

## LARGE DEVIATIONS OF POISSON TELECOM PROCESSES

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### Abstract

We study large-deviation probabilities of Telecom processes appearing as limits in a critical regime of the infinite-source Poisson model elaborated by I. Kaj and M. Taqqu. We examine three different regimes of large deviations (LD) depending on the deviation level. A Telecom process  $(Y_t)_{t \geq 0}$  scales as  $t^{1/\gamma}$ , where  $t$  denotes time and  $\gamma \in (1, 2)$  is the key parameter of  $Y$ . We must distinguish moderate LD  $\mathbb{P}(Y_t \geq y_t)$  with  $t^{1/\gamma} \ll y_t \ll t$ , intermediate LD with  $y_t \approx t$ , and ultralarge LD with  $y_t \gg t$ . The results we obtain essentially depend on another parameter of  $Y$ , namely the resource distribution. We solve completely the cases of moderate and intermediate LD (the latter being the most technical one), whereas the ultralarge deviation asymptotics is found for the case of regularly varying distribution tails. In all the cases considered, the large-deviation level is essentially reached by the minimal necessary number of ‘service processes’.

*Keywords:* Large-deviation probabilities; Telecom process; Poisson random measure; teletraffic; workload

2020 Mathematics Subject Classification: Primary 60F10  
Secondary 60F05; 60G57; 60E07

### 1. Introduction: Telecom processes

#### 1.1. A service system

Telecom processes originate from a remarkable work by I. Kaj and M. S. Taqqu [9], who handled the limit behavior of ‘teletraffic systems’ by using the language of integral representations as a unifying technique. Their article represented a wave of interest in the subject; see, e.g., [8, 10, 12, 13, 14], and the surveys with further references [5, 6, 7], to mention just a few. The simplicity of the dependence mechanism used in the model enables us to get a clear understanding both of long-range dependence in one case, and independent increments in other cases.

The work of the system represents a collection of *service processes* or *sessions*, using telecommunication terminology. Every process starts at some time  $s$ , lasts  $u$  units of time, and occupies  $r$  resource units (synonyms for resource are *reward*, *transmission rate*, etc.). The amount of occupied resources  $r$  remains constant during every service process.

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Received 5 August 2021; revision received 8 April 2022.

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The formal model of the service system is based on Poisson random measures. Let  $\mathcal{R} := \{(s, u, r)\} = \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+$ . Every point  $(s, u, r)$  corresponds to a possible service process with starting time  $s$ , duration  $u$ , and required resources  $r$ .

The system is characterized by the following parameters:

- $\lambda > 0$ : the *arrival intensity* of service processes;
- $F_U(du)$ : the distribution of service duration;
- $F_R(dr)$ : the distribution of the amount of resources required.

We can assume that  $\mathbb{P}(R > 0) = \mathbb{P}(U > 0) = 1$  without loss of generality.

Define on  $\mathcal{R}$  an intensity measure  $\mu(ds, du, dr) := \lambda ds F_U(du) F_R(dr)$ . Let  $N$  be a Poisson random measure with intensity  $\mu$ . We can consider the samples of  $N$  (sets of triplets  $(s, u, r)$ , each triplet corresponding to a service process) as variants (sample paths) of the work for the system.

The *instant workload* of the system at time  $t$  is  $W^\circ(t) := \int_{\mathcal{R}} r \mathbf{1}_{\{s \leq t \leq s+u\}} dN$ . This is essentially the sum of occupied resources over the processes active at time  $t$ . The *integral workload* over the interval  $[0, t]$  is

$$\begin{aligned} W^*(t) &:= \int_0^t W^\circ(\tau) d\tau = \int_{\mathcal{R}} r \int_0^t \mathbf{1}_{\{s \leq \tau \leq s+u\}} d\tau dN \\ &= \int_{\mathcal{R}} r \cdot |[s, s+u] \cap [0, t]| dN := \int_{\mathcal{R}} r \ell_t(s, u) dN. \end{aligned}$$

Here,  $|\cdot|$  stands for the length of an interval, and the kernel

$$\ell_t(s, u) := |[s, s+u] \cap [0, t]| \tag{1}$$

will be used often in the following.

Notice that  $W^\circ(\cdot)$  is a stationary process, and its integral  $W^*(\cdot)$  is a process with stationary increments.

### 1.2. Limit theorems for the workload

The main object of theoretical interest is the behavior of the integral workload as a process (function of time) observed on long time intervals.

In order to obtain a meaningful limit, we must scale (contract) the time, center the workload process, and divide it by an appropriate scalar factor.

Centering and scaling by an appropriate factor  $b$  leads to a *normalized integral workload process*  $Z_a(t) := (W^*(at) - \mathbb{E} R \cdot \mathbb{E} U \cdot a\lambda t)/b$ ,  $b = b(a, \lambda)$ .

In order to obtain a limit theorem, we usually assume that either the variables  $R$  and  $U$  have finite variance, or their distributions have regular tails. More precisely, either

$$\mathbb{P}(U > u) \sim \frac{c_U}{u^\gamma} \quad \text{as } u \rightarrow \infty, \quad 1 < \gamma < 2, \quad c_U > 0, \tag{2}$$

or  $\mathbb{E}U^2 < \infty$ . In the latter case we formally set  $\gamma := 2$ .

Analogously, either

$$\mathbb{P}(R > r) \sim \frac{c_R}{r^\delta} \quad \text{as } r \rightarrow \infty, \quad 1 < \delta < 2, \quad c_R > 0, \tag{3}$$

or  $\mathbb{E}R^2 < \infty$ . In the latter case we formally set  $\delta := 2$ .

Notice that in all these cases  $\mathbb{E} R < \infty$ ,  $\mathbb{E} U < \infty$ , and the workload processes are correctly defined.

The behavior of the service system crucially depends on the parameters  $\gamma, \delta \in (1, 2]$ .

It is remarkable that a simple tuning of the three parameters  $\lambda, \gamma$ , and  $\delta$  may lead to different limiting processes for  $Z_a$ ; namely, we can obtain

- Wiener process;
- fractional Brownian motion with index  $H \in (1/2, 1)$ ;
- centered Lévy stable process with positive spectrum;
- stable Telecom process;
- Poisson Telecom process.

While the first three processes present a core of the classical theory of stochastic processes, the Telecom processes remain almost unstudied. In this article we focus on some key properties of the Poisson Telecom process.

For the full panorama of related limit theorems we refer to [11, Chapter 3], and recall here only one result concerning the Poisson Telecom process (cf. [11, Theorem 13.16]) related to the case of *critical intensity*,  $\lambda/a^{\gamma-1} \rightarrow L, 0 < L < \infty$ .

**Theorem 1.** *Assume that conditions (2) and (3) hold with some  $1 < \gamma < \delta \leq 2$ . Let  $a, \lambda \rightarrow \infty$  so that the critical intensity condition holds. Let  $Q := Lc_U \gamma$ . Then, with scaling  $b := a$  the finite-dimensional distributions of the process  $Z_a$  converge to those of the Poisson Telecom process  $Y_{Q,\gamma}$  admitting an integral representation  $Y_{Q,\gamma}(t) = \int_{\mathcal{R}} r \ell_t(s, u) \bar{N}_{Q,\gamma}(ds, du, dr)$ . Here,  $\ell_t(s, u)$  is the kernel defined in (1), and  $\bar{N}_{Q,\gamma}$  is a centered Poisson random measure of intensity  $Q\mu_\gamma$ , where*

$$\mu_\gamma(ds, du, dr) := \frac{ds du}{u^{\gamma+1}} F_R(dr).$$

Poisson Telecom processes were introduced in [3] and placed into a more general picture in [9]. For further studies on this subject we refer to [1, 4]. In accordance with its role in the limit theorem, the process  $(Y_{Q,\gamma}(t))_{t \geq 0}$  has stationary increments. It is, however, not self-similar like other limiting processes in the same model, such as Wiener processes, fractional Brownian motion, or strictly stable Lévy processes.

It is well known that the process  $Y_{Q,\gamma}$  is correctly defined if  $\mathbb{E}(R^\gamma) < \infty$ . In the rest of the paper we make only this assumption on  $R$  and *do not* assume any tail regularity of  $R$  like the one required in (3). The only notable exception is the ultralarge-deviation case (Theorem 5) where the tail regularity appears to be essential.

## 2. Main results

### 2.1. A limit theorem for Telecom process

At large time scales the Poisson Telecom process essentially behaves as a  $\gamma$ -stable Lévy process. This fact is basically known, but we present it here for completeness of exposition. The analogy with a stable law will also guide us (to some extent and within a certain range) in the subsequent studies of large-deviation probabilities.

**Proposition 1.** *We have a weak convergence*

$$(\mathbb{E}(R^\gamma)t)^{-1/\gamma} Y_{Q,\gamma}(t) \Rightarrow S_{Q,\gamma} \quad \text{as } t \rightarrow \infty, \tag{4}$$

where  $S_{Q,\gamma}$  is a centered strictly  $\gamma$ -stable random variable with positive spectrum, i.e.

$$\mathbb{E} \exp\{i\theta S_{Q,\gamma}\} = \exp\left\{Q \int_0^\infty \frac{e^{i\theta u} - 1 - i\theta u}{u^{\gamma+1}} du\right\}, \quad \theta \in \mathbb{R}.$$

### 2.2. Large deviations

According to Proposition 1, the large-deviation probabilities are  $\mathbb{P}(Y_{Q,\gamma}(t) \geq y_t)$  as  $y_t \gg t^{1/\gamma}$ . Their behavior may be different in different zones of  $y_t$  and may depend on the distribution of  $R$ . Actually, three large-deviation zones emerge naturally:

- Moderate large deviations:  $t^{1/\gamma} \ll y_t \ll t$ . This case is completely explored in Section 2.2.1. The large-deviation probabilities behave exactly as those of the limiting stable process from Proposition 1.
- Intermediate large deviations:  $y_t = \kappa t$ . This case is explored in Section 2.2.2. The decay order of the large-deviation probabilities is still the same as for the limiting stable process but the corresponding constants are different. The study is quite involved, especially due to the tricky dependence of these new emerging constants on the distribution of  $R$ .
- Ultralarge deviations:  $y_t \gg t$ . This case is partially considered in Section 2.2.3. Here, the large-deviation probabilities are determined by the tail probabilities of the underlying random variable  $R$ . Our result is limited by one of the most natural cases, regular behavior of the tails. Solving the case of light tails in sufficient generality remains a challenging problem.

We present specific results in the following subsections.

#### 2.2.1. Moderate large deviations

**Theorem 2.** *Let  $y_t$  be such that  $t^{1/\gamma} \ll y_t \ll t$ . Then*

$$\mathbb{P}(Y_{Q,\gamma}(t) \geq y_t) = Dt y_t^{-\gamma} (1 + o(1)) \quad \text{as } t \rightarrow \infty, \tag{5}$$

where  $D := Q\mathbb{E}(R^\gamma)/\gamma$ .

This result should be compared with the limit theorem (4) because for the levels  $\rho_t$  satisfying  $1 \ll \rho_t \ll t^{(\gamma-1)/\gamma}$ , (5) yields

$$\begin{aligned} \mathbb{P}((\mathbb{E}(R^\gamma)t)^{-1/\gamma} Y_{Q,\gamma}(t) \geq \rho_t) &= \mathbb{P}(Y_{Q,\gamma}(t) \geq (\mathbb{E}(R^\gamma)t)^{1/\gamma} \rho_t) \\ &\sim Dt(\mathbb{E}(R^\gamma)t)^{-1} \rho_t^{-\gamma} = \frac{Q}{\gamma} \rho_t^{-\gamma} \sim \mathbb{P}(S_{Q,\gamma} \geq \rho_t). \end{aligned}$$

In other words, the moderate large-deviation probabilities are equivalent to those of the limiting distribution.

Using the terminology of the background service system, moderate deviation is attained by a unique heavy service process. We will stress this fact later in the proof.

2.2.2. *Intermediate large deviations* The following result describes the situation on the upper boundary of the moderate deviations zone.

**Theorem 3.** *Let  $\kappa > 0$  be such that  $\mathbb{P}(R \geq \kappa) > 0$  and*

$$\mathbb{P}(R = \kappa) = 0. \tag{6}$$

Let  $y_t := \kappa t$ . Then  $\mathbb{P}(Y_{Q,\gamma}(t) \geq y_t) = QD_1^{(1)}(\kappa)t^{-(\gamma-1)}(1 + o(1))$  as  $t \rightarrow \infty$ , where

$$D_1^{(1)}(\kappa) := \frac{\kappa^{-\gamma}}{\gamma} \mathbb{E}(R^\gamma \mathbf{1}_{\{R \geq \kappa\}}) + \frac{(2 - \gamma)\kappa^{1-\gamma}}{(\gamma - 1)\gamma} \mathbb{E}(R^{\gamma-1} \mathbf{1}_{\{R \geq \kappa\}}).$$

**Remark 1.** There is a certain continuity between the asymptotic expressions in the zones of moderate and intermediate large deviations, as in the intermediate case we let  $\kappa \rightarrow 0$ . Indeed, by plugging formally  $y_t := \kappa t$  into (5) we obtain the asymptotics  $Q\mathbb{E}(R^\gamma)\gamma^{-1}\kappa^{-\gamma}t^{-(\gamma-1)}$ , while the first term in the definition of  $D_1^{(1)}(\kappa)$  taken alone provides almost the same asymptotics,  $Q\mathbb{E}(R^\gamma \mathbf{1}_{\{R \geq \kappa\}})\gamma^{-1}\kappa^{-\gamma}t^{-(\gamma-1)}$ , given that  $\mathbb{E}(R^\gamma \mathbf{1}_{\{R \geq \kappa\}})$  tends to  $\mathbb{E}(R^\gamma)$  as  $\kappa \rightarrow 0$ . Moreover, when  $\kappa$  goes to zero, the second term in the definition of  $D_1^{(1)}(\kappa)$  is negligible with respect to the first one because it contains an extra degree of  $\kappa$ .

**Remark 2.** If (6) does not hold, the decay order of large deviations will be the same but the expression for the corresponding constant becomes more involved and less explicit.

The attentive reader will notice that Theorem 3 does not work for large  $\kappa$  if the distribution of  $R$  is compactly supported. Indeed, in this case the large-deviation asymptotics will be different, as the next result shows. In terms of the service system, it handles the case when the large deviation can be attained by accumulation of  $n$  heavy service processes but cannot be attained by  $(n - 1)$  such processes.

**Theorem 4.** Let  $\kappa > 0$ . Assume that there is a positive integer  $n$  such that  $\mathbb{P}(R \geq \kappa/n) > 0$  but

$$\mathbb{P}\left(R \geq \frac{\kappa}{n - \zeta}\right) = 0 \quad \text{for some } \zeta \in (0, 1). \tag{7}$$

Assume that

$$\mathbb{P}(R_1 + \dots + R_n = \kappa) = 0, \tag{8}$$

where  $R_1, \dots, R_n$  are independent copies of  $R$ .

Let  $y_t := \kappa t$ . Then  $\mathbb{P}(Y_{Q,\gamma}(t) \geq y_t) = Q^n D_1^{(n)}(\kappa)t^{-(\gamma-1)n}(1 + o(1))$  as  $t \rightarrow \infty$ , where  $D_1^{(n)}(\kappa)$  is some finite positive constant depending on  $n, \kappa$ , and on the law of  $R$ .

**Remark 3.** The explicit form of  $D_1^{(n)}(\kappa)$  is given in (31).

**Remark 4.** Theorem 4 does not cover the critical case  $\zeta = 1$ , where we have  $\mathbb{P}(R \geq \kappa/(n - 1)) = 0$  but  $\mathbb{P}(R \geq \kappa/(n - 1) - \varepsilon) > 0$  for all  $\varepsilon > 0$ . In this case, the assertion of the theorem may not hold because the large-deviation probability behavior depends on that of the upper tail,  $\mathbb{P}(R \in [\kappa/(n - 1) - \varepsilon, \kappa/(n - 1)])$ , as  $\varepsilon \rightarrow 0$ .

### 2.2.3. Ultralarge deviations

**Theorem 5.** Let  $y_t \gg t$ . Assume that the tail probability function  $\bar{F}_R(r) := \mathbb{P}(R \geq r)$  is regularly varying of negative order  $-m$ , where  $m > \gamma$ . Then

$$\mathbb{P}(Y_{Q,\gamma}(t) \geq y_t) = QDt^{-(\gamma-1)}\bar{F}_R(y_t/t)(1 + o(1))$$

as  $t \rightarrow \infty$ , where

$$D := \frac{m(m - 1)}{\gamma(\gamma - 1)(m - \gamma + 1)(m - \gamma)}.$$

As in Theorem 2, the workload's large deviation is attained by a unique long and heavy service process.

We stress that the parameter  $m$  in Theorem 5 is allowed to be arbitrarily large. In particular, the case  $m > 2$  corresponds to the notation  $\delta = 2$  used in the introduction. Therefore, we are not allowed to identify  $m$  with  $\delta$ .

On the other hand, considering  $m < \gamma$  is meaningless because for such a case our Telecom process  $Y_{Q,\gamma}$  is simply not correctly defined.

### 2.3. Concluding remark

The challenging case of light tails of the distribution of  $R$  not covered by Theorem 5 remains beyond the scope of this article. Unlike all our results, here the workload's large deviation may be achieved through many overlapping heavy service processes.

Consider a 'toy', but not completely trivial, example of deterministic  $R$ . Let  $\mathbb{P}(R = 1) = 1$ . Then the contribution of any one service process is bounded by  $t$ . Therefore, in order to reach an ultralarge deviation level  $y_t \gg t$  we essentially need  $y_t/t \rightarrow \infty$  long service processes of maximal length order  $t$ . By quantifying this idea, we get an exponential probability decay,

$$\ln \mathbb{P}(Y_{Q,\gamma}(t) \geq y_t) \sim -\frac{y_t}{t} \ln \left( \frac{C y_t}{t^{2-\gamma}} \right), \quad C = \frac{(\gamma - 1)\gamma}{eQ},$$

which is very different from the results of this article both in its form and in its nature.

We expect that for a sufficiently large class of distributions with light tails the results in the spirit of the classical large-deviation theory [2] might be relevant. This could be a subject of subsequent work.

## 3. Proofs

### 3.1. Preliminaries

Let us introduce two auxiliary intensity measures that play a central role in the whole paper. The first one corresponds to the kernel  $\ell_t$ , namely

$$\mu_t^{(\ell)}(A) := \int_{\mathbb{R}} \int_{\mathbb{R}_+} \mathbf{1}_{\{\ell_t(s,u) \in A\}} \frac{du}{u^{\gamma+1}} ds, \quad A \in \mathcal{B}([0, t]).$$

Recall that  $\ell_t$  denotes the time length of a service process restricted to the interval  $[0, t]$ . These lengths form a Poisson random point process (or Poisson random measure), and  $\mu_t^{(\ell)}$  is the corresponding intensity (or mean measure).

The second measure is the 'distribution' of the product  $r\ell_t(s, u)$ ,

$$\begin{aligned} \mu_t^{(\ell,r)}(A) &:= \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \mathbf{1}_{\{r\ell \in A\}} \mu_t^{(\ell)}(d\ell) F_R(dr) \\ &= \mu_\gamma \{(s, u, r): r\ell_t(s, u) \in A\}, \quad A \in \mathcal{B}(\mathbb{R}_+). \end{aligned}$$

Here, the product  $r\ell_t$  represents the contribution of a service process to the integral workload of the system on the time interval  $[0, t]$ . Again, these contributions form a Poisson random point process, and  $\mu_t^{(\ell,r)}$  is the corresponding intensity.

Notice that the total mass of both measures is infinite due to the presence of infinitely many very short service processes.

A simple variable change  $v = r\ell_t(s, u)$  in the definition of  $Y_{Q,\gamma}(t)$  yields

$$Y_{Q,\gamma}(t) = \int_{\mathbb{R}_+} v \tilde{N}_{Q,\gamma}(dv), \quad (9)$$

where  $\tilde{N}_{Q,\gamma}$  is a centered Poisson measure with intensity  $Q\mu_t^{(\ell,r)}$ . Therefore, the properties of  $\mu_t^{(\ell,r)}$  determine those of  $Y_{Q,\gamma}(t)$ .

As a first step, we give an explicit formula for the intermediate measure  $\mu_t^{(\ell)}$ . First, by definition, we have  $\mu_t^{(\ell)}(t, \infty) = 0$ . Next, let us fix an  $\ell_0 \in (0, t]$  and find  $\mu_t^{(\ell)}[\ell_0, t]$ . In fact,  $\ell_t(s, u) \geq \ell_0$  if and only if  $u \geq \ell_0$  and  $s \in [\ell_0 - u, t - \ell_0]$ . Therefore,

$$\mu_t^{(\ell)}[\ell_0, t] = \int_{\ell_0}^{\infty} (t - 2\ell_0 + u) \frac{du}{u^{\gamma+1}} = \frac{t\ell_0^{-\gamma}}{\gamma} + \frac{2 - \gamma}{(\gamma - 1)\gamma} \ell_0^{1-\gamma}. \tag{10}$$

It follows that the measure  $\mu_t^{(\ell)}$  has an atom with weight  $t^{-(\gamma-1)}/(\gamma - 1)\gamma$  at the right boundary point  $t$ , and a density

$$\frac{d\mu_t^{(\ell)}}{d\ell}(\ell) = t\ell^{-1-\gamma} + \frac{2 - \gamma}{\gamma} \ell^{-\gamma}, \quad 0 < \ell < t.$$

For each  $\ell_0 > 0$ , (10) also yields the bound

$$\mu_t^{(\ell)}[\ell_0, \infty) = \mu_t^{(\ell)}[\ell_0, t] \leq \frac{t\ell_0^{-\gamma}}{\gamma} \left(1 + \frac{2 - \gamma}{\gamma - 1}\right) = \frac{t\ell_0^{-\gamma}}{\gamma(\gamma - 1)}. \tag{11}$$

Finally, consider the asymptotic behavior of

$$\mu_t^{(\ell,r)}[y_t, \infty) = \int_{\mathbb{R}_+} \mu_t^{(\ell)}\left[\frac{y_t}{r}, t\right] F_R(dr). \tag{12}$$

Assume that  $y_t \rightarrow \infty$  but  $y_t/t \rightarrow 0$ . Then it follows from (10) that, for every fixed  $r$ ,

$$\mu_t^{(\ell)}\left[\frac{y_t}{r}, t\right] = \frac{ty_t^{-\gamma} r^\gamma}{\gamma} (1 + o(1)). \tag{13}$$

By using (11), we also have an integrable majorant with respect to the law  $F_R$ :

$$\mu_t^{(\ell)}\left[\frac{y_t}{r}, t\right] \leq \frac{ty_t^{-\gamma} r^\gamma}{\gamma(\gamma - 1)}.$$

By integrating this estimate in (12) we obtain

$$\mu_t^{(\ell,r)}[y_t, \infty) \leq \frac{\mathbb{E}(R^\gamma)}{\gamma(\gamma - 1)} ty_t^{-\gamma}. \tag{14}$$

Furthermore, by Lebesgue's dominated convergence theorem, (12) and (13) yield

$$\mu_t^{(\ell,r)}[y_t, \infty) = y_t^{-\gamma} t \int_{\mathbb{R}_+} \frac{r^\gamma}{\gamma} F_R(dr) (1 + o(1)) = \frac{\mathbb{E}(R^\gamma)}{\gamma} ty_t^{-\gamma} (1 + o(1)). \tag{15}$$

### 3.2. Proof of Proposition 1

*Proof.* Consider the integral representation (9). According to a general criterion of the weak convergence of Poisson integrals to a stable law [11, Corollary 8.5], it is enough to check that, for each fixed  $\rho > 0$ ,

$$Q\mu_t^{(\ell,r)}\{v: (\mathbb{E}(R^\gamma)t)^{-1/\gamma} v \geq \rho\} = Q \frac{\rho^{-\gamma}}{\gamma} (1 + o(1)), \tag{16}$$

combined with the uniform bound

$$\sup_{t>0} \sup_{\rho>0} \rho^\gamma \mu_t^{(\ell,r)} \{v: (\mathbb{E}(R^\gamma)t)^{-1/\gamma} v \geq \rho\} < \infty. \tag{17}$$

Indeed, by substituting  $y_t := \rho(\mathbb{E}(R^\gamma)t)^{1/\gamma}$  in (15) we obtain (16), and by making the same substitution in (14) we obtain (17). □

### 3.3. A decomposition

Take some  $v_0 > 0$  and split the integral representation (9) into three parts:

$$\begin{aligned} Y_{Q,\gamma}(t) &= \int_0^{v_0} v \tilde{N}_{Q,\gamma}(dv) + \int_{v_0}^\infty v N_{Q,\gamma}(dv) - Q \int_{v_0}^\infty v \mu_t^{(\ell,r)}(dv) \\ &:= Y^\circ(t) + Y^\dagger(t) - E_t, \end{aligned} \tag{18}$$

where  $N_{Q,\gamma}$  is the corresponding *non-centered* Poisson random measure and  $E_t$  is the centering deterministic function.

The variance of  $Y^\circ(t)$  admits the upper bound

$$\text{Var } Y^\circ(t) = Q \int_0^{v_0} v^2 \mu_t^{(\ell,r)}(dv) = 2Q \int_0^{v_0} v \mu_t^{(\ell,r)}[v, v_0] dv \leq 2Q \int_0^{v_0} v \mu_t^{(\ell,r)}[v, \infty) dv.$$

Using (14) we get

$$\text{Var } Y^\circ(t) \leq \frac{2Qt}{\gamma(\gamma - 1)} \mathbb{E}(R^\gamma) \int_0^{v_0} v^{1-\gamma} dv = D_2 t v_0^{2-\gamma}, \tag{19}$$

where

$$D_2 := \frac{2Q}{\gamma(\gamma - 1)(2 - \gamma)} \mathbb{E}(R^\gamma).$$

Similarly, the centering term admits the bound

$$0 \leq E_t \leq Q \int_{v_0}^\infty \mu_t^{(\ell,r)}[v, \infty) dv + Q v_0 \mu_t^{(\ell,r)}[v_0, \infty) \leq D_1 t v_0^{1-\gamma}, \tag{20}$$

where

$$D_1 := \frac{Q}{(\gamma - 1)^2} \mathbb{E}(R^\gamma).$$

### 3.4. A lower bound for large deviations

We will give a lower bound for large-deviation probabilities  $\mathbb{P}(Y_{Q,\gamma}(t) \geq y_t)$  with  $y_t \gg t^{1/\gamma}$ . Let  $h, \varepsilon$  be small positive numbers. Define  $v_0 := hy_t$  and consider the corresponding decomposition (18).

First of all, notice that  $E_t$  is negligible at the range  $y_t$  because, by (20),  $E_t \leq D_1 t (hy_t)^{1-\gamma} = D_1 h^{1-\gamma} (t^{-1/\gamma} y_t)^{-\gamma} y_t = o(y_t)$ . Therefore, we may, and do, assume  $t$  to be so large that  $E_t \leq \varepsilon y_t$ .

Using (19), by Chebyshev’s inequality we have

$$\mathbb{P}(|Y^\circ(t)| \geq \varepsilon y_t) \leq \frac{\text{Var } Y^\circ(t)}{(\varepsilon y_t)^2} \leq \frac{D_2 t (hy_t)^{2-\gamma}}{(\varepsilon y_t)^2} = \frac{D_2 h^{2-\gamma}}{\varepsilon^2} (t^{-1/\gamma} y_t)^{-\gamma} \rightarrow 0. \tag{21}$$



It is also useful to notice that, for each fixed  $\rho > 0$  and all large  $t$ ,

$$\begin{aligned} \mu_t^{(\ell,r)}[v_0, \infty) &= \mu_t^{(\ell,r)}[hy_t, \infty) = \mu_t^{(\ell,r)}[h(t^{-1/\gamma}y_t)t^{1/\gamma}, \infty) \\ &\leq \mu_t^{(\ell,r)}[h\rho t^{1/\gamma}, \infty) \leq \frac{\mathbb{E}(R^\gamma)}{\gamma(\gamma - 1)}(h\rho)^{-\gamma}, \end{aligned}$$

where we used (14) at the last step. Letting  $\rho \rightarrow \infty$  we get  $\mu_t^{(\ell,r)}[v_0, \infty) \rightarrow 0$  as  $t \rightarrow \infty$ .

Using the basic properties of Poisson random measure, we may proceed now with the required lower bound as follows:

$$\begin{aligned} \mathbb{P}(Y_{Q,\gamma}(t) \geq y_t) &\geq \mathbb{P}(|Y^\circ(t)| \leq \varepsilon y_t, Y^\dagger(t) \geq (1 + 2\varepsilon)y_t) \\ &\geq \mathbb{P}(|Y^\circ(t)| \leq \varepsilon y_t)\mathbb{P}(Y^\dagger(t) \geq (1 + 2\varepsilon)y_t; N_{Q,\gamma}[v_0, \infty) = 1) \\ &= \mathbb{P}(|Y^\circ(t)| \leq \varepsilon y_t) \exp\{-Q\mu_t^{(\ell,r)}[v_0, \infty)\} Q\mu_t^{(\ell,r)}[(1 + 2\varepsilon)y_t, \infty). \end{aligned}$$

The idea behind this bound is to take a single service process providing a substantial large-deviation workload and to suppress other contributions.

As we have just seen, the first two factors tend to one, thus

$$\mathbb{P}(Y_{Q,\gamma}(t) \geq y_t) \geq Q\mu_t^{(\ell,r)}[(1 + 2\varepsilon)y_t, \infty)(1 + o(1)). \tag{22}$$

### 3.5. An upper bound for large deviations

Starting again with the representation in (18), using  $E_t \geq 0$  and (21) we have

$$\begin{aligned} \mathbb{P}(Y_{Q,\gamma}(t) \geq y_t) &\leq \mathbb{P}(Y^\circ(t) \geq \varepsilon y_t) + \mathbb{P}(N_{Q,\gamma}[v_0, \infty) \geq 2) + \mathbb{P}(Y^\dagger(t) \geq (1 - \varepsilon)y_t; N_{Q,\gamma}[v_0, \infty) = 1) \\ &= \mathbb{P}(Y^\circ(t) \geq \varepsilon y_t) + \mathbb{P}(N_{Q,\gamma}[v_0, \infty) \geq 2) + \mathbb{P}(N_{Q,\gamma}[(1 - \varepsilon)y_t, \infty) = 1) \\ &\leq \frac{D_2 th^{2-\gamma}}{\varepsilon^2 y_t^\gamma} + \frac{1}{2}(Q\mu_t^{(\ell,r)}[v_0, \infty))^2 + Q\mu_t^{(\ell,r)}[(1 - \varepsilon)y_t, \infty). \end{aligned} \tag{23}$$

Here, the last term is the main one. Recall that almost the same expression also shows up in the lower bound.

### 3.6. Proof of Theorem 2

*Proof.* Recall that, according to (15), in the zone under consideration,  $t^{1/\gamma} \ll y_t \ll t$ , it is true that

$$\mu_t^{(\ell,r)}[y_t, \infty) = \frac{\mathbb{E}(R^\gamma)}{\gamma} t y_t^{-\gamma} (1 + o(1)), \tag{24}$$

and we have similar representations with  $y_t$  replaced by either  $(1 + 2\varepsilon)y_t$ ,  $(1 - \varepsilon)y_t$ , or  $v_0 = hy_t$ .

In view of (24), the lower estimate (22) yields

$$\liminf_{t \rightarrow \infty} \frac{\mathbb{P}(Y_{Q,\gamma}(t) \geq y_t)}{t y_t^{-\gamma}} \geq \frac{Q\mathbb{E}(R^\gamma)}{\gamma} (1 + 2\varepsilon)^{-\gamma},$$

while the upper estimate (23) yields

$$\limsup_{t \rightarrow \infty} \frac{\mathbb{P}(Y_{Q,\gamma}(t) \geq y_t)}{ty_t^{-\gamma}} \leq \frac{D_2 h^{2-\gamma}}{\varepsilon^2} + \frac{Q\mathbb{E}(R^\gamma)}{\gamma} (1-\varepsilon)^{-\gamma},$$

because the second term in (23) has a lower order of magnitude.

First letting  $h \rightarrow 0$ , then  $\varepsilon \rightarrow 0$ , we obtain

$$\lim_{t \rightarrow \infty} \frac{\mathbb{P}(Y_{Q,\gamma}(t) \geq y_t)}{ty_t^{-\gamma}} = \frac{Q\mathbb{E}(R^\gamma)}{\gamma},$$

as required.  $\square$

### 3.7. Proof of Theorem 3

*Proof.* The proof goes along the same lines as in the moderate-deviation case, except for the evaluation of  $\mu_t^{(\ell,r)}[y_t, \infty)$ . Instead of (24), we have the following non-asymptotic exact formula. According to (10), for  $y_t = \kappa t$  we have

$$\begin{aligned} \mu_t^{(\ell,r)}[y_t, \infty) &= \int_0^\infty \mu_t^{(\ell)} \left[ \frac{y_t}{r}, \infty \right) F_R(dr) \\ &= \int_\kappa^\infty \left( t \frac{(y_t/r)^{-\gamma}}{\gamma} + \frac{2-\gamma}{(\gamma-1)\gamma} (y_t/r)^{1-\gamma} \right) F_R(dr) \\ &= \int_\kappa^\infty \left( \frac{\kappa^{-\gamma}}{\gamma} r^\gamma + \frac{(2-\gamma)\kappa^{1-\gamma}}{(\gamma-1)\gamma} r^{\gamma-1} \right) F_R(dr) t^{-(\gamma-1)} \\ &= \left( \frac{\kappa^{-\gamma}}{\gamma} \mathbb{E}(R^\gamma \mathbf{1}_{\{R \geq \kappa\}}) + \frac{(2-\gamma)\kappa^{1-\gamma}}{(\gamma-1)\gamma} \mathbb{E}(R^{\gamma-1} \mathbf{1}_{\{R \geq \kappa\}}) \right) t^{-(\gamma-1)} \\ &= D_1^{(1)}(\kappa) t^{-(\gamma-1)}. \end{aligned}$$

The latter constant is positive due to the assumption  $\mathbb{P}(R \geq \kappa) > 0$ .

For the lower bound, the estimate in (22) yields

$$\mathbb{P}(Y_{Q,\gamma}(t) \geq \kappa t) \geq QD_1^{(1)}((1+2\varepsilon)\kappa) t^{-(\gamma-1)} (1+o(1)).$$

Letting  $\varepsilon \searrow 0$  and using (6), we have

$$\lim_{\varepsilon \searrow 0} D_1^{(1)}((1+2\varepsilon)\kappa) = \frac{\kappa^{-\gamma}}{\gamma} \mathbb{E}(R^\gamma \mathbf{1}_{\{R > \kappa\}}) + \frac{(2-\gamma)\kappa^{1-\gamma}}{(\gamma-1)\gamma} \mathbb{E}(R^{\gamma-1} \mathbf{1}_{\{R > \kappa\}}) = D_1^{(1)}(\kappa).$$

Therefore,  $\mathbb{P}(Y_{Q,\gamma}(t) \geq \kappa t) \geq QD_1^{(1)}(\kappa) t^{-(\gamma-1)} (1+o(1))$ , as required.

For the upper bound, the estimate in (23) with  $y_t = \kappa t$  yields

$$\mathbb{P}(Y_{Q,\gamma}(t) \geq \kappa t) \leq \left( \frac{D_2 h^{2-\gamma}}{\varepsilon^2 \kappa^\gamma} + QD_1^{(1)}((1-\varepsilon)\kappa) \right) t^{-(\gamma-1)} (1+o(1)).$$

First letting  $h \searrow 0$ , we get rid of the first term and obtain

$$\mathbb{P}(Y_{Q,\gamma}(t) \geq \kappa t) \leq QD_1^{(1)}((1-\varepsilon)\kappa) t^{-(\gamma-1)} (1+o(1)).$$

Letting  $\varepsilon \searrow 0$ , we have  $\lim_{\varepsilon \searrow 0} QD_1^{(1)}((1-\varepsilon)\kappa) = D_1^{(1)}(\kappa)$ . Therefore,  $\mathbb{P}(Y_{Q,\gamma}(t) \geq \kappa t) \leq QD_1^{(1)}(\kappa) t^{-(\gamma-1)} (1+o(1))$ , as required.  $\square$

### 3.8. Proof of Theorem 4

In the setting of this theorem, the large-deviation probabilities decay faster with  $t$  than Chebyshev’s inequality (21) suggests. Therefore, we need the finer estimate for  $Y^\circ(t)$  given in the following lemma.

**Lemma 1.** *For every  $M, \varepsilon > 0$  there exist  $h > 0$  and  $C = C(h, \varepsilon) > 0$  such that, for all  $t > 0$ ,  $\mathbb{P}(Y^\circ(t) \geq \varepsilon t) \leq Ct^{-(\gamma-1)M}$ , where  $Y^\circ(t)$  is defined by (18) with the splitting point  $v_0 := ht$ .*

*Proof.* We start with some calculations valid for arbitrary  $v_0$ . We have the following formula for the exponential moment of the centered Poisson integral:

$$\mathbb{E} \exp(\lambda Y^\circ(t)) = \exp \left\{ \int_0^{v_0} (e^{\lambda v} - 1 - \lambda v) \mu_t^{(\ell,r)}(dv) \right\}.$$

We can split the integration domain here into two parts:  $[0, v_0/2]$  and  $(v_0/2, v_0]$ . For the second one we have

$$\int_{\frac{v_0}{2}}^{v_0} (e^{\lambda v} - 1 - \lambda v) \mu_t^{(\ell,r)}(dv) \leq e^{\lambda v_0} \cdot \mu_t^{(\ell,r)} \left[ \frac{v_0}{2}, v_0 \right] \leq D_3 e^{\lambda v_0} t v_0^{-\gamma}, \quad D_3 := \frac{2^\gamma \mathbb{E}(R^\gamma)}{\gamma(\gamma - 1)}. \tag{25}$$

At the last step we used (14).

For the first zone, by using the inequality  $e^x - 1 - x \leq x^2 e^x$  and (14) we have

$$\begin{aligned} \int_0^{\frac{v_0}{2}} (e^{\lambda v} - 1 - \lambda v) \mu_t^{(\ell,r)}(dv) &\leq \int_0^{\frac{v_0}{2}} \lambda^2 v^2 e^{\lambda v} \mu_t^{(\ell,r)}(dv) \\ &\leq 2e^{\lambda v_0/2} \lambda^2 \int_0^{\frac{v_0}{2}} v \mu_t^{(\ell,r)}[v, v_0/2] dv \\ &\leq \frac{2\mathbb{E}(R^\gamma)}{\gamma(\gamma - 1)} e^{\lambda v_0/2} \lambda^2 t \int_0^{\frac{v_0}{2}} v^{1-\gamma} dv \\ &= D_4 e^{\lambda v_0/2} \lambda^2 t v_0^{2-\gamma}, \quad D_4 := \frac{2^{\gamma-1} \mathbb{E}(R^\gamma)}{\gamma(\gamma - 1)(2 - \gamma)}. \end{aligned}$$

Next, using the inequality  $e^{\frac{x}{2}} x^2 \leq 3e^x$ , we have

$$\int_0^{\frac{v_0}{2}} (e^{\lambda v} - 1 - \lambda v) \mu_t^{(\ell,r)}(dv) \leq 3D_4 e^{\lambda v_0} t v_0^{-\gamma}. \tag{26}$$

Summing (25) and (26), we obtain  $\mathbb{E} \exp(\lambda Y^\circ(t)) \leq \exp\{(D_3 + 3D_4) t v_0^{-\gamma} e^{\lambda v_0}\} := \exp\{A e^{\lambda v_0}\}$ , where  $A := (D_3 + 3D_4) t v_0^{-\gamma}$ . For every real  $z$ , by the exponential Chebyshev inequality we have

$$\mathbb{P}(Y^\circ(t) \geq z) \leq \inf_{\lambda > 0} \exp(A e^{\lambda v_0} - \lambda z). \tag{27}$$

If  $z > A v_0$ , the minimum on the right-hand side is attained at the point  $\lambda = (1/v_0) \log(z/Av_0)$ . By plugging this value into (27) we obtain

$$\mathbb{P}(Y^\circ(t) \geq z) \leq \exp \left( \frac{z}{v_0} \right) \left( \frac{Av_0}{z} \right)^{\frac{z}{v_0}} = \exp \left( \frac{z}{v_0} \right) \left( (D_3 + 3D_4) \frac{t v_0^{1-\gamma}}{z} \right)^{\frac{z}{v_0}}. \tag{28}$$

Letting  $z := \varepsilon t$ ,  $v_0 := ht$  yields  $\mathbb{P}(Y^\circ(t) \geq \varepsilon t) \leq Ct^{-\frac{\varepsilon}{h}(\gamma-1)}$ , where  $C$  depends only on  $\varepsilon$  and  $h$ . Choosing  $h < \frac{\varepsilon}{M}$ , we get the result. □

Now we can proceed to the proof of the theorem.

*Proof of Theorem 4.* We start with the upper bound. Let  $\eta := (1 - \zeta)\kappa/(n - \zeta)$ . Since  $\zeta \in (0, 1)$ , we have  $\eta > 0$ . It also follows from the definition that  $\kappa/(n - \zeta) = (\kappa - \eta)/(n - 1)$ . Therefore, we may rewrite (7) as

$$\mathbb{P}\left(R \geq \frac{\kappa - \eta}{n - 1}\right) = 0. \quad (29)$$

Let  $\varepsilon \in (0, \eta)$ . We now use the decomposition in (18) with the splitting point  $v_0 = ht$ , where  $h$  is small number. More precisely, by using Lemma 1 with  $M = n + 1$  we find a small  $h > 0$  such that

$$\mathbb{P}(Y^\circ(t) \geq \varepsilon t) \leq Ct^{-(\gamma-1)(n+1)}. \quad (30)$$

Taking into account  $E_t \geq 0$ , we get the bound

$$\mathbb{P}(Y_{Q,\gamma}(t) \geq \kappa t) \leq \mathbb{P}(Y^\circ(t) \geq \varepsilon t) + \mathbb{P}(Y^\dagger(t) \geq (\kappa - \varepsilon)t).$$

By (30), the first term is negligible compared with the decay order  $t^{-(\gamma-1)n}$  announced in the theorem. Let us write  $N_0 := N_{Q,\gamma}[v_0, \infty)$ , which is a Poissonian random variable with intensity  $\mu_0 := Q\mu_t^{(\ell,r)}[v_0, \infty)$ , and apply the following bound to the second term:

$$\begin{aligned} \mathbb{P}(Y^\dagger(t) \geq (\kappa - \varepsilon)t) &\leq \mathbb{P}(N_0 > n) + \mathbb{P}(Y^\dagger(t) \geq (\kappa - \varepsilon)t; N_0 = n) \\ &\quad + \mathbb{P}(Y^\dagger(t) \geq (\kappa - \varepsilon)t, N_0 \leq n - 1). \end{aligned}$$

For the first term, an elementary bound for the Poisson tail works, namely

$$\mathbb{P}(N_0 > n) = e^{-\mu_0} \sum_{j=0}^{\infty} \frac{\mu_0^{n+1+j}}{(n+1+j)!} \leq e^{-\mu_0} \frac{\mu_0^{n+1}}{(n+1)!} \sum_{j=0}^{\infty} \frac{\mu_0^j}{j!} \leq \frac{\mu_0^{n+1}}{(n+1)!},$$

where we used that  $(n+1+j)! \geq (n+1)!j!$ . Notice that by (14) with  $y_t := v_0 = ht$  we have

$$\mu_0 \leq \frac{Q\mathbb{E}(R^\gamma)}{\gamma(\gamma-1)} t(ht)^{-\gamma} = \frac{Q\mathbb{E}(R^\gamma)h^{-\gamma}}{\gamma(\gamma-1)} t^{-(\gamma-1)},$$

and hence  $\mathbb{P}(N_0 > n) = O(t^{-(\gamma-1)(n+1)})$  is negligible compared to the term  $t^{-(\gamma-1)n}$  in the theorem's assertion.

Further, by using (29) and the definition of the measure  $\mu_t^{(\ell,r)}$ , we see that

$$\mu_t^{(\ell,r)}\left[\frac{(\kappa - \varepsilon)t}{n - 1}, \infty\right) \leq \mu_t^{(\ell,r)}\left[\frac{(\kappa - \eta)t}{n - 1}, \infty\right) = 0,$$

which implies  $\mathbb{P}(Y^\dagger(t) \geq (\kappa - \varepsilon)t, N_0 \leq n - 1) = 0$ , because here the Poissonian integral  $Y^\dagger(t)$  is a sum of not more than  $n - 1$  terms each being strictly smaller than  $(\kappa - \varepsilon)t/(n - 1)$ .

For  $A \in \mathcal{B}([v_0, \infty))$ , we write  $N_A := N_{Q,\gamma}(A)$  with intensity  $\mu_A := Q\mu_t^{(\ell,r)}(A)$  and  $\nu_t^{(\ell,r)}(A) := \mathbb{P}(N_A = 1 \mid N_0 = 1)$ , which is a measure on  $[v_0, \infty)$ . We therefore have  $\nu_t^{(\ell,r)}(A) = e^{-\mu_A} \mu_A \cdot e^{\mu_A - \mu_0} \cdot (e^{\mu_0}/\mu_0) = \mu_A/\mu_0$ .

The remaining Poissonian integral with a fixed number of points admits the following representation:

$$\begin{aligned}
 &\mathbb{P}(Y^\dagger(t) \geq (\kappa - \varepsilon)t; N_0 = n) \\
 &= \mathbb{P}(Y^\dagger(t) \geq (\kappa - \varepsilon)t \mid N_0 = n)\mathbb{P}(N_0 = n) \\
 &= e^{-\mu_0} \frac{\mu_0^n}{n!} \int_{[v_0, \infty)^n} \mathbf{1}_{\{v_1 + \dots + v_n \geq (\kappa - \varepsilon)t\}} \prod_{j=1}^n v_j^{(l,r)}(dv_j) \\
 &\leq e^{-\mu_0} \frac{Q^n}{n!} \int_{\mathbb{R}_+^n} \mathbf{1}_{\{v_1 + \dots + v_n \geq (\kappa - \varepsilon)t\}} \prod_{j=1}^n \mu_t^{(\ell,r)}(dv_j) \\
 &= e^{-\mu_0} \frac{Q^n}{n!} \int_{[0,t]^n} \int_{\mathbb{R}_+^n} \mathbf{1}_{\{\ell_1 r_1 + \dots + \ell_n r_n \geq (\kappa - \varepsilon)t\}} \prod_{j=1}^n \mu_t^{(\ell)}(d\ell_j) \prod_{j=1}^n F_R(dr_j) \\
 &= e^{-\mu_0} \frac{Q^n}{n!} \int_{[0,1]^n} \mathbb{P}(s_1 R_1 + \dots + s_n R_n \geq \kappa - \varepsilon) \prod_{j=1}^n \nu(ds_j) t^{-(\gamma-1)n}, \\
 &:= e^{-\mu_0} Q^n D_1^{(n)}(\kappa - \varepsilon) t^{-(\gamma-1)n},
 \end{aligned}$$

where  $R_1, \dots, R_n$  are independent and identically distributed variables with distribution  $F_R$  and, according to (10),  $\nu$  is a measure on  $[0, 1]$  having the atom  $1/(\gamma(\gamma - 1))$  at 1 and the density

$$\frac{d\nu}{ds}(s) = s^{-(\gamma+1)} + \frac{2 - \gamma}{\gamma} s^{-\gamma}, \quad 0 < s < 1.$$

Notice also that the constant  $D_1^{(n)}(\kappa - \varepsilon)$  is finite although the measure  $\nu$  is infinite at each neighborhood of zero. The reason is that the probability we integrate vanishes if, for some  $i$ , we have  $s_i < s_* := (n - 1)(\eta - \varepsilon)/(\kappa - \eta)$  where  $\eta > 0$  satisfies (29). Indeed, in this case we have

$$\begin{aligned}
 \mathbb{P}(s_1 R_1 + \dots + s_n R_n \geq \kappa - \varepsilon) &\leq \mathbb{P}((s_* + (n - 1)) \max_{1 \leq j \leq n} R_j \geq \kappa - \varepsilon) \\
 &\leq n \mathbb{P}\left(R \geq \frac{\kappa - \varepsilon}{s_* + (n - 1)}\right) = n \mathbb{P}\left(R \geq \frac{\kappa - \eta}{n - 1}\right) = 0.
 \end{aligned}$$

We summarize our findings as  $\mathbb{P}(Y_{Q,\gamma}(t) \geq \kappa t) \leq Q^n D_1^{(n)}(\kappa - \varepsilon) t^{-(\gamma-1)n} (1 + o(1))$ .

Letting  $\varepsilon \searrow 0$ , we obtain  $\mathbb{P}(Y_{Q,\gamma}(t) \geq \kappa t) \leq Q^n D_1^{(n)}(\kappa) t^{-(\gamma-1)n} (1 + o(1))$ , where

$$D_1^{(n)}(\kappa) := \lim_{\varepsilon \rightarrow 0} D_1^{(n)}(\kappa - \varepsilon) = \frac{1}{n!} \int_{[0,1]^n} \mathbb{P}(s_1 R_1 + \dots + s_n R_n \geq \kappa) \prod_{j=1}^n \nu(ds_j). \tag{31}$$

It is easy to see that for  $n = 1$  we obtain the same value of  $D_1^{(1)}(\kappa)$  as in Theorem 3.

For the lower bound, first, notice that  $E_t$  in (18) is still negligible because by (20) we have  $E_t \leq D_1 t v_0^{1-\gamma} = D_1 t (ht)^{1-\gamma} = O(t^{2-\gamma}) = o(t)$ . Hence, for every fixed small  $\varepsilon$  we may and do assume that  $E_t \leq \varepsilon t$  for large  $t$ .

Second, using (19), by Chebyshev’s inequality we have, as  $t \rightarrow \infty$ ,

$$\mathbb{P}(|Y^\circ(t)| \geq \varepsilon t) \leq \frac{\text{Var } Y^\circ(t)}{(\varepsilon t)^2} \leq \frac{D_2 t (ht)^{2-\gamma}}{(\varepsilon t)^2} = \frac{D_2 h^2 t^{-\gamma}}{\varepsilon^2} t^{-(\gamma-1)} \rightarrow 0.$$

Therefore, we may proceed towards the required lower bound as follows:

$$\begin{aligned} \mathbb{P}(Y_{Q,\gamma}(t) \geq \kappa t) &\geq \mathbb{P}(|Y^\circ(t)| \leq \varepsilon t, Y^\dagger(t) \geq (\kappa + 2\varepsilon)t) \\ &\geq \mathbb{P}(|Y^\circ(t)| \leq \varepsilon t) \mathbb{P}(Y^\dagger(t) \geq (\kappa + 2\varepsilon)t; N_0 = n) \\ &= (1 + o(1)) \mathbb{P}(Y^\dagger(t) \geq (\kappa + 2\varepsilon)t; N_0 = n). \end{aligned}$$

The idea behind this bound is to focus on  $n$  service processes providing a substantial large-deviation workload and to suppress other contributions.

Furthermore, by using the expression obtained while working on the upper bound,

$$\begin{aligned} &\mathbb{P}(Y^\dagger(t) \geq (\kappa + 2\varepsilon)t; N_0 = n) \\ &= \frac{Q^n}{n!} \int_{[0,1]^n} \mathbb{P}(s_1 R_1 + \dots + s_n R_n \geq \kappa + 2\varepsilon) \prod_{j=1}^n \nu(ds_j) t^{-(\gamma-1)n} (1 + o(1)) \\ &:= Q^n D_1^{(n)}(\kappa + 2\varepsilon) t^{-(\gamma-1)n} (1 + o(1)). \end{aligned}$$

By letting  $\varepsilon \searrow 0$ , we obtain

$$\begin{aligned} \lim_{\varepsilon \searrow 0} D_1^{(n)}(\kappa + 2\varepsilon) &= \frac{1}{n!} \int_{[0,1]^n} \mathbb{P}(s_1 R_1 + \dots + s_n R_n > \kappa) \prod_{j=1}^n \nu(ds_j) \\ &= \frac{1}{n!} \int_{[0,1]^n} \mathbb{P}(s_1 R_1 + \dots + s_n R_n \geq \kappa) \prod_{j=1}^n \nu(ds_j) = D_1^{(n)}(\kappa). \end{aligned}$$

For the non-obvious passage we used the following lemma. □

**Lemma 2.** Assume that (8) holds. Then

$$\nu^n\{\mathbf{s} := (s_1, \dots, s_n): \mathbb{P}(s_1 R_1 + \dots + s_n R_n = \kappa) > 0\} = 0. \tag{32}$$

The required lower bound  $\mathbb{P}(Y_{Q,\gamma}(t) \geq \kappa t) \geq Q^n D_1^{(n)}(\kappa) t^{-(\gamma-1)n} (1 + o(1))$  now follows from the previous estimates. The proof is complete once the lemma is proved. □

*Proof of Lemma 2.* Let  $r_1, \dots, r_n$  be a sequence of atoms of the distribution  $F_R$ , so that  $\mathbb{P}(R = r_j) > 0, 1 \leq j \leq n$ . Define  $F = F(r_1, \dots, r_n) := \{\mathbf{s} \in [0, 1]^n: s_1 r_1 + \dots + s_n r_n = \kappa\}$ .

For every subset of integers  $J \subset \{1, \dots, n\}$  let

$$B_J := \{\mathbf{s} \in [0, 1]^n: s_j \in [0, 1), j \in J; s_j = 1, j \notin J\},$$

and notice that  $[0, 1]^n = \bigcup_J B_J$ . Let  $F_J := F \cap B_J = \{\mathbf{s} \in B_J: \sum_{j \in J} s_j r_j = \kappa - \sum_{j \notin J} r_j\}$ . If  $J$  is not empty, then  $\nu^n(F_J) = 0$  because  $\nu$  is absolutely continuous on  $[0, 1)$ .

If  $J$  is empty, then  $B_J = \{(1, \dots, 1)\}$  is a singleton and  $F_J = \emptyset$ , because otherwise  $\sum_{j=1}^n r_j = \kappa$  which would contradict (8). We conclude that  $v^n(F(r_1, \dots, r_n)) = \sum_J v^n(F_J) = 0$ . Since  $\{s: \mathbb{P}(s_1R_1 + \dots + s_nR_n = \kappa) > 0\} \subset \bigcup_{r_1, \dots, r_n} F(r_1, \dots, r_n)$  and the union is countable, we obtain (32).  $\square$

### 3.9. Proof of Theorem 5

*Proof.* For the upper bound, we take a small  $\varepsilon > 0$ , use the decomposition in (18) with  $v_0 := hy_t$  (a small  $h = h(\varepsilon)$  will be specified later on), and start with the usual bound

$$\begin{aligned} \mathbb{P}(Y_{Q,\gamma}(t) \geq y_t) &\leq \mathbb{P}(Y^\circ(t) \geq \varepsilon y_t) + \mathbb{P}(Y^\dagger(t) \geq (1 - \varepsilon)y_t) \\ &\leq \mathbb{P}(Y^\circ(t) \geq \varepsilon y_t) + \mathbb{P}(Y^\dagger(t) \geq (1 - \varepsilon)y_t; N_0 = 1) + P(N_0 \geq 2). \end{aligned}$$

To show that the first term is negligible compared to  $\bar{F}_R(y_t/t)$ , we use the estimate in (28) with  $z := \varepsilon y_t$ ,  $v_0 := hy_t$  and obtain, for some  $C = C(\varepsilon, h)$ ,  $\mathbb{P}(Y^\circ(t) \geq \varepsilon y_t) \leq C(ty_t^{-\gamma})^{\frac{\varepsilon}{h}} \leq C y_t^{-(\gamma-1)\frac{\varepsilon}{h}} \ll \bar{F}_R(y_t) \leq \bar{F}_R(y_t/t)$  whenever  $h$  is chosen so small that  $(\gamma - 1)\varepsilon/h > m$ .

Subsequent evaluation of  $Y^\dagger(t)$  requires analysis of the measure  $\mu_t^{(\ell,r)}$ . By using (12) and (10) we obtain

$$\begin{aligned} \mu_t^{(\ell,r)}[v, \infty) &= \int_{v/t}^\infty \mu_t^{(\ell)}\left[\frac{v}{r}, t\right] F_R(dr) \\ &= \int_{v/t}^\infty \left(\frac{t(r/v)^\gamma}{\gamma} + \frac{2 - \gamma}{(\gamma - 1)\gamma} (r/v)^{\gamma-1}\right) F_R(dr) \\ &= \frac{tv^{-\gamma}}{\gamma} \int_{v/t}^\infty r^\gamma F_R(dr) + \frac{(2 - \gamma)v^{1-\gamma}}{(\gamma - 1)\gamma} \int_{v/t}^\infty r^{\gamma-1} F_R(dr). \end{aligned}$$

Since the tail of  $F_R$  is regularly varying, we have the following asymptotics for the integrals as  $z \rightarrow \infty$ :

$$\begin{aligned} \int_z^\infty r^\gamma F_R(dr) &= \frac{mz^\gamma}{m - \gamma} \bar{F}_R(z)(1 + o(1)), \\ \int_z^\infty r^{\gamma-1} F_R(dr) &= \frac{mz^{\gamma-1}}{m - \gamma + 1} \bar{F}_R(z)(1 + o(1)). \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} \mu_t^{(\ell,r)}[v, \infty) &= t^{-(\gamma-1)} \left[ \frac{m}{\gamma(m - \gamma)} + \frac{(2 - \gamma)m}{(\gamma - 1)\gamma(m - \gamma + 1)} \right] \bar{F}_R(v/t)(1 + o(1)) \\ &= \frac{m(m - 1)}{\gamma(\gamma - 1)(m - \gamma + 1)(m - \gamma)} t^{-(\gamma-1)} \bar{F}_R(v/t)(1 + o(1)) \\ &= Dt^{-(\gamma-1)} \bar{F}_R(v/t)(1 + o(1)), \quad \text{as } v \gg t. \end{aligned} \tag{33}$$

Now the evaluation of  $Y^\dagger$  is straightforward. Indeed, by (33),

$$\begin{aligned} \mathbb{P}(Y^\dagger(t) \geq (1 - \varepsilon)y_t; N_0 = 1) &\leq Q\mu_t^{(\ell, r)}[(1 - \varepsilon)y_t, \infty) \\ &= QDt^{\gamma-1} \bar{F}_R((1 - \varepsilon)y_t/t)(1 + o(1)) \\ &= QDt^{\gamma-1} \bar{F}_R(y_t/t)(1 - \varepsilon)^{-m}(1 + o(1)), \\ P(N_0 \geq 2) &\leq Q^2 \mu_t^{(\ell, r)}[hy_t, \infty)^2 = Q^2(Dt^{-(\gamma-1)} \bar{F}_R(hy_t/t))^2(1 + o(1)) \\ &= Q^2(Dt^{-(\gamma-1)} \bar{F}_R(y_t/t)h^{-m})^2(1 + o(1)) \\ &\ll t^{-(\gamma-1)} \bar{F}_R(y_t/t). \end{aligned}$$

By combining these estimates and letting  $\varepsilon \rightarrow 0$  we obtain the desired bound,

$$\mathbb{P}(Y_{Q, \gamma}(t) \geq y_t) \leq QDt^{-(\gamma-1)} \bar{F}_R(y_t/t)(1 + o(1)).$$

For the lower bound, since  $y_t \gg t \gg t^{1/\gamma}$ , all the bounds from Section 3.4 apply. For every  $\varepsilon > 0$ , the inequality in (22) along with (33) yield

$$\begin{aligned} \mathbb{P}(Y_{Q, \gamma}(t) \geq y_t) &\geq Q\mu_t^{(\ell, r)}[(1 + 2\varepsilon)y_t, \infty)(1 + o(1)) \\ &= QDt^{-(\gamma-1)}(1 + 2\varepsilon)^{-m} \bar{F}_R(y_t/t)(1 + o(1)), \end{aligned}$$

and letting  $\varepsilon \rightarrow 0$  we get the desired bound,  $\mathbb{P}(Y_{Q, \gamma}(t) \geq y_t) \geq QDt^{-(\gamma-1)} \bar{F}_R(y_t/t)(1 + o(1))$ .  $\square$

### Acknowledgement

We are very grateful to two anonymous referees for careful reading and useful advice.

### Funding information

This work was supported by Russian Science Foundation grant 21-11-00047.

### Competing interests

There were no competing interests to declare which arose during the preparation or publication process of this article.

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