lk\tau) = (\infty', k), \quad \phi_1(Lgt^i k\tau) = (i', k),$$

for any $k \in K$. Clearly, ϕ_1 is an isomorphism from X' to $X_{31}(p+1, p)$.

Step 2. Show that $Cos(\widetilde{A}_2; \langle k't, a\rangle, \langle t, a\rangle g\tau\langle k't, a\rangle) \cong X_{32}(6, 5)$.

Note that L is adjacent to $Lgt^j(-j, -j^2, j^3)\tau$ with $j \in \mathbb{F}_p$. Moreover, for any $k \in K$,

$$\{Lgt^i, Lgt^j k\tau\} \in E(X') \quad \text{if and only if } \{L, Lgt^j kt^{-i} g\tau\} \in E(X').$$

By computation,

$$Lgt^j kt^{-i} g\tau = Lgt^{j-i} gk^{t^{-i}g}\tau = Lgt^{(i-j)^{-1}}(j-i, -(j-i)^2, -(j-i)^3)^{gt^{(i-j)^{-1}}} k^{t^{-i}g}\tau.$$

Therefore,

$$(j-i, -(j-i)^2, -(j-i)^3)^{gt^{(i-j)^{-1}}} k^{t^{-i}g} = (-(i-j)^{-1}, -(i-j)^{-2}, (i-j)^{-3}),$$

that is,

$$\begin{aligned} k = (&3(j-i) - 3(j-i)^{-3}, i(j-i) - i(j-i)^{-3} + 2(j-i)^2 + (j-i)^{-2},\\ &i^2(j-i) - i^2(j-i)^{-3} - i(j-i)^2 + 2i(j-i)^{-2} + (j-i)^3 + 2(j-i)^{-1}). \end{aligned}$$

Since $i \neq j$ and $i, j \in \mathbb{F}_5$, it follows that $(i-j)^2 = \pm 1$. Then

$$k = (0, \pm 2, \pm 2(i+j)) \quad \text{for } (i-j)^2 = \mp 1.$$

Set $X' := Cos(\widetilde{A}_2; \langle k't, a\rangle, \langle t, a\rangle g\tau\langle k't, a\rangle)$. Define a map $\phi_2\colon V(X') \to V(X_{32}(6, 5))$ by the rule

$$\phi_2(Lk) = (\infty, k), \quad \phi_2(Lgt^i k) = (i, k),$$
$$\phi_2(Lk\tau) = (\infty', k), \quad \phi_2(Lgt^i k\tau) = (i', k),$$

for any $k \in K$. Obviously, ϕ_2 is an isomorphism from the graph X' to $X_{32}(6, 5)$. □

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3.4. G is of affine type. In this subsection, we assume that either $G \cong \mathrm{AGL}(m,2) \cong \mathbb{Z}_2^m \rtimes \mathrm{GL}(m,2)$ with $m \geq 3$ or $G \cong \mathbb{Z}_2^4 \rtimes A_7$. With the same notation as before, $\widetilde{A}/K = A = G \times \langle\sigma\rangle$, where σ is the involution exchanging every pair i and i'. By Propositions 2.3 and 2.6, we get either $C_{\widetilde{G}}(K) = \widetilde{G}$ or $C_{\widetilde{G}}(K)/K \cong \mathbb{Z}_2^m$ with $m \geq 3$.

When $C_{\widetilde{G}}(K) = \widetilde{G}$, the same discussion as Lemma 3.1 shows that there exist no connected covers occurring.

When $C_{\widetilde{G}}(K)/K \cong \mathbb{Z}_2^m$, by checking Proposition 2.6, we get $m = 3$ and either $p = 7$ or $p^3 \equiv 1 \bmod 7$. Thus, $Y = K_{8,8} - 8K_2$ and $\widetilde{A}/K \cong \mathrm{AGL}(3,2) \times \mathbb{Z}_2 \cong (\mathbb{Z}_2^3 \rtimes \mathrm{GL}(3,2)) \times \mathbb{Z}_2$. In what follows, the cases either $p = 7$ or p is an odd prime and $p^3 \equiv 1 \bmod 7$ will be dealt with in Lemma 3.7 and the case $p = 2$ will be dealt with in Lemma 3.8.

Lemma 3.7. *There exist no covers when either $p = 7$ or p is an odd prime and $p^3 \equiv 1 \bmod 7$.*

Proof. Let F be a fiber. Since $(|\widetilde{A} : \widetilde{A}_F|, |K|) = (16, p^3) = 1$ for both cases, it follows that K has a complement in $\widetilde{A}$. Thus, we may set

$$\widetilde{G} = K \rtimes (L \rtimes T), \quad \widetilde{A} = K \rtimes ((L \rtimes T) \times \langle\tau\rangle),$$

where $L \cong \mathbb{Z}_2^3$, $[K, L] = 1$, $T \cong \mathrm{GL}(3,2) \cong \mathrm{PSL}(2,7)$ and τ is an involution, which is a lift of σ.

Take $\widetilde{u} \in F := f^{-1}(0)$, where 0 is the zero vector of L. Set $H := \widetilde{G}_{\widetilde{u}} = \widetilde{A}_{\widetilde{u}} \leq K \rtimes T$. So, X is isomorphic to a coset graph $X' := X(\widetilde{A}; H, D)$, where $D = H\tau\ell k_1 H$ for some $\ell \in L \setminus \{0\}$ and $k_1 \in K$. Therefore, D corresponds to a suborbit of $\widetilde{A}$ of length seven relative to H.

Suppose that the representations of $\widetilde{G}$ on the two biparts are equivalent. Then there exists an $\widetilde{u}'$ in the other bipart such that $\widetilde{G}_{\widetilde{u}} = \widetilde{G}_{\widetilde{u}'} = H \cong \mathrm{PSL}(2,7)$. Then $|H\ell k_1 H|/|H| = 7$ for some nontrivial elements ℓ and k_1, that is, $|(\ell k_1)^H| = 7$. This forces $H \cong \mathrm{PSL}(2,7)$ having an orbit of length seven in its conjugacy action on K. However, this is impossible by Proposition 2.7.

From now on, suppose that the two representations of $\widetilde{G}$ on the two biparts are inequivalent. In particular, $[K, \tau] \neq 1$. Suppose that $p^3 \equiv 1 \bmod 7$. Then there is only one conjugacy class of $\mathrm{PSL}(2,7)$ in KT. In this case, two representations of $\widetilde{G}$ on two biparts are equivalent. Therefore, we let $p = 7$.

By Proposition 2.6, $\mathrm{GL}(3,7)$ has only one conjugacy class of subgroups isomorphic to $\mathrm{PSL}(2,7)$. So, we may fix a matrix representation ϕ of T in $\mathrm{GL}(3,7)$ as follows:

$$a_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mapsto a = \begin{pmatrix} 0 & 0 & 2 \\ 0 & -1 & 0 \\ 4 & 0 & 0 \end{pmatrix},$$

$$b_1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \mapsto b = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 4 & 1 & 1 \end{pmatrix},$$

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where $\langle a_1, b_1\rangle = \mathrm{SL}(2,7)$ and $\phi(e) = 1$ for the center involution e. Then $C_{\mathrm{GL}(3,7)}(\mathrm{PSL}(2,7)) = Z(\mathrm{GL}(3,7))$. Since $[\tau, T] = 1$, the element τ is the center involution of $\mathrm{GL}(3,p)$, which implies that $k^\tau = k^{-1}$ for any $k \in K$.

Acting on $V(3,2)$, we have $G_0 = \langle a, b\rangle$ and H is the lift of G_0. Then we turn to the group H. Since $H \le K \rtimes T$ and since there is only one conjugacy class of involutions in $K \rtimes T$, we may assume that $H = \langle a, bk_2\rangle$ for some $k_2 = (x_2, y_2, z_2) \in K$. As $H \cong \mathrm{PSL}(2,7)$, the generators of H should satisfy

$$a^2 = 1,\ (bk_2)^7 = 1,\ (abk_2)^3 = 1,\ ((bk_2)^4a)^4 = 1. \tag{3.7}$$

From the last two equations of (3.7),

$$2x_2 - y_2 + z_2 = 0, \quad 2x_2 - y_2 + 2z_2 = 0,$$

forcing $y_2 = 2x_2$ and $z_2 = 0$. Thus, $H = \langle a, bk_2\rangle$, where $k_2 = (x_2, 2x_2, 0)$.

Since the length of the orbit of H containing the vertex $H\tau\ell k_1$ is seven, every involution in H should fix a point in the orbit and every Sylow 7-subgroup of H should be transitive on the orbit. Taking this into account, we get the following.

(1) $H\tau\ell k_1 a = H\tau\ell k_1$, which forces $\ell^a = \ell$ and $k_1^a = k_1$, where $k_1 = (x_1, y_1, z_1) \in K$. From

$$(4z_1, -y_1, 2x_1) = k_1^a = k_1 = (x_1, y_1, z_1),$$

we get $y_1 = 0$ and $z_1 = 2x_1$. Hence, $k_1 = (x_1, 0, 2x_1)$.

(2) The $\langle bk_2\rangle$-orbit containing $H\tau\ell k_1$ is

$$\triangle := \{H\tau\ell k_1,\ H\tau\ell^{b^i} k_1^{b^i} (k_2^2)^{\sum_{j=0}^{i-1} b^j} : 1 \le i \le 6\},$$

that is,

$$\begin{gathered} H\tau\ell k_1, \quad H\tau\ell^b(2x_1 + 2x_2, 2x_1 + 4x_2, 2x_1), \quad H\tau\ell^{b^2}(5x_1 + x_2, 4x_1 + x_2, 2x_1), \\ H\tau\ell^{b^3}(3x_1 + 4x_2, -x_1 - 2x_2, 2x_1), \quad H\tau\ell^{b^4}(3x_1 + 4x_2, x_1 + 2x_2, 2x_1), \\ H\tau\ell^{b^5}(-2x_1 + x_2, 3x_1 - x_2, 2x_1), \quad H\tau\ell^{b^6}(2x_1 + 2x_2, -2x_1 + 3x_2, 2x_1). \end{gathered}$$

(3) The images of a acting on those points are

$$\begin{gathered} H\tau\ell k_1, \quad H\tau\ell^{ba}(x_1, -2x_1 - 4x_2, 4x_1 + 4x_2), \quad H\tau\ell^{b^2a}(x_1, -4x_1 - x_2, 3x_1 + 2x_2), \\ H\tau\ell^{b^3a}(x_1, x_1 + 2x_2, -x_1 + x_2), \quad H\tau\ell^{b^4a}(x_1, -x_1 - 2x_2, -x_1 + x_2), \\ H\tau\ell^{b^5a}(x_1, -3x_1 + x_2, 3x_1 + 2x_2), \quad H\tau\ell^{b^6a}(x_1, 2x_1 - 3x_2, 4x_1 + 4x_2). \end{gathered}$$

Since a preserves the set $\triangle$ setwise, by comparing (2) and (3), one may get that $x_1 = -2x_2$. Thus, $k_1 = (-2x_2, 0, -4x_2)$ and $k_2 = (x_2, 2x_2, 0)$. Moreover, $a^{\tau k_1} = a$ and

$$(bk_2)^{\tau k_1} = k_1^{-1}bk_2^{-1}k_1 = b((k_1^{-1})^b k_2^{-1}k_1 = b((4,4,4) + (-1,-2,0) + (-2,0,-4))) = bk_2.$$

Therefore, $[\tau k_1, H] = 1$. Finally,

$$\langle D\rangle = \langle a, bk_2, \ell\tau k_1\rangle \le \langle a, bk_2, L, \tau k_1\rangle = (L \rtimes \langle a, bk_2\rangle) \times \langle \tau k_1\rangle < \widetilde{A},$$

contradicting the connectedness of X. □

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LEMMA 3.8. *If $p = 2$, then $X \cong X_4(8, 2)$.*

PROOF. Let $C = C_{\widetilde{G}}(K)$. Then C acts regularly on $V(X)$ and $C/K \cong \mathbb{Z}_2^3$. Now C is an extension of K by $\mathbb{Z}_2^3$ and so it has exponent either 2 or 4. Let $T = \widetilde{G}_{\widetilde{v}}$ for some $\widetilde{v} \in V(X)$. Then $T \cong \mathrm{GL}(3, 2) \cong \mathrm{PSL}(2, 7)$ and $\widetilde{G} = C \rtimes T$. Since C/K is elementary abelian, we get $\Phi(C) \leq K$. Since T normalizes C, it normalizes $\Phi(C)$. On the other hand, since T acts on K nontrivially and T is simple, K is a minimal normal subgroup in $\widetilde{G}$. It follows that $\Phi(C)$ is trivial or K. Thus, C is isomorphic to either $\mathbb{Z}_2^6$ or a 2-group generated by three elements of order four. Suppose that the latter case happens, that is, $\Phi(C) = K$. A direct checking from a classification of groups of order 2^6 (see [19]) shows that C cannot be nonabelian. Therefore, it should be $C \cong \mathbb{Z}_4^3$ or $\mathbb{Z}_2^6$.

Recall our conditions

$$\widetilde{A} = \widetilde{G}\langle\tau\rangle = (C_{\widetilde{G}}(K) \rtimes T)\langle\tau\rangle,$$

where $\tau^2 \in K$, $T = \widetilde{G}_{\widetilde{u}} \cong \mathrm{PSL}(2, 7)$ for some vertex $\widetilde{u} \in V(X)$, $\widetilde{G} = C \rtimes T$, $C = C_{\widetilde{G}}(K)$ and τ is a lift of σ. Then we prove the lemma by the following five steps.

(1) Show that $[K, \tau] = 1$. Consider the group $M = \langle K, T, \tau\rangle$. Suppose that $[K, \tau] \neq 1$. Then $\mathrm{PSL}(2, 7) \times \mathbb{Z}_2 \cong M/K = M/C_M(K) \leq \mathrm{GL}(3, 2)$, which is a contradiction.

(2) Show that $\tau^2 = 1$. Since $[\sigma, G] = 1$, for any $t \in T$, we may set $t^\tau = tk$ for some $k \in K$. Then $t^{\tau^2} = (tk)^\tau = tk^2 = t$, which means that $[\tau^2, T] = 1$. Since $\tau^2 \in K$ and T has no fixed nonzero elements in K, we get $\tau^2 = 1$.

(3) Show that $C \cong \mathbb{Z}_2^6$. To the contrary, suppose that $C \cong \mathbb{Z}_4^3$. Then T can be identified with a subgroup of $\mathrm{Aut}(C)$. By using Magma [2], we may compute that $\mathrm{Aut}(C)$ has only one conjugacy class of subgroups isomorphic to $\mathrm{GL}(3, 2)$. Therefore, we may fix a matrix representation of T in $\mathrm{Aut}(C)$. Pick two elements in $\mathrm{Aut}(C)$:

$$a = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} -1 & -1 & 2 \\ -1 & 1 & 1 \\ 2 & 1 & 2 \end{pmatrix}.$$

Then $T := \langle a, b\rangle \cong \mathrm{GL}(3, 2)$. Note that we are working in the ring $\mathbb{Z}_4$.

Suppose that $a^\tau = ak_1$ and $b^\tau = bk_2$, where $k_1 = (x_1, y_1, z_1)$, $k_2 = (x_2, y_2, z_2) \in K$. Since ak_1 and bk_2 should satisfy the defining relations of $\mathrm{GL}(3, 2)$,

$$(ak_1)^2 = k_1^a k_1 = 1, \quad ((ab)^\tau)^3 = (ak_1bk_2)^3 = (abk_1^b k_2)^3 = (k_1^b k_2)^{I+ab+(ab)^2} = 1,$$

which implies that $z_1 = 0$ and $x_1 + x_2 + z_2 = 0$.

Assume that $X \cong Cos(\widetilde{A}; T, D)$, where $D = T\tau\ell T$ for some $\ell = (x, y, z) \in C \setminus K$. It follows that T has an orbit of length seven in its conjugacy action on $C \setminus K$, where the involution a should fix a point in this orbit and $\langle b\rangle$ acts transitively on it.

Without loss of generality, suppose that $T\tau\ell = T\tau\ell a$, which is equivalent to $T\tau\ell = T\tau\ell^a k_1$. Therefore, $\ell^a = \ell k_1$, that is,

$$z = 2x + x_1, \quad 2z = 0, \quad 2y = y_1. \tag{3.8}$$

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By (3.8), the other six points in the $\langle b\rangle$-orbit Δ including $T\tau\ell$ are

$$\begin{aligned}
T\tau\ell b &= T\tau(-x-y+x_2, -x+y+z+y_2, 2x+y+z_2),\\
T\tau\ell b^2 &= T\tau(2x+2y+z+y_2, 2x-y+z+x_2+z_2, x+y+z+y_2+z_2),\\
T\tau\ell b^3 &= T\tau(2x+y+y_2+z_2, x+2y+z+y_2+x_2, y+z+x_2),\\
T\tau\ell b^4 &= T\tau(x-y+z+z_2, -x+2y+y_2+z_2, x+2y-z+x_2+y_2+z_2),\\
T\tau\ell b^5 &= T\tau(2x-y-z+x_2+y_2, -x+y+x_2+y_2+z_2, -x+y_2) \quad \text{and}\\
T\tau\ell b^6 &= T\tau(x+z+x_2+z_2, 2y-z+z_2, x-y+x_2+y_2).
\end{aligned} \tag{3.9}$$

As a fixes $\triangle$ setwise, by (3.8) and the equation $x_1+x_2+z_2=0$,

$$T\tau\ell ba = T\tau\ell^{ba}k_1k_2^a = T\tau(-x+2y, x+y-z+y_2, 2x+y+z_2) \in \triangle. \tag{3.10}$$

Comparing (3.9) and (3.10), one may get $\ell \in K$, which is a contradiction.

(4) Show that $[\tau, C]=1$. Since C is regular on both $\widetilde{U}$ and $\widetilde{U}'$ and $C \rtimes \langle\tau\rangle$ acts regularly on $V(X)$, we may identify $\widetilde{U}$ with C and $\widetilde{U}'$ with $C\tau$. Suppose that $X_1(1)=\{\tau c_i \mid c_i \in C, 1 \le i \le 7\}$, the neighborhood of 1 with size seven. Then, for any $1 \le i \le 7$, τc_i is adjacent to $\tau c_i \tau c_i = c_i^\tau c_i \in K$, as $[\overline{\tau}, \overline{c_i}] = \overline{1}$ in G. Since each τc_i is adjacent to just one vertex in the fiber K, that is, $\{1\}$, we have $c_i^\tau c_i = 1$, that is, $c_i^\tau = c_i$. From the connectedness of X, we get that C can be generated by c_i with $1 \le i \le 7$ and thus $[C, \tau]=1$.

(5) Show that $X \cong X_4(8,2)$.

Since $\widetilde{G} = C \rtimes T \cong \mathbb{Z}_2^6 \rtimes \mathrm{GL}(3,2)$, T has an isomorphism to $\mathrm{GL}(6,2)$. To describe these isomorphisms, let $\Omega = \mathrm{PG}(2,2)$ be the two-dimensional projective space over the field $\mathbb{F}_2$, while we identify Ω with $V(3,2) \setminus \{0\}$. Let χ_Δ denote the characteristic function of Δ, that is, $\chi_\Delta(i)=1$ for $i \in \Delta$ and $\chi_\Delta(i)=0$ for $i \notin \Delta$. Then the set $V=V(\Omega)$ of all characteristic functions χ_Δ, where $\Delta \in P(\Omega)$, forms a seven-dimensional vector space over $\mathbb{F}_2$ with the rule: $(a\chi_\Delta + b\chi_\Gamma)(i) = a\chi_\Delta(i) + b\chi_\Gamma(i)$ for any $a, b \in \mathbb{F}_2$ and $\chi_\Delta, \chi_\Gamma \in V(\Omega)$. Clearly, a natural basis for $V(\Omega)$ is the set of characteristic functions $\chi_{\{i\}}$ for all $i \in \Omega$. Moreover, V can be defined as a T-module, called a *permutation module*, where the action of $g \in T$ is defined by $(\chi^g)(i) = \chi(i^{g^{-1}})$ for all $i \in \Omega$ (see [27]).

For $i=0,1,2$, let V_i be the subspace of $V(\Omega)$ generated by the characteristic functions of all i-dimensional subspaces of $\mathrm{PG}(2,2)$. Then $V_0 = V(\Omega)$, $V_2 = I$, where $I = \langle \sum_{i\in\Omega} \chi_{\{i\}} \rangle$, and V_i is a T-submodule. Choose a basis $\{\alpha_1, \alpha_2, \alpha_3\}$ for $V(3,2)$. Then $\{\chi_{\{\alpha_i\}} + V_1 \mid 1 \le i \le 3\}$ (respectively $\{\chi_{\{\alpha_i, \alpha_j, \alpha_i+\alpha_j\}} + V_2 \mid i \ne j, 1 \le i, j \le 3\}$) is a basis for the irreducible quotient T-module V_0/V_1 (respectively V_1/V_2). Therefore, the T-module V_0/V_2 of dimension six has the irreducible T-submodule V_1/V_2 of dimension three, which is the unique faithful minimal T-submodule of $\overline{V}$ by [21, Theorem 5.1]. Consider the affine transformation group $\mathrm{AGL}(6,2)$ of the linear vector space V_0/V_2. Then T can be viewed as a subgroup in $\mathrm{AGL}(6,2)$, while K is exactly V_1/V_2.

Let every characteristic function in V_0 be presented as a seven-dimensional vector $(x_1, x_2, \ldots, x_7)$ over $\mathbb{F}_2$, whose vector components are indexed in order by

$$\{(1,0,0), (0,1,0), (0,0,1), (1,1,0), (1,0,1), (0,1,1), (1,1,1)\}.$$

Let $T = \langle a, b\rangle \cong \mathrm{GL}(3,2)$, where

$$a = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Then, via its action of $V(3,2)$, the actions of a and b on $V(\Omega)$ are given by

$$(x_1, x_2, x_3, x_4, x_5, x_6, x_7)^a = (x_1, x_2, x_5, x_4, x_3, x_7, x_6),$$
$$(x_1, x_2, x_3, x_4, x_5, x_6, x_7)^b = (x_5, x_3, x_4, x_1, x_6, x_7, x_2).$$

Since $\langle b\rangle$ is a Sylow 7-subgroup, we may set $a^\tau = ak_1$ and $b^\tau = b$, where $k_1 \in K$. Then ak_1 and b satisfy the defining relations of $\mathrm{GL}(3,2)$:

$$\begin{gathered}(ak_1)^2 = 1, \quad (ak_1b)^3 = k_1^{b(ab)^2} k_1^{b(ab)} k_1^b = 1, \\ (b^4ak_1)^4 = k_1^{(b^4a)^3} k_1^{(b^4a)^2} k_1^{b^4a} k_1 = 1.\end{gathered} \tag{3.11}$$

Solving (3.11), we get $k_1 = (0, x, x, x, x, 0, 0) + V_2$.

First, let $x = 1$. Suppose that $X \cong X(\widetilde{A}, T, D)$, where D corresponds to a suborbit of $\widetilde{A}$ of length seven relative to T. Since a should fix a point in D, we may assume that $T\tau ca = T\tau c$, so that $D = T\tau cT$, for $c = (x_1', \cdots x_7') + V_2 \in C \setminus K$. Then $T = Tca^\tau c = Tcak_1c = Tc^ak_1c$, that is, $c^ak_1c \in V_2$. However,

$$c^ack_1 = (0, 1, x_3' + x_5' + 1, 1, 1 + x_3' + x_5', x_6' + x_7', x_6' + x_7') \notin V_2.$$

Secondly, let $x = 0$. Then $k_1 = 0$ and so $[\tau, T] = 1$. In other words, τ is a central involution of $\widetilde{A}$ and so our graph X is a canonical double covering of a cover of the complete graph of order eight with the covering transformation group $\mathbb{Z}_2^3$ and whose fiber-preserving automorphism group acts 2-arc-transitively. This covering graph has been determined in [8] and is just the homomorphism image of $X_4(8,2)$ by mapping every pair (i, i') to one vertex. □

Combining the lemmas in Sections 3.1–3.4, we complete a proof of Theorem 1.1.

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SHAOFEI DU, School of Mathematical Sciences,
Capital Normal University, Beijing 100048, PR China
e-mail: dushf@cnu.edu.cn

WENQIN XU, School of Mathematical Sciences,
Capital Normal University, Beijing 100048, PR China
e-mail: wenqinxu85@163.com