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RESEARCH ARTICLE

p-Adic estimates of abelian Artin L-functions on curves

Joe Kramer-Miller

Department of Mathematics, Lehigh University, 17 Memorial Drive East, Bethlehem, PA 18015, United States;
E-mail: jjk221@lehigh.edu.

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Abstract

The purpose of this article is to prove a 'Newton over Hodge' result for finite characters on curves. Let X be a smooth proper curve over a finite field $\mathbb{F}_q$ of characteristic $p \geq 3$ and let $V \subset X$ be an affine curve. Consider a nontrivial finite character $\rho : \pi_1^{et}(V) \to \mathbb{C}^\times$. In this article, we prove a lower bound on the Newton polygon of the L-function $L(\rho, s)$. The estimate depends on monodromy invariants of ρ: the Swan conductor and the local exponents. Under certain nondegeneracy assumptions, this lower bound agrees with the irregular Hodge filtration introduced by Deligne. In particular, our result further demonstrates Deligne's prediction that the irregular Hodge filtration would force p-adic bounds on L-functions. As a corollary, we obtain estimates on the Newton polygon of a curve with a cyclic action in terms of monodromy invariants.

Contents

1. Introduction

Let p be a prime with $p \geq 3$ and let $q = p^a$. Let X be a smooth proper curve of genus g defined over $\mathbb{F}_q$ with function field $K(X)$. We define G_X to be the absolute Galois group of $K(X)$. Let $\rho : G_X \to \mathbb{C}^\times$ be a nontrivial continuous character. The L-function associated to ρ is defined by

$$L(\rho, s) = \prod \frac{1}{1 - \rho(Frob_x)s^{\deg(x)}}, \tag{1}$$

with the product taken over all closed points $x \in X$ where ρ is unramified. By the Weil conjectures for curves [31], we know that

$$L(\rho, s) = \prod_{i=1}^{d}(1 - \alpha_i s) \in \overline{\mathbb{Z}}[s].$$

It is then natural to ask what we can say about the algebraic integers α_i. The Riemann hypothesis for curves tell us that $|\alpha_i|_\infty = \sqrt{q}$ for each Archimedean place. Furthermore, we know that the α_i are ℓ-adic units for any prime $\ell \neq p$. This leaves us with the question: What are the p-adic valuations of the α_i?

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The purpose of this article is to study the p-adic properties of $L(\rho, s)$. We prove a 'Newton over Hodge' result. This is in the vein of a celebrated theorem of Mazur [20], which compares the Newton and Hodge polygons of an algebraic variety over $\mathbb{F}_q$. Our result differs from Mazur's in that we study cohomology with coefficients in a local system. Our Hodge bound is defined using two monodromy invariants: the *Swan conductor* and the *tame exponents*. The representation ρ is analogous to a rank 1 differential equation on a Riemann surface with regular singularities twisted by an exponential differential equation (i.e., a weight 0 twisted Hodge module in the language of Esnault, Sabbah and Yu [9]). In this context one may define an irregular Hodge polygon [8, 9]. The irregular Hodge polygon agrees with the Hodge polygon we define under certain nondegeneracy hypotheses. Our result thus gives further credence to the philosophy that characteristic 0 Hodge-type phenomena force p-adic bounds on lisse sheaves in characteristic p.

1.1. Statement of main results

To state our main result, we first introduce some monodromy invariants. The character ρ factors uniquely as $\rho = \rho^{wild} \otimes \chi$, where $\left|Im\left(\rho^{wild}\right)\right| = p^n$ and $|Im(\chi)| = N$ with $\gcd(N, p) = 1$.

1. (Local) Let $Q \in X$ be a closed point. After increasing q we may assume that Q is an $\mathbb{F}_q$-point. Let u_Q be a local parameter at Q. Then ρ restricts to a local representation $\rho_Q : G_Q \to \mathbb{C}^\times$, where G_Q is the absolute Galois group of $\mathbb{F}_q((u_Q))$. We let ρ_Q^{wild} (resp., χ_Q) denote the restriction of ρ^{wild} (resp., χ) to G_Q.
   (a) (Swan conductors) Let $I_Q \subset G_Q$ be the inertia subgroup at Q. There is a decreasing filtration of subgroups I_Q^s on I_Q, indexed by real numbers $s \geq 0$. The *Swan conductor* at Q is the infimum of all s such that $I_Q^s \subset \ker(\rho_Q)$ [13, Chapter 1]. We denote the Swan conductor by s_Q. Note that $s_Q = 0$ if and only if ρ_Q^{wild} is unramified.
   (b) (Tame exponents) After increasing q we may assume that χ_Q is totally ramified at Q. There exists $e_Q \in \frac{1}{q-1}\mathbb{Z}$ such that G_Q acts on $t_Q^{e_Q}$ by χ_Q. Note that e_Q is unique up to addition by an integer.
      - The *exponent* of χ at Q is the equivalence class $\mathbf{e}_Q$ of e_Q in $\frac{1}{q-1}\mathbb{Z}/\mathbb{Z}$.
      - We define ϵ_Q to be the unique integer between 0 and $q-2$ such that $\frac{\epsilon_Q}{q-1} \in \mathbf{e}_Q$.
      - Write $\epsilon_Q = e_{Q,0} + e_{Q,1}p + \cdots + e_{Q,a-1}p^{a-1}$, where $0 \leq e_{Q,i} \leq p-1$. We define $\omega_Q = \sum e_{Q,i}$, the sum of the p-adic digits of ϵ_Q. Note that $\omega_Q = 0$ if and only if χ_Q is unramified.

   We refer to the tuple $R_Q = \left(s_Q, \mathbf{e}_Q, \epsilon_Q, \omega_Q\right)$ as a *ramification datum* and $T_Q = \left(\mathbf{e}_Q, \epsilon_Q, \omega_Q\right)$ as a *tame ramification datum*. We define the sets

$$S_Q = \begin{cases} \emptyset & s_Q = 0, \\ \left\{\frac{1}{s_Q}, \ldots, \frac{s_Q - 1}{s_Q}\right\}, & s_Q \neq 0 \text{ and } \omega_Q = 0, \\ \left\{\frac{1}{s_Q} - \frac{\omega_Q}{a s_Q (p-1)}, \ldots, \frac{s_Q}{s_Q} - \frac{\omega_Q}{a s_Q (p-1)}\right\}, & s_Q \neq 0 \text{ and } \omega_Q \neq 0. \end{cases}$$

2. (Global) Let $\tau_1, \ldots, \tau_{\mathbf{m}}$ be the points at which ρ ramifies and let $\mathbf{n} \leq \mathbf{m}$ be such that $\tau_1, \ldots, \tau_{\mathbf{n}}$ are the points at which χ ramifies. We define

$$\Omega_\rho = \frac{1}{a(p-1)} \sum_{i=1}^{\mathbf{n}} \omega_{\tau_i}.$$

   This is a global invariant built up from the p-adic properties of the local exponents. One can show that $\Omega_\rho \in \mathbb{Z}_{\geq 0}$ (see Section 5.3.2).

Using these invariants, we define the Hodge polygon $HP(\rho)$ to be the polygon whose slopes are

$$\{\underbrace{0, \ldots, 0}_{g-1+\mathbf{m}-\Omega_\rho}\} \sqcup \{\underbrace{1, \ldots, 1}_{g-1+\mathbf{m}-\mathbf{n}+\Omega_\rho}\} \sqcup \left(\bigsqcup_{i=1}^{\mathbf{m}} S_{\tau_i}\right),$$

where $\sqcup$ denotes a disjoint union. We can now state our main result:

Theorem 1.1. *The q-adic Newton polygon $NP_q(L(\rho, s))$ lies above the Hodge polygon $HP(\rho)$.*

Remark 1.2. It is worth mentioning that $HP(\rho)$ and $NP_q(L(\rho, s))$ have the same endpoints. To see this, first note that the x-coordinates of the endpoints of both polygons are $g-1+\mathbf{m}+\sum s_Q$. For $NP_q(L(\rho, s))$ this follows from the Euler–Poincaré formula [13, Section 2.3.1], and for $HP(\rho)$ it is clear from the definition. Next let $\left(s'_{\tau_i}, \mathbf{e}'_{\tau_i}, \epsilon'_{\tau_i}, \omega'_{\tau_i}\right)$ be the ramification datum associated to ρ^{-1} at τ_i. Then we have $s'_{\tau_i} = s_{\tau_i}$ and $\mathbf{e}'_{\tau_i} = -\mathbf{e}_{\tau_i}$. From this we see that $\omega'_{\tau_i} = a(p-1) - \omega_{\tau_i}$ for $1 \leq i \leq \mathbf{n}$ and $\omega'_{\tau_i} = 0$ for $i > \mathbf{n}$, which implies $\Omega_{\rho^{-1}} = \mathbf{n} - \Omega_\rho$. Thus, for every slope α of $HP(\rho)$, there is a corresponding slope $1-\alpha$ of $HP\left(\rho^{-1}\right)$. Similarly, by Poincaré duality we know that for every slope α of $NP_q(L(\rho, s))$, there is a corresponding slope $1-\alpha$ of $NP_q\left(L\left(\rho^{-1}, s\right)\right)$. It follows that the y-coordinates of the endpoints of $HP(\rho) \sqcup HP\left(\rho^{-1}\right)$ and $NP_q(L(\rho, s)) \sqcup NP_q\left(L\left(\rho^{-1}, s\right)\right)$ agree. By applying Theorem 1.1 to ρ and ρ^{-1}, we see that the endpoints of $HP(\rho)$ and $NP_q(L(\rho, s))$ are the same.

Remark 1.3. When ρ factors through an Artin–Schreier cover, Theorem 1.1 is due to previous work of the author [15].

Remark 1.4. The only other case where parts of Theorem 1.1 were previously known is when $X = \mathbb{P}^1$ and ρ is unramified outside of $\mathbb{G}_m$. Work of Adolphson and Sperber [2, 3] studies the case when $|Im(\rho)| = pN$ and $\gcd(p, N) = 1$. We note that the work of Adolphson and Sperber treats the case of higher-dimensional tori as well. These groundbreaking methods were applied to the case when ρ is totally wild by Liu and Wei in [18], introducing ideas from Artin–Schreier–Witt theory. For ρ with arbitrary image there are some results by Liu [17], under strict conditions on the wild part of ρ (this case corresponds to Heilbronn sums).

To the best of our knowledge, Theorem 1.1 was completely unknown outside of the situations described in Remarks 1.3 and 1.4.

Example 1.5. Let $X = \mathbb{P}^1_{\mathbb{F}_q}$ and let $\tau_1, \ldots, \tau_4$ be the points where ρ ramifies. Assume $|Im(\rho)| = 2p^n$ and that ρ is totally ramified at each τ_i (i.e., the inertia group at τ_i is equal to $Im(\rho)$). Let $f : E \to X$ be the genus 1 curve over which χ trivialises and let $v_i = f^{-1}(\tau_i)$. Consider the restriction $\rho_E = \rho|_{G_E}$. Let $(s_i, \mathbf{e}_i, \epsilon_i, \omega_i)$ be the ramification datum of ρ at τ_i and let $\left(s'_i, \mathbf{e}'_i, \epsilon'_i, \omega'_i\right)$ be the ramification datum of ρ_E at v_i. By Theorem 1.1 we know that $NP_q(L(\rho_E, s))$ lies above

$$HP(\rho_E) = \{0,0,0,0\} \sqcup \{1,1,1,1\} \sqcup \left(\bigsqcup_{i=1}^{4} \left\{\frac{1}{2s_i}, \ldots, \frac{2s_i-1}{2s_i}\right\}\right).$$

This follows by recognising that $s'_i = 2s_i$ and $\omega'_i = 0$. The factorisation $L(\rho_E, s) = L(\rho, s) L\left(\rho^{wild}, s\right)$ corresponds to a ‘decomposition’ of $HP(\rho_E)$ into two Hodge polygons, one giving a lower bound for $NP_q(L(\rho, s))$ and the other for $NP_q\left(L\left(\rho^{wild}, s\right)\right)$. We have $\omega_i = \frac{a(p-1)}{2}$ for each i and $\Omega_\rho = 2$. This allows us to compute the Hodge polygons as

$$HP(\rho) = \{0,1\} \sqcup \left(\bigsqcup_{i=1}^{4} \left\{\frac{1}{2s_i}, \frac{3}{2s_i}, \ldots, \frac{2s_i-1}{2s_i}\right\}\right),$$

$$HP\left(\rho^{wild}\right) = \{0,0,0\} \sqcup \{1,1,1\} \sqcup \left(\bigsqcup_{i=1}^{4} \left\{\frac{1}{s_i}, \ldots, \frac{s_i-1}{s_i}\right\}\right),$$

so that $HP(\rho_E) = HP(\rho) \sqcup HP\left(\rho^{wild}\right)$. More generally, we will obtain similar decompositions of the Hodge bounds as long as $Im(\chi) \mid p-1$.

1.1.1. Newton polygons of abelian covers of curves

Theorem 1.1 also has interesting consequences for Newton polygons of cyclic covers of curves. Let $G = \mathbb{Z}/Np^n\mathbb{Z}$, where N is coprime to p. Let $f : C \to X$ be a G-cover rami-

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fied over $\tau_1, \ldots, \tau_{\mathbf{m}}$. We let $H^1_{cris}(X)$ $\left(\text{resp., } H^1_{cris}(C)\right)$ be the crystalline cohomology of X (resp., C). For a character ρ of G, we let $H^1_{cris}(C)^{\rho}$ be the ρ-isotypical subspace for the action of G on $H^1_{cris}(C)$. Let NP_C $\left(\text{resp. } NP_X \text{ and } NP_C^{\rho}\right)$ denote the q-adic Newton polygon of $\det\left(1 - s\mathrm{F} \mid H^1_{cris}(C)\right)$ $\left(\text{resp., } \det\left(1 - s\mathrm{F} \mid H^1_{cris}(X)\right) \text{ and } \det\left(1 - s\mathrm{F} \mid H^1_{cris}(C)^{\rho}\right)\right)$. We are interested in the following question: To what extent can we determine NP_C from NP_X and the ramification invariants of f? The most basic result is the Riemann–Hurwitz formula, which determines the dimension of $H^1_{cris}(C)$ from $H^1_{cris}(X)$ and the ramification invariants. When $N = 1$, there is also the Deuring–Shafarevich formula [7], which determines the number of slope 0 segments in NP_C. In general, however, a precise formula for the slopes of NP_C seems impossible. Instead, the best we may hope for are estimates. To connect this problem to Theorem 1.1, recall the decomposition

$$\det\left(1 - s\mathrm{F} \mid H^1_{cris}(C)\right) = \det\left(1 - s\mathrm{F} \mid H^1_{cris}(X)\right) \prod_{\rho} \det\left(1 - s\mathrm{F} \mid H^1_{cris}(C)^{\rho}\right), \tag{2}$$

where ρ varies over the nontrivial characters $\mathbb{Z}/Np^n\mathbb{Z} \to \mathbb{C}^{\times}$. By the Lefschetz trace formula we know $L(\rho, s) = \det\left(1 - s\mathrm{F} \mid H^1_{cris}(C)^{\rho}\right)$. Thus, Theorem 1.1 gives lower bounds for NP_C using equation (2).

Consider the case when $N = 1$, so that $G = \mathbb{Z}/p^n\mathbb{Z}$. Let r_i be the ramification index of a point of C above τ_i and define

$$\Omega = \sum_{i=1}^{\mathbf{m}} p^{n-r_i}\left(p^{r_i} - 1\right).$$

For $j = 1, \ldots, n$, let C_j be the cover of X corresponding to the subgroup $p^{n-j}\mathbb{Z}/p^n\mathbb{Z} \subset G$. Fix a point $x_i(j) \in C_j$ above τ_i; this gives a local field extension of $\mathbb{F}_q\left(\left(t_{\tau_i}\right)\right)$. We let $s_{\tau_i}(j)$ denote the largest upper numbering ramification break of this extension.

Corollary 1.6. *The Newton polygon NP_C lies above the polygon whose slopes are the multiset*

$$NP_X \sqcup \{\underbrace{0, \ldots, 0}_{(p^n-1)(g-1)+\Omega}, \underbrace{1, \ldots, 1}_{(p^n-1)(g-1)+\Omega}\} \sqcup \left(\bigsqcup_{i=1}^{\mathbf{m}} \bigsqcup_{j=1}^{n} p^{j-1}(p-1) \times \left\{\frac{1}{s_{\tau_i}(j)}, \ldots, \frac{s_{\tau_i}(j) - 1}{s_{\tau_i}(j)}\right\}\right),$$

where we take $\left\{\frac{1}{s_{\tau_i}(j)}, \ldots, \frac{s_{\tau_i}(j)-1}{s_{\tau_i}(j)}\right\}$ *to be the empty set when* $s_{\tau_i}(j) = 0$.

Remark 1.7. When $N > 1$, we can obtain a complicated bound for NP_C from Theorem 1.1 and equation (2). Alternatively, we can replace X with the intermediate curve X^{tame} satisfying $Gal\left(C/X^{tame}\right) = \mathbb{Z}/p^n\mathbb{Z}$ and then apply Corollary 1.6 to the cover $C \to X^{tame}$ to obtain a bound. Both bounds are the same.

1.2. Outline of proof

The classical approaches to studying p-adic properties of exponential sums on tori no longer work when one considers more general curves. Instead, we have to expand on the methods developed in earlier work of the author on exponential sums on curves [15]. We use the Monsky trace formula (see Section 7.1). This trace formula allows us to compute $L(\rho, s)$ by studying Fredholm determinants of certain operators. More precisely, let $V = X - \{\tau_1, \ldots, \tau_{\mathbf{m}}\}$ and let $\overline{B}$ be the coordinate ring of V. Let L be a finite extension of $\mathbb{Q}_p$ whose residue field is $\mathbb{F}_q$ such that the image of ρ is contained in $L^{\times}$. Let $B^{\dagger}$ be the ring of *integral overconvergent functions* on a formal lifting of $\overline{B}$ over $\mathcal{O}_L$ (see Section 3). For example, if $V = \mathbb{A}^1$, then $B^{\dagger} = \mathcal{O}_L\langle t\rangle^{\dagger}$ (i.e., $B^{\dagger}$ is the ring of power series with integral coefficients that converge beyond the closed unit disc). Choose an endomorphism $\sigma : B^\dagger \to B^\dagger$ that lifts the q-power Frobenius of $\overline{B}$. Using σ, we define an operator $U_q : B^\dagger \to B^\dagger$, which is the composition of a trace map $Tr : B^\dagger \to \sigma\left(B^\dagger\right)$ with $\frac{1}{q}\sigma^{-1}$.

The Galois representation ρ corresponds to a unit-root overconvergent F-crystal of rank 1. This is a $B^\dagger$-module $M = B^\dagger e_0$ and a $B^\dagger$-linear isomorphism $\varphi : M \otimes_\sigma B^\dagger \to M$. Note that this F-crystal is determined entirely by $\alpha \in B^\dagger$ satisfying $\varphi(e_0 \otimes 1) = \alpha e_0$. We refer to α as the Frobenius structure of M. In our specific setup (see Section 3), the Monsky trace formula can be written as

$$L(\rho, s) = \frac{\det\left(1 - sU_q \circ \alpha \mid B^\dagger\right)}{\det\left(1 - sqU_q \circ \alpha \mid B^\dagger\right)},$$

where we regard α as the 'multiplication by α' map on $B^\dagger$. Thus, we need to understand the operator $U_q \circ \alpha$. Let us outline how we study this operator.

1.2.1. Lifting the Frobenius endomorphism

Both U_q and α depend on the choice of Frobenius endomorphism σ. When $V = \mathbb{G}_m$, the ring $B^\dagger$ is $\mathcal{O}_L\langle t\rangle^\dagger$, and the natural choice for σ sends t to t^q. However, no such natural choice exists for higher-genus curves. Our approach is to pick a convenient mapping $\eta : X \to \mathbb{P}^1$ and then pull back the Frobenius $t \mapsto t^q$ along η. We take η to be a tamely ramified map that is étale outside of $\{0, 1, \infty\}$. We may further assume that $\eta(\tau_i) \in \{0, \infty\}$ and the ramification index of every point in $\eta^{-1}(1)$ is $p - 1$ (see Lemma 3.1). This leaves us with two types of local Frobenius endomorphisms. For $Q \in X$ with $\eta(Q) \in \{0, \infty\}$, we may take the local parameter at Q to look like $u_Q = t^{\pm\frac{1}{e_Q}}$, where e_Q is the ramification index at Q. In particular, the Frobenius endomorphism sends $u_Q \mapsto u_Q^q$. If $\eta(Q) = 1$, we take the local parameter to look like $u_Q = \sqrt[p-1]{t-1}$. Thus, the Frobenius endomorphism sends $u_Q \mapsto \sqrt[p-1]{\left(u_Q^{p-1}+1\right)^p - 1}$. In Section 4 we study U_q for both types of local Frobenius endomorphisms, and in Section 5.2 we study the local versions of the Frobenius structure α.

1.2.2. The problem of *a*th roots of $U_q \circ \alpha$

To obtain the correct estimates of $\det\left(1 - sU_q \circ \alpha \mid B^\dagger\right)$, it is necessary to work with an ath root of $U_q \circ \alpha$. That is, we need an element $\alpha_0 \in B^\dagger$ and a U_p operator (this is analogous to the U_q operator, but for liftings of the p-power endomorphism) such that $\left(U_p \circ \alpha_0\right)^a = U_q \circ \alpha$. However, this ath root is only guaranteed to exist if the order of $Im(\chi)$ divides $p - 1$ (see Section 5.1). This presents a major technical obstacle. The solution is to consider $\rho^{wild} \otimes \bigoplus_{j=0}^{a-1} \chi^{\otimes p^j}$, which is a restriction of scalars of ρ. The L-functions of each summand are Galois conjugate, and thus have the same Newton polygon. We can then study an operator $U_p \circ N$, where N is the Frobenius structure of the F-crystal associated to $\rho^{wild} \otimes \bigoplus_{j=0}^{a-1} \chi^{\otimes p^j}$. This is similar to the idea used in Adolphson and Sperber's study of twisted exponential sums on tori [2]. They present it in an ad hoc manner, but the underlying idea is to study $\rho^{wild} \otimes \bigoplus_{j=0}^{a-1} \chi^{\otimes p^j}$ in lieu of ρ.

1.2.3. Global to local computations

When V is $\mathbb{G}_m$ or $\mathbb{A}^1$, the ring $B^\dagger$ is just $\mathcal{O}_L\langle t\rangle^\dagger$ or $\mathcal{O}_L\left\langle t, t^{-1}\right\rangle^\dagger$. In both cases, it is relatively easy to study operators on $B^\dagger$. The situation is more complex for higher-genus curves. Our approach to make sense of $B^\dagger$ is to 'expand' each function around the τ_i (and some other auxiliary points). Namely, let $t_i \in B^\dagger$ be a function whose reduction in $\overline{B}$ has a simple zero at τ_i. We let $\mathcal{O}_{\mathcal{E}_i^\dagger}$ be the ring of formal Laurent series in t_i that converge on an annulus $r < |t_i|_p < 1$ (i.e., the bounded Robba ring). Any

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$f \in B^\dagger$ has a Laurent expansion in t_i, and our overconvergence condition implies this expansion lies in $\mathcal{O}_{\mathcal{E}_i^\dagger}$. We obtain an injection

$$B^\dagger \hookrightarrow \bigoplus_{i=1}^{\mathbf{m}} \mathcal{O}_{\mathcal{E}_i^\dagger}. \tag{3}$$

The operator $U_p \circ N$ extends to an operator on each summand. By carefully keeping track of the image of $B^\dagger$, we are able to compute on each summand (see Section 7.2). This lets us compute $U_p \circ N$ on the bounded Robba ring, which ostensibly looks like a ring of functions on $\mathbb{G}_m$. We are thus able to compute $U_p \circ N$ by studying local Frobenius structures and local U_p operators.

1.2.4. Comparing Frobenius structures and Ω_ρ

In Section 5.2 we study the shape of the unit-root F-crystal associated to ρ when we localise at a ramified point τ_i. We show that the localised unit-root F-crystal has a particularly nice Frobenius structure, which depends on the ramification datum. However, these well-behaved local Frobenius structures do not patch together to give a well-behaved global Frobenius structure. This is a major technical obstacle. When comparing local and global Frobenius structures, we end up having to 'twist' the image of formula (3). This process explains the invariant Ω_ρ occurring in Theorem 1.1 – it arises by 'averaging' the local exponents for each $\rho^{wild} \otimes \chi^{\otimes p^i}$. This invariant is essentially absent in the work of Adolphson and Sperber, since $\Omega_\rho = 1$ if $V = \mathbb{G}_m$. It is also absent in the author's previous work, where the local exponents were all zero.

1.3. Further remarks

Pinning down the exact Newton polygon of a covering of a curve, as well as the Newton polygon of the isotypical constituents, is a fascinating question. A general answer seems impossible, but one can certainly hope for results that hold generically. If the genus of X and the monodromy invariants from Section 1.1 are specified, what is the Newton polygon for a generic character? We believe the bound from Theorem 1.1 should only be generically attained if $N \mid p-1$ and there are some congruence relations between p and the Swan conductors. When ρ factors through an Artin–Schreier cover, this is known by combining work of the author [15] with work of Booher and Pries [5]. The next step would be to study the case arising from a cyclic cover whose degree divides $p(p-1)$ (or even allowing higher powers of p). When $N \nmid p-1$, the bound from Theorem 1.1 has too many slope 0 segments. The issue is that a generic tame cyclic cover of degree N is not ordinary, even if X is ordinary [6]. Even when $X = \mathbb{P}^1$, the study of Newton polygons for tame cyclic covers is already a complicated topic (e.g., [16]). The author plans to return to these questions at a later time. It would also be interesting to prove Hodge bounds for representations with positive weight. In recent work, Fresán, Sabbah and Yu use irregular Hodge theory to study the p-adic slopes of symmetric powers of Kloosterman sums [10]. Not much is known beyond this case.

2. Notation

2.1. Conventions

The following conventions will be used throughout the article. We let $\mathbb{F}_q$ be an extension of $\mathbb{F}_p$ with $a = [\mathbb{F}_q : \mathbb{F}_p]$. It is enough to prove Theorem 1.1 after replacing q with a larger power of p. In particular, we increase q throughout the article if it simplifies arguments. We will frequently have families of things indexed by $i = 0, \dots, a-1$ (e.g., the p-adic digits $e_{Q,i}$ of ϵ_Q from Section 1.1). It will be convenient to have the indices 'wrap around' modulo a. That is, we take $e_{Q,a}$ to be $e_{Q,0}$, $e_{Q,a+1}$ to be $e_{Q,1}$ and so forth.

Let L_0 be the unramified extension of $\mathbb{Q}_p$ whose residue field is $\mathbb{F}_q$. Let E be a finite totally ramified extension of $\mathbb{Q}_p$ of degree e and set $L = E \otimes_{\mathbb{Q}_p} L_0$. Define $\mathcal{O}_L$ (resp., $\mathcal{O}_E$) to be the ring of integers of L (resp., E) and let $\mathfrak{m}$ be the maximal ideal of $\mathcal{O}_L$. We let $\pi_\circ$ be a uniformising element of E. Fix

$\pi = (-p)^{\frac{1}{p-1}}$, and for any positive rational number s we set $\pi_s = \pi^{\frac{1}{s}}$. We will assume that E is large enough to contain $\pi_{s_{\tau_i}}$ for each $i = 1, \ldots, \mathbf{m}$. We also assume that E is large enough to contain the image of ρ^{wild} (i.e., E contains enough pth-power roots of unity). Define ν to be the endomorphism id $\otimes$ Frob of L, where Frob is the p-Frobenius automorphism of L_0. For any E-algebra R and $x \in R$, we obtain an operator $R \to R$ sending $r \mapsto xr$. By abuse of notation, we will refer to this operator as x. Finally, for any ring R with valuation $v : R \to \mathbb{R}$ and any $x \in R$ with $v(x) > 0$, we let $v_x(\cdot)$ denote the normalisation of v satisfying $v_x(x) = 1$.

2.2. Frobenius endomorphisms

Let $\overline{A}$ be an $\mathbb{F}_q$-algebra, let A be an $\mathcal{O}_L$-algebra with $A \otimes_{\mathcal{O}_L} \mathbb{F}_q = \overline{A}$ and let $\mathcal{A} = A \otimes_{\mathcal{O}_L} L$. A p-Frobenius endomorphism (resp., q-Frobenius endomorphism) of A is a ring endomorphism $\nu : A \to A$ (resp., $\sigma : A \to A$) that extends the map ν (resp., $\nu^a = id$) on $\mathcal{O}_L$ defined in Section 2.1 and reduces to the pth-power map (resp., qth-power map) of $\overline{A}$. Note that ν (resp., σ) extends to a map $\nu : \mathcal{A} \to \mathcal{A}$ (resp., $\sigma : \mathcal{A} \to \mathcal{A}$), which we refer to as a p-Frobenius endomorphism (resp., q-Frobenius endomorphism) of $\mathcal{A}$. For a square matrix $M = (m_{i,j})$ with entries in $\mathcal{A}$, we take M^{ν^k} to mean the matrix $\left(m_{i,j}^{\nu^k}\right)$ and we define $M^{\nu^{a-1}+\cdots+\nu+1}$ by $M^{\nu^{a-1}} \cdots M^{\nu} M$.

2.3. Definitions of local rings

We begin by defining some rings and modules which will be used throughout this article. Define the L-algebras

$$\mathcal{E}_t = \left\{ \sum_{-\infty}^{\infty} a_n t^n \;\middle|\; \begin{array}{l} \text{We have } a_n \in L, \ \lim_{n\to-\infty} v_p(a_n) = \infty, \\ \text{and } v_p(a_n) \text{ is bounded below.} \end{array} \right\},$$

$$\mathcal{E}_t^\dagger = \left\{ \sum_{-\infty}^{\infty} a_n t^n \in \mathcal{E} \;\middle|\; \begin{array}{l} \text{There exists } m > 0 \text{ such that} \\ v_p(a_n) \geq -mn \text{ for } n \ll 0. \end{array} \right\}.$$

We refer to $\mathcal{E}_t$ $\left(\text{resp., } \mathcal{E}_t^\dagger\right)$ as the Amice ring (resp., the bounded Robba ring) over L with parameter t. We will often omit the t in the subscript if there is no ambiguity. Note that $\mathcal{E}^\dagger$ and $\mathcal{E}$ are local fields with residue field $\mathbb{F}_q((t))$. The valuation v_p on L extends to the Gauss valuation on each of these fields. We define $\mathcal{O}_\mathcal{E}$ (resp., $\mathcal{O}_{\mathcal{E}^\dagger}$) to be the subring of $\mathcal{E}$ $\left(\text{resp., } \mathcal{E}^\dagger\right)$ consisting of formal Laurent series with coefficients in $\mathcal{O}_L$. Let $u \in \mathcal{O}_{\mathcal{E}^\dagger}$ be such that the reduction of u in $\mathbb{F}_q((t))$ is a uniformising element. Then we have $\mathcal{E}_u = \mathcal{E}$ $\left(\text{resp., } \mathcal{E}_u^\dagger = \mathcal{E}^\dagger\right)$. In particular, we see that u is a different parameter of $\mathcal{E}$. Note that if $\nu : \mathcal{E} \to \mathcal{E}$ is any p-Frobenius endomorphism, we have $\mathcal{E}^{\nu=1} = E$. For $m \in \mathbb{Z}$, we define the L-vector space of truncated Laurent series

$$\mathcal{E}^{\leq m} = \left\{ \sum_{-\infty}^{\infty} a_n t^n \in \mathcal{E} \;\middle|\; a_n = 0 \text{ for all } n > m \right\}.$$

The space $\mathcal{E}^{\leq 0}$ is a ring, and $\mathcal{E}^{\leq m}$ is an $\mathcal{E}^{\leq 0}$-module. There is a natural projection $\mathcal{E} \to \mathcal{E}^{\leq m}$ given by truncating the Laurent series. Finally, we define the following $\mathcal{O}_L$-algebra:

$$\mathcal{O}_{\mathcal{E}(0,r]} = \left\{ \sum_{-\infty}^{\infty} a_n t^n \in \mathcal{O}_\mathcal{E} \;\middle|\; \lim_{n\to-\infty} v_p(a_n) + rn = \infty \right\}.$$

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Set $\mathcal{E}(0,r] = \mathcal{O}_{\mathcal{E}(0,r]} \otimes_{\mathcal{O}_L} L$. Note that $\mathcal{E}(0,r]$ is the ring of bounded functions on the closed annulus $0 < v_p(t) \leq r$. In particular, we have $\mathcal{E}^\dagger = \bigcup_{r>0} \mathcal{E}(0,r]$.

2.4. *Matrix notation*

For any $c_0, \ldots, c_{a-1} \in \mathcal{E}$, we define the following $a \times a$ matrices:

$$\mathbf{diag}\left(c_0, \ldots, c_{a-1}\right) = \begin{pmatrix} c_0 & & \\ & \ddots & \\ & & c_{a-1} \end{pmatrix},$$

$$\mathbf{cyc}\left(c_0, \ldots, c_{a-1}\right) = \begin{pmatrix} & c_0 & & \\ & & \ddots & \\ & & & c_{a-2} \\ c_{a-1} & & & \end{pmatrix},$$

$$\mathbf{tcyc}\left(c_0, \ldots, c_{a-1}\right) = \mathbf{cyc}\left(c_0, \ldots, c_{a-1}\right)^{\mathrm{T}}.$$

3. Global setup

We now introduce the global setup, which closely follows [15, Section 3]. We adopt the notation from Section 1.1. Our main goal is to choose a Frobenius endomorphism on a lift of an affine subspace of X. We require two things from this Frobenius endomorphism. First, we want an endomorphism that behaves reasonably with respect to certain local parameters. Second, it should make the Monsky trace formula satisfy a certain form (see Section 7.1). We find this Frobenius endomorphism by bootstrapping from the standard Frobenius endomorphism on the projective line.

3.1. *Mapping to* $\mathbb{P}^1$

Lemma 3.1. *After increasing q, there exists a tamely ramified morphism $\eta : X \to \mathbb{P}^1_{\mathbb{F}_q}$, ramified only above* $0, 1,$ *and* ∞*, such that* $\tau_1, \ldots, \tau_{\mathbf{m}} \in \eta^{-1}(\{0,\infty\})$ *and each* $P \in \eta^{-1}(1)$ *has ramification index* $p-1$.

Proof. This is [15, Lemma 3.1]. □

3.2. *Basic setup*

Write $\mathbb{P}^1_{\mathbb{F}_q} = \mathrm{Proj}\left(\mathbb{F}_q[x_1, x_2]\right)$ and let $\bar{t} = \frac{x_1}{x_2}$ be a parameter at 0. Fix a morphism η as in Lemma 3.1. For $* \in \{0, 1, \infty\}$, we let $\left\{P_{*,1}, \ldots, P_{*,r_*}\right\} = \eta^{-1}(*)$ and set $W = \eta^{-1}(\{0,1,\infty\})$. Again, we will increase q so that each $P_{*,i}$ is defined over $\mathbb{F}_q$. Fix $Q = P_{*,i} \in W$. We define e_Q to be the ramification index of Q over $*$. From Lemma 3.1, if $* = 1$ we have $e_Q = p-1$ for $1 \leq i \leq r_1$, so that $r_1(p-1) = \deg(\eta)$. Also, by the Riemann–Hurwitz formula,

$$(g-1) + (r_0 + r_1 + r_\infty) = \deg(\eta) - g + 1, \tag{4}$$

where g denotes the genus of X. Let $U = \mathbb{P}^1_{\mathbb{F}_q} - \{0,1,\infty\}$ and $V = X - W$. Then $\eta : V \to U$ is a finite étale map of degree $\deg(\eta)$. Let $\overline{B}$ $\left(\text{resp., } \overline{A}\right)$ be the $\mathbb{F}_q$-algebra such that $V = \mathrm{Spec}\left(\overline{B}\right)$ $\left(\text{resp., } U = \mathrm{Spec}\left(\overline{A}\right)\right)$.

Let $\mathbb{P}^1_{\mathcal{O}_L}$ be the projective line over $\mathrm{Spec}(\mathcal{O}_L)$ and let $\mathbf{P}^1_{\mathcal{O}_L}$ be the formal projective line over $\mathrm{Spf}(\mathcal{O}_L)$. Let t be a global parameter of $\mathbf{P}^1_{\mathcal{O}_L}$ lifting $\bar{t}$. By the deformation theory of tame coverings

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[11, Theorem 4.3.2], there exists a tame cover $\mathbf{X} \to \mathbf{P}^1_{\mathcal{O}_L}$ whose special fibre is η, and by formal GAGA [26, Tag 09ZT] there exists a morphism of smooth curves $\mathbb{X} \to \mathbb{P}^1_{\mathcal{O}_L}$ whose formal completion is $\mathbf{X} \to \mathbf{P}^1_{\mathcal{O}_L}$.

Define the functions $t_0 = t$, $t_\infty = \frac{1}{t}$ and $t_1 = t - 1$. Let $[*]$ denote the $\mathcal{O}_L$-point of $\mathbb{P}^1_{\mathcal{O}_L}$ given by $t_* = 0$. For $Q = P_{*,i}$, let $[Q]$ be a point of $\eta^{-1}([*])$ that reduces to Q in the special fibre. Note that such a point exists because $Q \in \eta^{-1}(*)$, but it is not necessarily unique. Let $\mathbb{U} = \mathbb{P}^1_{\mathcal{O}_L} - \{[0],[1],[\infty]\}$ and $\mathbb{V} = \mathbb{X} - \{[R]\}_{R\in W}$. We define $\mathbf{U} = \mathbf{P}^1_{\mathcal{O}_L} - \{0,1,\infty\}$ and $\mathbf{V} = \mathbf{X} - \{R\}_{R\in W}$. Then $\mathbf{U}$ (resp., $\mathbf{V}$) is the formal completion of $\mathbb{U}$ (resp., $\mathbb{V}$). We let $\mathcal{U}^{rig}$ (resp., $\mathcal{V}^{rig}$) be the rigid analytic fibre of $\mathbf{U}$ (resp., $\mathbf{V}$). Let $\widehat{A}$ $\left(\text{resp., } \widehat{\mathcal{A}}\right)$ be the ring of functions $\mathcal{O}_{\mathbf{U}}(\mathbf{U})$ (resp., $\mathcal{O}_{\mathcal{U}^{rig}}(\mathcal{U}^{rig})$) and let $\widehat{B}$ $\left(\text{resp., } \widehat{\mathcal{B}}\right)$ be the ring of functions $\mathcal{O}_{\mathbf{V}}(\mathbf{V})$ (resp., $\mathcal{O}_{\mathcal{V}^{rig}}(\mathcal{V}^{rig})$).

3.3. *Local parameters and overconvergent rings*

For $Q = P_{*,i}$, let w_Q be a rational function on $\mathbb{X}$ that has a simple zero at Q. Let $\mathcal{E}_*$ (resp., $\mathcal{E}_Q$) be the Amice ring over L with parameter t_* (resp., w_Q). By expanding functions in terms of the t_* and w_Q, we obtain the following diagrams:

$$\begin{tikzcd} \widehat{B} \arrow[r] & \bigoplus_{Q\in W} \mathcal{O}_{\mathcal{E}_Q} \\ \widehat{A} \arrow[r] \arrow[u] & \bigoplus_{*\in\{0,1,\infty\}} \mathcal{O}_{\mathcal{E}_*}, \arrow[u] \end{tikzcd} \qquad \begin{tikzcd} \widehat{\mathcal{B}} \arrow[r] & \bigoplus_{Q\in W} \mathcal{E}_Q \\ \widehat{\mathcal{A}} \arrow[r] \arrow[u] & \bigoplus_{*\in\{0,1,\infty\}} \mathcal{E}_*. \arrow[u] \end{tikzcd} \tag{5}$$

We let $A^\dagger$ (resp., $B^\dagger$) be the subring of $\widehat{A}$ $\left(\text{resp., } \widehat{B}\right)$ consisting of functions that are overconvergent in the tube $]*[$ for each $* \in \{0,1,\infty\}$ (resp., $]Q[$ for all $Q \in W$). In particular, $B^\dagger$ fits into the following Cartesian diagram:

$$\begin{tikzcd} B^\dagger \arrow[r] \arrow[d] & \bigoplus_{Q\in W} \mathcal{O}^\dagger_{\mathcal{E}_Q} \arrow[d] \\ \widehat{B} \arrow[r] & \bigoplus_{Q\in W} \mathcal{O}_{\mathcal{E}_Q}. \end{tikzcd} \tag{6}$$

Note that $A^\dagger$ (resp., $B^\dagger$) is the weak completion of A (resp., B) in the sense of [23, Section 2]. In particular, we have $A^\dagger = \mathcal{O}_L\left\langle t, t^{-1}, \frac{1}{t-1}\right\rangle^\dagger$ and $B^\dagger$ is a finite étale $A^\dagger$-algebra. Finally, we define $\mathcal{A}^\dagger$ $\left(\text{resp., } \mathcal{B}^\dagger\right)$ to be $A^\dagger \otimes \mathbb{Q}_p$ (resp., $B^\dagger \otimes \mathbb{Q}_p$). Then $\mathcal{A}^\dagger$ $\left(\text{resp., } \mathcal{B}^\dagger\right)$ is equal to the functions in $\widehat{\mathcal{A}}$ $\left(\text{resp., } \widehat{\mathcal{B}}\right)$ that are overconvergent in the tube $]*[$ for each $* \in \{0,1,\infty\}$ (resp., $]R[$ for all $R \in W$).

The extension $\mathcal{E}^\dagger_Q \big/ \mathcal{E}^\dagger_*$ is an unramified extension of local fields and thus completely determined by the residual extension. By our assumption on the tameness of η, we know that this residual extension is tame and can be written as $\mathbb{F}_q\left(\left(t_*^{\frac{1}{e_Q}}\right)\right) \Big/ \mathbb{F}_q((t_*))$. Since $\mathcal{O}_{\mathcal{E}^\dagger_Q}$ is Henselian [19, Proposition 3.2], there exists a parameter u_Q of $\mathcal{E}^\dagger_Q$ such that $u_Q^{e_Q} = t_*$. We remark that u_Q will be defined on an annulus inside the disc $]Q[$, and in general it will not extend to a function on the whole disc.

We will need to consider functions in $\mathcal{B}^\dagger$ with a precise radius of overconvergence in terms of the parameters u_Q. Let $\mathbf{r} = (r_Q)_{Q\in W}$ be a tuple of positive rational numbers. We define $\mathcal{B}(0,\mathbf{r}]$ to be the

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subring of functions in $\mathcal{B}^\dagger$ that overconverge in the annulus $0 < v_p\left(u_Q\right) \leq r_Q$.[1] More precisely, $\mathcal{B}(0,\mathbf{r}]$ fits into the following Cartesian diagram:

$$\begin{tikzcd} \mathcal{B}(0,\mathbf{r}] \arrow[r] \arrow[d] & \bigoplus_{Q\in W} \mathcal{E}_Q\left(0, r_Q\right] \arrow[d] \\ \widehat{\mathcal{B}} \arrow[r] & \bigoplus_{Q\in W} \mathcal{E}_Q. \end{tikzcd} \tag{7}$$

Note that $\mathcal{B}^\dagger$ is the union over all $\mathcal{B}(0,\mathbf{r}]$.

3.4. Global Frobenius and U_p operators

Let $\nu : \mathcal{A}^\dagger \to \mathcal{A}^\dagger$ be the p-Frobenius endomorphism that restricts to ν on L and sends t to t^p. Let $\sigma = \nu^a$. For $* \in \{0,1,\infty\}$, we may extend ν to a p-Frobenius endomorphism of $\mathcal{E}_*^\dagger$, which we refer to as ν_*. In terms of the parameters t_*, these endomorphisms are given as follows:

$$t_0 \mapsto t_0^p, \quad t_\infty \mapsto t_\infty^p, \quad t_1 \mapsto (t_1+1)^p - 1.$$

Since the map $\widehat{A} \to \widehat{B}$ is étale and both rings are p-adically complete, we may extend both σ and ν to maps $\sigma,\nu : \widehat{B} \to \widehat{B}$. This extends to a p-Frobenius endomorphism ν_Q of $\mathcal{E}_Q$, which makes the diagrams (5) p-Frobenius equivariant. Furthermore, since ν_Q extends ν_*, we know that ν_Q induces a p-Frobenius endomorphism of $\mathcal{E}_Q^\dagger$. It follows from diagram (6) that σ and ν restrict to endomorphisms $\sigma,\nu : \mathcal{B}^\dagger \to \mathcal{B}^\dagger$. The p-Frobenius endomorphisms ν_Q can be described as follows:

1. When $*$ is 0 or ∞, have $u_Q^{\nu_Q} = u_Q^p$, since $t_*^{\nu_*} = t_*^p$ and $u_Q^{e_Q} = t_*$.
2. When $* = 1$, we have $u_Q^{\nu_Q} = \sqrt[p-1]{\left(u_Q^{p-1}+1\right)^p - 1}$, since $t_1^{\nu_1} = (t_1+1)^p - 1$ and $u_Q^{p-1} = t_1$.

Following [28, Section 3], there is a trace map $Tr_0 : \mathcal{B}^\dagger \to \nu\left(\mathcal{B}^\dagger\right)$ $\left(\text{resp., } Tr : \mathcal{B}^\dagger \to \sigma\left(\mathcal{B}^\dagger\right)\right)$. We may define the U_p operator on $\mathcal{B}^\dagger$:

$$\begin{aligned} U_p : \mathcal{B}^\dagger &\to \mathcal{B}^\dagger \\ x &\mapsto \frac{1}{p}\nu^{-1}(Tr_0(x)). \end{aligned}$$

Similarly, we define $U_q = \frac{1}{q}\sigma^{-1} \circ Tr$, so that $U_p^a = U_q$. Note that U_p is E-linear and U_q is L-linear. Both U_p and U_q extend to operators on $\mathcal{E}_Q^\dagger$.

[1] Our definition of $\mathcal{B}(0,\mathbf{r}]$ is somewhat nonstandard. Typically one measures 'overconvergence' with global functions. However, if the r_Q are small enough, the spaces of functions we obtain are essentially the same. For example, consider the modular-curve case (i.e., $X = X_0(N)$ and V is the ordinary locus). One typically looks at affinoid spaces $\mathcal{X}_0(N)^{r'}$ of the form $0 \leq v_p\left(E_{p-1}\right) < r'$, where E_{p-1} is the weight $p-1$ Eisenstein series (i.e., a lift of the Hasse invariant). If r' is sufficiently small, we have $\mathcal{O}_{\mathcal{X}_0(N)}\left(\mathcal{X}_0(N)^{r'}\right) = \mathcal{B}(0,\mathbf{r}']$, where $\mathbf{r}'$ is the tuple with r' in each coordinate. This follows from the following two facts. First, note that E_{p-1} is a parameter of $\mathcal{E}_Q^\dagger$ for each supersingular point $Q \in X$, since the Hasse invariant has simple zeros at supersingular points. Second, if $u \in \mathcal{E}_t^\dagger$ is a parameter and $u \in \mathcal{E}_t(0,r]$, then we have $\mathcal{E}_u(0,r_0] = \mathcal{E}_t(0,r_0]$ for any $r_0 < r$.

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4. Local U_p operators

Let ν be a p-Frobenius endomorphism of $\mathcal{E}^\dagger$ (see Section 2.2). We define U_p to be the map

$$\frac{1}{p}\nu^{-1} \circ \mathrm{Tr}_{\mathcal{E}^\dagger/\nu(\mathcal{E}^\dagger)} : \mathcal{E}^\dagger \to \mathcal{E}^\dagger.$$

Note that U_p is ν^{-1}-semilinear (i.e., $U_p(y^\nu x) = yU_p(x)$ for all $y \in \mathcal{E}^\dagger$). In this section we will study U_p for the p-Frobenius endomorphisms of $\mathcal{E}^\dagger$ appearing in Section 3.4.

4.1. Type 1: $t \mapsto t^p$

First consider the p-Frobenius endomorphism $\nu : \mathcal{E}^\dagger \to \mathcal{E}^\dagger$ sending t to t^p. We see that $U_p\left(t^i\right) = 0$ if $p \nmid i$ and $U_p\left(t^i\right) = t^{\frac{i}{p}}$ if $p \mid i$. Thus, for $s > 0$ we have

$$U_p\left(\mathcal{O}_{\mathcal{E}}^s\right) \subset \mathcal{O}_{\mathcal{E}}^{\frac{s}{p}} \quad \text{and} \quad U_p\left(\mathcal{O}_L\left[\left[\pi_s t^{-1}\right]\right]\right) \subset \mathcal{O}_L\left[\left[\pi_s^p t^{-1}\right]\right]. \tag{8}$$

4.1.1. Local estimates

Let $R = (s, \mathbf{e}, \epsilon, \omega)$ be a ramification datum and let $e_0, \dots, e_{a-1}$ be the p-adic digits of ϵ as in Section 1.1. For $j = 0, \dots, a-1$, we define

$$q(\mathbf{e}, j) = -\sum_{i=0}^{a-1}(i+1)e_{i+j}.$$

Note that

$$q(\mathbf{e}, j) - q(\mathbf{e}, j+1) = ae_j - \omega. \tag{9}$$

Let $t_i^n \in \bigoplus_{j=0}^{a-1}\mathcal{E}^\dagger$ denote the element that has t^n in the ith coordinate and zero in the other coordinates. We then define the spaces

$$\mathcal{D}_{\mathbf{e},s}^{(j)} = \pi_{as}^{q(\mathbf{e},j)}\pi_s^p t_j^{-1}\mathcal{O}_L\left[\left[\pi_s^p t_j^{-1}\right]\right] \oplus \mathcal{O}_L\left[\left[t_j\right]\right],$$
$$\mathcal{D}_{\mathbf{e},s} = \bigoplus_{j=0}^{a-1}\mathcal{D}_{\mathbf{e},s}^{(j)} \subset \bigoplus_{j=0}^{a-1}\mathcal{E}^\dagger.$$

We know $-q(\mathbf{e}, i) \le a(p-1)$, which implies $\pi_{as}^{q(\mathbf{e},j)}\pi_s^p \in \mathcal{O}_L$. In particular,

$$\mathcal{D}_{\mathbf{e},s} \subset \bigoplus_{j=0}^{a-1}\mathcal{O}_{\mathcal{E}^\dagger}.$$

Proposition 4.1. *Let ν be the p-Frobenius endomorphism that sends $t \mapsto t^p$. Set $\alpha \in \mathcal{O}_L\left[\left[\pi_s t^{-1}\right]\right]$ and set $A = \mathbf{tcyc}\left(\alpha t^{-e_0}, \dots, \alpha t^{-e_{a-1}}\right)$. Then*

$$U_p \circ A\left(\mathcal{D}_{\mathbf{e},s}\right) \subset \mathcal{D}_{\mathbf{e},s}, \tag{10}$$

$$U_p \circ A\left(\pi_{as}^{q(\mathbf{e},j)}\pi_s^{np}t_j^{-n}\right) \subset \pi_s^{n(p-1)}\pi_{as}^{-\omega}\mathcal{D}_{\mathbf{e},s}, \tag{11}$$

for $n \ge 1$ and $0 \le j \le a-1$.

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Proof. For $n \geq 1$ we have $A\left(t_j^{-n}\right) = \alpha t_{j+1}^{-n-e_{j+1}}$. Then from equation (9) we have

$$\begin{aligned} A\left(\pi_{as}^{q(\mathbf{e},j)}\pi_s^{pn}t_j^{-n}\right) &= \pi_{as}^{q(\mathbf{e},j)}\pi_s^{pn}t_{j+1}^{-n-e_{j+1}}\alpha \\ &= \pi_{as}^{q(\mathbf{e},j+1)}\pi_s^{n(p-1)}\pi_{as}^{-\omega}\cdot\left(\pi_s^{n+e_{j+1}}t_{j+1}^{-n-e_{j+1}}\alpha\right). \end{aligned}$$

Note that $\pi_s^{n+e_{j+1}}t_{j+1}^{-n-e_{j+1}}\alpha \in \mathcal{O}_L\left[\left[\pi_s t_{j+1}^{-1}\right]\right]$. Then formula (11) follows from formula (8). To prove formula (10), we need to make sure $U_p \circ A\left(t_j^n\right) \in \mathcal{D}_{\mathbf{e},s}$ for $n \geq 0$, which can be done by a similar argument. □

4.2. *Type 2:* $t \mapsto \sqrt[p-1]{\left(t^{p-1}+1\right)^p - 1}$

Next, consider the p-Frobenius endomorphism $\nu : \mathcal{E}^\dagger \to \mathcal{E}^\dagger$ that sends t to $\sqrt[p-1]{\left(t^{p-1}+1\right)^p - 1}$. Define the following sequence of numbers:

$$b(n) = \begin{cases} \left\lfloor \frac{-n-1}{p-1} \right\rfloor, & n \leq -1, \\ 0, & n \geq 0. \end{cases}$$

We then define the space

$$\mathcal{D} = \prod_{n\in\mathbb{Z}} p^{b(n)}t^n\mathcal{O}_L, \tag{12}$$

which we regard as a sub- $\mathcal{O}_L$-module of $\mathcal{O}_{\mathcal{E}^\dagger}$.

Proposition 4.2. *Let ν be the p-Frobenius endomorphism of $\mathcal{E}^\dagger$ that sends t to $\sqrt[p-1]{\left(t^{p-1}+1\right)^p - 1}$. For all $n \in \mathbb{Z}_{\geq 0}$ and $0 \leq k \leq p-1$, we have*

$$\begin{aligned} U_p\left(p^{b(-k-np)}t^{-k-np}\right) &\in p^n\mathcal{D}, \\ U_p(\mathcal{D}) &\subset \mathcal{D}. \end{aligned}$$

Proof. See [15, Proposition 4.4]. □

5. Unit-root F-crystals

5.1. *F-crystals and p-adic representations*

For this subsection, we let $\overline{S}$ be either Spec $\left(\mathbb{F}_q((t))\right)$ or a smooth, irreducible affine $\mathbb{F}_q$-scheme Spec $\left(\overline{R}\right)$. We let $S = \mathrm{Spec}(R)$ be a flat $\mathcal{O}_L$-scheme whose special fibre is $\overline{S}$ and assume that R is p-adically complete – for example, if $\overline{S} = \mathrm{Spec}\left(\mathbb{F}_q((t))\right)$, then we may take $R = \mathcal{O}_{\mathcal{E}}$. Fix a p-Frobenius endomorphism ν on R (as in Section 2.2). Then $\sigma = \nu^a$ is a q-Frobenius endomorphism.

Definition 5.1. A φ-module for σ over R is a locally free R-module M equipped with a σ-semilinear endomorphism $\varphi : M \to M$. That is, we have $\varphi(cm) = \sigma(c)\varphi(m)$ for $c \in R$.

Definition 5.2. A unit-root F-crystal M over $\overline{S}$ is a φ-module such that $\sigma^*\varphi : R \otimes_\sigma M \to M$ is an isomorphism. The rank of M is defined as the rank of the underlying R-module.

Theorem 5.3 (Katz [12, Section 4].). *There is an equivalence of categories*

$$\left\{\text{rank } d \text{ unit-root } F\text{-crystals over } \overline{S}\right\} \longleftrightarrow \left\{\text{continuous representations } \psi : \pi_1^{et}\left(\overline{S}\right) \to GL_d(\mathcal{O}_L)\right\}.$$

Let us describe a certain case of this correspondence. Let $\overline{S}_1 \to \overline{S}$ be a finite étale cover and assume that ψ comes from a map $\psi_0 : Gal\left(\overline{S}_1/\overline{S}\right) \to GL_d(\mathcal{O}_E)$. This cover deforms into a finite étale map of affine schemes $S_1 = \mathrm{Spec}(R_1) \to S$. Both ν and σ extend to R_1 and commute with the action of $Gal\left(\overline{S}_1/\overline{S}\right)$ (see, e.g., [27, Section 2.6]). Let V_0 be a free $\mathcal{O}_E$-module of rank d on which $Gal\left(\overline{S}_1/\overline{S}\right)$ acts via ψ_0 and let $V = V_0 \otimes_{\mathcal{O}_E} \mathcal{O}_L$. The unit-root F-crystal associated to ψ is $M_\psi = \left(R_1 \otimes_{\mathcal{O}_L} V\right)^{Gal\left(\overline{S}_1/\overline{S}\right)}$ with $\varphi = \sigma \otimes_{\mathcal{O}_L} id$. There is a map

$$\left(S_1 \otimes_{\mathcal{O}_E} V_0\right) \to \left(S_1 \otimes_{\mathcal{O}_L} V\right),$$

which is Galois equivariant. In particular, the map φ has an ath root $\varphi_0 = \nu \otimes_{\mathcal{O}_E} id$.

Now make the additional assumption that M_ψ is free as an R-module. Let $e_1, \ldots, e_d$ be a basis of M_ψ as an R-module and let $\mathbf{e} = [e_1, \ldots, e_d]$. Then $\varphi(\mathbf{e}) = \alpha\mathbf{e}$ (resp., $\varphi_0(\mathbf{e}) = \alpha_0\mathbf{e}$), where $\alpha, \alpha_0 \in GL_d(R)$. We refer to the matrix α (resp., α_0) as a *Frobenius structure* (resp., *p-Frobenius structure*) of M and to the matrix α^{T} (resp., α_0^{T}) as a *dual Frobenius structure* (resp., *dual p-Frobenius structure*) of M. We have the relation $\alpha^{\mathrm{T}} = \left(\alpha_0^{\mathrm{T}}\right)^{\nu^{a-1}+\cdots+\nu+1}$ – recall from Section 2.2 that $\left(\alpha_0^{\mathrm{T}}\right)^{\nu^{a-1}+\cdots+\nu+1} = \left(\alpha_0^{\mathrm{T}}\right)^{\nu^{a-1}} \cdots \left(\alpha_0^{\mathrm{T}}\right)^{\nu} \alpha_0^{\mathrm{T}}$. If $\mathbf{e}' = b\mathbf{e}$ and $\varphi(\mathbf{e}') = \alpha'\mathbf{e}'$ (resp., $\varphi(e_1) = \alpha_0' e_1$) with $\alpha', \alpha_0', b \in GL_d(R)$, then we have $(\alpha')^{\mathrm{T}} = (b^\sigma)^{\mathrm{T}}\alpha^{\mathrm{T}}\left(b^{-1}\right)^{\mathrm{T}}$ $\left(\text{resp., } \left(\alpha_0'\right)^{\mathrm{T}} = (b^\nu)^{\mathrm{T}}\alpha_0^{\mathrm{T}}\left(b^{-1}\right)^{\mathrm{T}}\right)$. In particular, a dual Frobenius structure (resp., dual p-Frobenius structure) of M is unique up to σ-skew conjugation (resp., ν-skew conjugation) by elements of $GL_d(R)$. We remark that if M_ψ has rank 1, then p-Frobenius structures (resp., Frobenius structures) are also dual p-Frobenius structures (resp., dual Frobenius structures).

5.2. *Local Frobenius structures*

We now restrict ourselves to the case when $\overline{S} = \mathrm{Spec}\left(\mathbb{F}_q((t))\right)$. In particular, unit-root F-crystals over $\overline{S}$ correspond to representations of $G_{\mathbb{F}_q((t))}$, the absolute Galois group of $\mathbb{F}_q((t))$. Note that since $\mathcal{O}_{\mathcal{E}}$ is a local ring, all locally free modules are free.

5.2.1. Unramified Artin–Schreier–Witt characters

Proposition 5.4. *Let ν be any p-Frobenius endomorphism of $\mathcal{O}_{\mathcal{E}}$ and let $\sigma = \nu^a$. Let $\psi : G_{\mathbb{F}_q((t))} \to \mathcal{O}_L^\times$ be a continuous character and let M_ψ be the corresponding unit-root F-crystal. Assume that $Im(\psi) \cong \mathbb{Z}/p^n\mathbb{Z}$ and that ψ is unramified. Then there exists a p-Frobenius structure α_0 of M_ψ with $\alpha_0 \in 1 + \mathfrak{m}$ (recall that $\mathfrak{m}$ is the maximal ideal of $\mathcal{O}_L$). Furthermore, if $c \in 1+\mathfrak{m}\mathcal{O}_{\mathcal{E}}$ is another p-Frobenius structure of M_ψ, there exists $b \in 1 + \mathfrak{m}\mathcal{O}_{\mathcal{E}}$ with $\alpha_0 = \frac{b^\nu}{b}c$.*

Proof. This is essentially the same as [15, Proposition 5.4]. □

5.2.2. Wild Artin–Schreier–Witt characters

A global version over $\mathbb{G}_m$ of the following result is commonplace in the literature (see, e.g., [30, Section 4.1] for the exponential-sum situation or [18]). However, to the best of our knowledge, the local version presented here does not appear anywhere.

Proposition 5.5. *Let ν be the p-Frobenius endomorphism of $\mathcal{O}_{\mathcal{E}}$ sending t to t^p and let $\sigma = \nu^a$. Let $\psi : G_{\mathbb{F}_q((t))} \to \mathcal{O}_L^\times$ be a continuous character and let M_ψ be the corresponding unit-root F-crystal. Assume that $Im(\psi) \cong \mathbb{Z}/p^n\mathbb{Z}$. Let K be the fixed field of* $\ker(\psi)$ *and let s be the Swan conductor of ψ. We assume that $\pi_s \in \mathcal{O}_E$. Then there exists a p-Frobenius structure E_r of ψ such that $E_r \in \mathcal{O}_L\left[\left[\pi_s t^{-1}\right]\right]$ and*

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$E_r \equiv 1 \bmod \mathfrak{m}$. *Furthermore, if* $c \in 1+\mathfrak{m}\mathcal{O}_{\mathcal{E}}$ *is another p-Frobenius structure, there exists* $b \in 1+\mathfrak{m}\mathcal{O}_{\mathcal{E}}$ *with* $E_r = \frac{b^\nu}{b}c$.

Proof. The extension of $K/\mathbb{F}_q((t))$ corresponds to an equivalence class of $W_n\left(\mathbb{F}_q((t))\right)/(\mathbf{Fr}-1)W_n\left(\mathbb{F}_q((t))\right)$; here $W_n\left(\mathbb{F}_q((t))\right)$ is the nth truncated Witt vectors and $\mathbf{Fr}$ is the Frobenius map. Following [14, Proposition 3.3], we may represent this equivalence class with

$$r(t) = \sum_{i=0}^{n-1}\sum_{j=0}^{s_i}\left[r_{i,j}t^{-j}\right]p^i, \quad r_{i,s_i} \neq 0,$$
$$s = \min_{i=0}^{n-1}\left\{p^{n-i}s_i\right\},$$

where $r_{i,j} \in \mathbb{F}_q$. Since $r(t) \in W_n\left(\mathbb{F}_q\left[t^{-1}\right]\right)$, the extension $K/\mathbb{F}_q((t))$ extends to finite étale $\mathbb{F}_q\left[t^{-1}\right]$-algebra B that fits into a commutative diagram

$$\begin{array}{ccc}
\mathrm{Spec}(K) & \longrightarrow & \mathrm{Spec}(B) \\
\downarrow & & \downarrow \\
\mathrm{Spec}\left(\mathbb{F}_q((t))\right) & \longrightarrow & \mathbb{P}^1 - \{0\} = \mathrm{Spec}\left(\mathbb{F}_q\left[t^{-1}\right]\right).
\end{array}$$

In particular, ψ extends to a representation $\psi^{ext} : Gal\left(B/\mathbb{F}_q\left[t^{-1}\right]\right) \to \mathcal{O}_L^\times$. This extension is uniquely defined by the following property: For $k \geq 1$ and $x \in \mathbb{P}^1\left(\mathbb{F}_{q^k}\right) - \{0\}$, we have

$$\psi^{ext}(Frob_x) = \zeta_{p^n}^{Tr_{W_n\left(\mathbb{F}_{q^k}\right)/W_n(\mathbb{F}_p)}(r([x]))}, \tag{13}$$

where $[x]$ denotes the Teichmüller lift of x in $W_n\left(\mathbb{F}_{q^k}\right)$ and ζ_{p^n} is a primitive p^nth root of unity. Let $\mathcal{O}_L\left\langle t^{-1}\right\rangle \subset \mathcal{O}_{\mathcal{E}}$ be the Tate algebra in t^{-1} with coefficients in $\mathcal{O}_L$. Note that ν restricts to a p-Frobenius endomorphism of $\mathcal{O}_L\left\langle t^{-1}\right\rangle$. All projective modules over $\mathcal{O}_L\left\langle t^{-1}\right\rangle$ are free, so that $M_{\psi^{ext}}$ is isomorphic to $\mathcal{O}_L\left\langle t^{-1}\right\rangle$ as an $\mathcal{O}_L\left\langle t^{-1}\right\rangle$-module. We see that $M_\psi = M_{\psi^{ext}} \otimes_{\mathcal{O}_L\langle t^{-1}\rangle} \mathcal{O}_{\mathcal{E}}$. In particular, any p-Frobenius structure of $M_{\psi^{ext}}$ is a p-Frobenius structure of M_ψ.

A series $\alpha_0 \in \mathcal{O}_L\left\langle t^{-1}\right\rangle$ is a p-Frobenius structure for $M_{\psi^{ext}}$ if for every $x \in \mathbb{P}^1\left(\mathbb{F}_{q^k}\right) - \{0\}$ we have

$$\prod_{i=0}^{ak-1}\alpha_0([x])^{\nu^i} = \psi^{ext}(Frob_x).$$

We let $E(x)$ denote the Artin–Hasse exponential and let γ_i be an element of $\mathbb{Z}_p\left[\zeta_{p^n}\right]$ with $E(\gamma_n) = \zeta_{p^n}^{p^{n-i}}$. Note that $v_p(\gamma_i) = \frac{1}{p^{i-1}(p-1)}$. Thus from equation (13) we see that

$$E_r = \prod_{i=0}^{n-1}\prod_{j=0}^{s_i}E\left(\left[r_{i,j}\right]t^{-j}\gamma_{n-i}\right)$$

is a p-Frobenius structure of $M_{\psi^{ext}}$. Since $E(x) \in \mathbb{Z}_p[[x]]$, it is clear that $E_r \in \mathcal{O}_L\left[\left[\pi_s t^{-1}\right]\right]$. □

5.2.3. Tame characters

Let $\psi : G_{\mathbb{F}_q((t))} \to \mathcal{O}_L^\times$ be a totally ramified tame character and let $T = (\mathbf{e}, \epsilon, \omega)$ be the corresponding tame ramification datum (see Section 1.1). Write $\epsilon = e_0 + \cdots + e_{a-1}p^{a-1}$ and define $\epsilon_j = \sum_{i=0}^{a-1} e_{i+j}p^i$.

Proposition 5.6. *The following hold:*

1. *The matrix* $C = \mathbf{diag}\,(t^{-\epsilon_0}, \ldots, t^{-\epsilon_{a-1}})$ *(resp.,* $C_0 = \mathbf{tcyc}\,(t^{-e_0}, \ldots, t^{-e_{a-1}})$*) is a dual Frobenius structure (resp., dual p-Frobenius structure) of* $\bigoplus_{j=0}^{a-1} \psi^{\otimes p^j}$ *and* $C = C_0^{\nu^{a-1}+\cdots+\nu+1}$.
2. *Let* $A = \mathbf{diag}\,(x_0, \ldots, x_{a-1})$ *(resp.,* $A_0 = \mathbf{tcyc}\,(y_0, \ldots, y_{a-1})$*) be another dual Frobenius structure (resp., dual p-Frobenius structure) of* $\bigoplus_{j=0}^{a-1} \psi^{\otimes p^j}$ *with* $A = A_0^{\nu^{a-1}+\cdots+\nu+1}$*. Then* $v_t\left(\overline{x_j}\right) = -\epsilon_j + n_j(q-1)$ *for some* $n_j \in \mathbb{Z}$ *(here* $\overline{x_j}$ *is the image of* x_j *in* $\mathbb{F}_q((t))$*). Furthermore, there exists* $B = \mathbf{diag}\,(b_0, \ldots, b_{a-1})$ *with* $v_t\left(\overline{b_j}\right) = n_j$ *such that* $B^{\sigma} A B^{-1} = C$ $\left(\textit{resp.}, B^{\nu} A_0 B^{-1} = C_0\right)$.

Proof. Let $G_{\mathbb{F}_q((t))}$ act on $\mathcal{L} = \bigoplus_{j=0}^{a-1} v_j \mathcal{O}_L$ via $\bigoplus_{j=0}^{a-1} \psi^{\otimes p^j}$. Let $u = t^{\frac{1}{q-1}}$ and let $\mathcal{E}'$ be the Amice ring over L with parameter u. The F-crystal associated to $\bigoplus_{j=0}^{a-1} \psi^{\otimes p^j}$ is $(\mathcal{O}_{\mathcal{E}'} \otimes \mathcal{L})^{G_{\mathbb{F}_q((t))}}$. In particular, we see that $\left\{u^{-\epsilon_j} \otimes v_j\right\}$ is a basis of $(\mathcal{O}_{\mathcal{E}'} \otimes \mathcal{L})^{G_{\mathbb{F}_q((t))}}$. The first part of the proposition follows from considering the action of ν and σ on this basis. To deduce the second part of the proposition, observe what happens when C and C_0 are skew-conjugated by a diagonal matrix. □

5.3. The F-crystal associated to ρ

We now continue with ρ from Section 1.1 and the setup from Section 3.

5.3.1. The Frobenius structure of ρ^{wild}

Let $\mathcal{L}$ be a rank 1 $\mathcal{O}_L$-module on which $\pi_1^{et}(V)$ acts through ρ^{wild}. Let $f : C \to X$ be the $\mathbb{Z}/p^n\mathbb{Z}$-cover that trivialises ρ^{wild}. Let $\overline{R}$ be the $\overline{B}$-algebra with $C \times_X V = \operatorname{Spec}\left(\overline{R}\right)$. We may deform $\overline{B} \to \overline{R}$ to a finite étale map $\widehat{B} \to \widehat{R}$. The F-crystal corresponding to ρ is the $\widehat{B}$-module $M = \left(\widehat{R} \otimes \mathcal{L}\right)^{Gal(C/X)}$. For each $Q \in W$ and $P \in f^{-1}(Q)$, we obtain a finite extension $\mathcal{E}_P^\dagger$ of $\mathcal{E}_Q^\dagger$; recall from Section 3.2 that $W = \eta^{-1}(\{0, 1, \infty\})$. As in Section 3.3, we may consider the ring of overconvergent functions $R^\dagger$, which makes the following diagram Cartesian:

$$\begin{array}{ccc} R^\dagger & \longrightarrow & \bigoplus_{P \in f^{-1}(W)} \mathcal{O}^\dagger_{\mathcal{E}_P} \\ \downarrow & & \downarrow \\ \widehat{R} & \longrightarrow & \bigoplus_{P \in f^{-1}(W)} \mathcal{O}_{\mathcal{E}_P}. \end{array}$$

Since the action of $Gal(C/X)$ (resp., ν) on $\bigoplus_{P \in f^{-1}(Q)} \mathcal{O}_{\mathcal{E}_P}$ preserves $\bigoplus_{P \in f^{-1}(Q)} \mathcal{O}_{\mathcal{E}_P^\dagger}$, we see that $Gal(C/X)$ (resp., ν) acts on $R^\dagger$ (see, e.g., [27, Section 2]). This gives the following proposition:

Proposition 5.7. *Let* $M^\dagger = (R^\dagger \otimes \mathcal{L})^{Gal(C/X)}$*. The map* $M^\dagger \otimes_{B^\dagger} \widehat{B} \to M$ *is a* ν*-equivariant isomorphism.*

Lemma 5.8. *The module* $M^\dagger$ *(resp.,* M*) is a free* $B^\dagger$*-module (resp.,* $\widehat{B}$*-module). Furthermore,* M *has a p-Frobenius structure* α_0 *contained in* $1 + \mathfrak{m}B^\dagger$.

Proof. The proof of this is identical to [15, Lemma 5.9]. □

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5.3.2. The Frobenius structure of $\bigoplus_{j=0}^{a-1} \chi^{\otimes p^j}$

By Kummer theory, there exists $\overline{f} \in \overline{B}^{\times}$ such that χ factors through the étale $\mathbb{Z}/(q-1)\mathbb{Z}$-cover $\operatorname{Spec}\left(\overline{B}\left[\overline{h}\right]\right) \to \operatorname{Spec}\left(\overline{B}\right)$, where $\overline{h} = \sqrt[q-1]{\overline{f}}$. Let $f \in B^{\dagger}$ be a lift of $\overline{f}$ and set $h = \sqrt[q-1]{f}$, so that $\operatorname{Spec}\left(B^{\dagger}[h]\right) \to \operatorname{Spec}\left(B^{\dagger}\right)$ is an étale $\mathbb{Z}/(q-1)\mathbb{Z}$-cover whose special fibre is $\operatorname{Spec}\left(\overline{B}\left[\overline{h}\right]\right) \to \operatorname{Spec}\left(\overline{B}\right)$. There exists $0 \le \Gamma < q-1$ such that $\chi(g) = \frac{(h^{\Gamma})^g}{h^{\Gamma}}$ for all $g \in \pi_1^{et}(V)$. Write the p-adic expansion $\Gamma = \gamma_0 + \cdots + \gamma_{a-1}p^{a-1}$ and define

$$\Gamma_j = \sum_{i=0}^{a-1} \gamma_{i+j} p^i.$$

Note that $\chi^{\otimes p^j}(g) = \frac{\left(h^{\Gamma_j}\right)^g}{h^{\Gamma_j}}$ for each j. This gives the following proposition:

Proposition 5.9. *The matrix $N = \mathbf{diag}\left(f^{-\Gamma_0}, \ldots, f^{-\Gamma_{a-1}}\right)$ (resp., $N_0 = \mathbf{tcyc}\left(f^{-\gamma_0}, \ldots, f^{-\gamma_{a-1}}\right)$) is a dual Frobenius structure (resp., dual p-Frobenius structure) of $\bigoplus_{j=0}^{a-1} \chi^{\otimes p^j}$ and $N = N_0^{\nu^{a-1}+\cdots+1}$.*

Set $Q \in W$. Recall from Section 1.1 that we associate a tame ramification datum $T_Q = \left(\mathbf{e}_Q, \epsilon_Q, \omega_Q\right)$ to Q and write $\epsilon_Q = \sum e_{Q,i}p^i$. The exponent of $\chi^{\otimes p^j}$ at $Q \in W$ is

$$\frac{\epsilon_{Q,j}}{q-1} \bmod \mathbb{Z}, \text{ where}$$
$$\epsilon_{Q,j} = \sum_{i=0}^{a-1} e_{Q,i+j}p^i.$$

By definition we have

$$-\operatorname{Div}\left(\overline{f}^{\Gamma_j}\right) = \sum_{Q \in W} \left(-\epsilon_{Q,j} + (q-1)n_{Q,j}\right)[Q],$$

with $n_{Q,j} \in \mathbb{Z}$. Since $0 \le \epsilon_{Q,j} \le q-2$ and $\sum_Q n_{Q,j} = \frac{\sum_Q \epsilon_{Q,j}}{q-1}$, we know

$$\sum_{Q \in W} n_{Q,j} \le \mathbf{m} \le r_0 + r_\infty, \tag{14}$$

where we recall that $\mathbf{m}$ is the number of points where ρ is ramified. We also have

$$\begin{aligned}\sum_{j=0}^{a-1} \sum_{Q \in W} n_{Q,j} &= \frac{1}{q-1} \sum_{Q \in W} \sum_{j=0}^{a-1} \epsilon_{Q,j} \\ &= a\Omega_\rho,\end{aligned} \tag{15}$$

where Ω_ρ is the monodromy invariant introduced in Section 1.1.

5.3.3. Comparing local and global Frobenius structures

We fix α_0 as in Lemma 5.8 and set $\alpha = \prod_{i=0}^{a-1} \alpha_0^{\nu^i}$. We also let N and N_0 be as in Proposition 5.9. In particular, αN (resp., $\alpha_0 N_0$) is a dual Frobenius structure (resp., dual p-Frobenius structure) of $\rho^{wild} \otimes \bigoplus_{j=0}^{a-1} \chi^{\otimes p^j}$. Set $Q \in W$ with $Q = P_{*,i}$. There is a map $\overline{B} \to \mathbb{F}_q\left(\left(u_Q\right)\right)$, where we expand each function on V in terms of the parameter u_Q. This gives a point $\operatorname{Spec}\left(\mathbb{F}_q\left(\left(u_Q\right)\right)\right) \to V$. By pulling back ρ along this point we

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obtain a local representation $\rho_Q : G_{\mathbb{F}_q((u_Q))} \to \mathcal{O}_L^\times$, where $G_{\mathbb{F}_q((u_Q))}$ is the absolute Galois group of $\mathbb{F}_q\left((u_Q)\right)$. We will compare $\alpha_0 N_0$ to the local dual p-Frobenius structures from Section 5.2.

There are three cases we need to consider. The first case is when $* = 1$. In this case ρ_Q^{wild} and χ_Q are both unramified. This is because ρ is only ramified at the points $\tau_1, \dots, \tau_{\mathbf{m}}$, and by Lemma 3.1 we have $\eta(\tau_i) \in \{0, \infty\}$. The second case is when $* \in \{0, \infty\}$ and ρ_Q^{wild} is unramified. The last case is when $* \in \{0,\infty\}$ and ρ_Q^{wild} is ramified. In each case, we will describe a dual p-Frobenius structure C_Q of $\rho_Q^{wild} \otimes \bigoplus_{j=0}^{a-1} \chi_Q^{\otimes p^j}$, an element $b_Q \in \mathcal{O}_{\mathcal{E}_Q}^\dagger$ and a diagonal matrix $M_Q \in GL_a\left(\mathcal{O}_{\mathcal{E}^\dagger}\right)$ satisfying

$$\begin{aligned}\left(b_Q M_Q\right)^\nu \alpha_0 N_0 \left(b_Q M_Q\right)^{-1} &= C_Q,\\ \left(b_Q M_Q\right)^\sigma \alpha N \left(b_Q M_Q\right)^{-1} &= C_Q^{\nu^{a-1}+\nu^{a-2}+\dots+1}.\end{aligned} \tag{16}$$

The dual p-Frobenius structure C_Q will be closely related to the dual p-Frobenius structures studied in Section 5.2. It is helpful for us to introduce the following rings:

$$\mathcal{R}_Q = \bigoplus_{j=0}^{a-1} \mathcal{E}_Q, \qquad \mathcal{O}_{\mathcal{R}_Q} = \bigoplus_{j=0}^{a-1} \mathcal{O}_{\mathcal{E}_Q},$$

$$\mathcal{R}_Q^\dagger = \bigoplus_{j=0}^{a-1} \mathcal{E}_Q^\dagger, \qquad \mathcal{O}_{\mathcal{R}_Q^\dagger} = \mathcal{R}_Q^\dagger \cap \mathcal{O}_{\mathcal{R}_Q}.$$

We define $u_{Q,j} \in \mathcal{R}_Q$ to have u_Q in the jth coordinate and zero in the other coordinates. For each Q we will define a subspace $\mathcal{O}_{\mathcal{R}_Q}^{con} \subset \mathcal{R}_Q^\dagger$ of elements satisfying some precise convergence conditions.

I. If $* = 1$, then ν_Q sends $u_Q \mapsto \sqrt[p-1]{\left(u_Q^{p-1}+1\right)^p - 1}$ (see the end of Section 3.4).

(Wild) As ρ_Q is unramified, we know from Proposition 5.4 that there exists $b_Q \in 1 + \mathfrak{m}\mathcal{O}_{\mathcal{E}_Q}^\dagger$ such that the dual p-Frobenius structure $c_Q = \frac{b_Q^\nu}{b_Q}\alpha_0$ of ρ_Q^{wild} lies in $1+\mathfrak{m}$.

(Tame) Since χ_Q is unramified, the exponent is zero. By Proposition 5.6, there exists $M_Q = \mathbf{diag}\left(m_{Q,0}, \dots, m_{Q,a-1}\right)$ with $v_{u_Q}\left(\overline{m_{Q,j}}\right) = n_{Q,j}$ such that $M_Q^\sigma N M_Q^{-1} = \mathbf{diag}\left(1, \dots, 1\right)$ and $M_Q^\nu N_0 M_Q^{-1} = \mathbf{tcyc}\left(1, \dots, 1\right)$.

(Both) We see that $C_Q = \mathbf{tcyc}\left(c_Q, \dots, c_Q\right)$ is a dual p-Frobenius structure of $\rho_Q^{wild} \otimes \bigoplus_{j=0}^{a-1} \chi_Q^{\otimes p^j}$ and that equation (16) holds. Define $\mathcal{O}_{\mathcal{R}_Q}^{con}$ to be $\bigoplus_{j=0}^{a-1} \mathcal{D}$ viewed as a subspace of $\mathcal{O}_{\mathcal{R}_Q^\dagger}$ (see equation (12) for the definition of $\mathcal{D}$). From Proposition 4.2 we have

$$\begin{aligned} U_p \circ C_Q\left(p^{b(k+pn)} u_{Q,j}^{-(k+pn)}\right) &\in p^n \mathcal{O}_{\mathcal{R}_Q}^{con},\\ U_p \circ C_Q\left(\mathcal{O}_{\mathcal{R}_Q}^{con}\right) &\subset \mathcal{O}_{\mathcal{R}_Q}^{con}.\end{aligned} \tag{17}$$

II. Next, consider the case where $*$ is 0 or ∞ and ρ_Q^{wild} is unramified. Then ν_Q sends $u_Q \mapsto u_Q^p$. We choose $\mathfrak{s}_Q \in \mathbb{Q}$ such that the following hold:

$$\begin{aligned} \pi_{\mathfrak{s}_Q} &\in \mathcal{O}_E,\\ \frac{1}{\mathfrak{s}_Q} - \frac{\omega_Q}{a\mathfrak{s}_Q(p-1)} &\geq 1.\end{aligned} \tag{18}$$

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(Wild) From Proposition 5.4 there exists $b_Q \in 1 + \mathfrak{m}\mathcal{O}^{\dagger}_{\mathcal{E}_Q}$ such that $c_Q = \frac{b_Q^{\nu}}{b_Q}\alpha_0 \in 1 + \mathfrak{m}$ is a dual p-Frobenius structure of ρ_Q^{wild}.

(Tame) By Proposition 5.6 there exists $M_Q = \mathbf{diag}\left(m_{Q,0}, \ldots, m_{Q,a-1}\right)$ with $v_{u_Q}\left(\overline{m_{Q,j}}\right) = n_{Q,j}$ such that $M_Q^{\sigma} N M_Q^{-1} = \mathbf{diag}\left(u_Q^{-\epsilon_{Q,0}}, \ldots, u_Q^{-\epsilon_{Q,a-1}}\right)$ and $M_Q^{\nu} N_0 M_Q^{-1} = \mathbf{tcyc}\left(u_Q^{-e_{Q,0}}, \ldots, u_Q^{-e_{Q,a-1}}\right)$.

(Both) We see that $C_Q = \mathbf{tcyc}\left(c_Q u_Q^{-e_{Q,0}}, \ldots, c_Q u_Q^{-e_{Q,a-1}}\right)$ is a dual p-Frobenius structure of $\rho_Q^{wild} \otimes \bigoplus_{j=0}^{a-1} \chi_Q^{\otimes p^j}$ and that equation (16) holds. Define $\mathcal{O}^{con}_{\mathcal{R}_Q}$ to be a copy of $\mathcal{D}_{\mathbf{e}_Q,\mathfrak{s}_Q}$ in $\mathcal{O}_{\mathcal{R}_Q^{\dagger}}$ (recall the definition of $\mathcal{D}_{\mathbf{e},s}$ from Section 4.1.1). From Proposition 4.1 we have

$$\begin{aligned} U_p \circ C_Q \left(\pi_{a\mathfrak{s}_Q}^{q(\mathbf{e}_Q,j)} \pi_{\mathfrak{s}_Q}^{pn} u_{Q,j}^{-n}\right) &\in \pi_{\mathfrak{s}_Q}^{n(p-1)} \pi_{a\mathfrak{s}_Q}^{-\omega_Q} \mathcal{O}^{con}_{\mathcal{R}_Q}, \\ U_p \circ C_Q \left(\mathcal{O}^{con}_{\mathcal{R}_Q}\right) &\subset \mathcal{O}^{con}_{\mathcal{R}_Q}. \end{aligned} \tag{19}$$

III. Finally, we consider the case when $*$ is 0 or ∞ and ρ_Q^{wild} is ramified. Then ν_Q sends $u_Q \mapsto u_Q^p$.

(Wild) By Proposition 5.5 there is $b_Q \in 1 + \mathfrak{m}\mathcal{O}^{\dagger}_{\mathcal{E}_Q}$ such that $c_Q = \frac{b_Q^{\nu}}{b_Q}\alpha_0 \in \mathcal{O}_L\left[\left[\pi_{s_Q} u_Q^{-1}\right]\right]$ is a dual p-Frobenius structure of ρ_Q^{wild} (recall that s_Q is the Swan conductor of ρ at Q). Note that $c_Q \equiv 1 \bmod \mathfrak{m}$.

(Tame) By Proposition 5.6 there exists $M_Q = \mathbf{diag}\left(m_{Q,0}, \ldots, m_{Q,a-1}\right)$ with $v_{u_Q}\left(\overline{m_{Q,j}}\right) = n_{Q,j}$ such that $M_Q^{\sigma} N M_Q^{-1} = \mathbf{diag}\left(u_Q^{-\epsilon_{Q,0}}, \ldots, u_Q^{-\epsilon_{Q,a-1}}\right)$ and $M_Q^{\nu} N_0 M_Q^{-1} = \mathbf{tcyc}\left(u_Q^{-e_{Q,0}}, \ldots, u_Q^{-e_{Q,a-1}}\right)$.

(Both) We see that $C_Q = \mathbf{tcyc}\left(c_Q u_Q^{-e_{Q,0}}, \ldots, c_Q u_Q^{-e_{Q,a-1}}\right)$ is a dual p-Frobenius structure of $\rho_Q^{wild} \otimes \bigoplus_{j=0}^{a-1} \chi_Q^{\otimes p^j}$ and that equation (16) holds. We define $\mathcal{O}^{con}_{\mathcal{R}_Q}$ to be a copy of $\mathcal{D}_{\mathbf{e}_Q,s_Q}$ in $\mathcal{O}_{\mathcal{R}_Q^{\dagger}}$. From Proposition 4.1 we see that

$$\begin{aligned} U_p \circ C_Q \left(\pi_{as_Q}^{q(\mathbf{e}_Q,j)} \pi_{s_Q}^{pn} u_{Q,j}^{-n}\right) &\in \pi_{s_Q}^{n(p-1)} \pi_{as}^{-\omega_Q} \mathcal{O}^{con}_{\mathcal{R}_Q}, \\ U_p \circ C_Q \left(\mathcal{O}^{con}_{\mathcal{R}_Q}\right) &\subset \mathcal{O}^{con}_{\mathcal{R}_Q}. \end{aligned} \tag{20}$$

5.3.4. Comparing global and semilocal Frobenius structures

We define the following spaces:

$$\mathcal{R} = \bigoplus_{Q \in W} \mathcal{R}_Q, \qquad \mathcal{R}^{\dagger} = \bigoplus_{Q \in W} \mathcal{R}_Q^{\dagger},$$

$$\mathcal{R}_Q^{trun} = \begin{cases} \bigoplus_{j=0}^{a-1} \mathcal{E}_Q^{\leq -1}, & \eta(Q) = 0, \infty, \\ \bigoplus_{j=0}^{a-1} \mathcal{E}_Q^{\leq -p}, & \eta(Q) = 1, \end{cases}$$

$$\mathcal{R}^{trun} = \bigoplus_{Q \in W} \mathcal{R}_Q^{trun}, \qquad \mathcal{O}^{con}_{\mathcal{R}} = \bigoplus_{Q \in W} \mathcal{O}^{con}_{\mathcal{R}_Q} \subset \mathcal{R}^{\dagger}.$$

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Define $\mathcal{O}_{\mathcal{R}}$ to be $\bigoplus_{Q\in W} \mathcal{O}_{\mathcal{R}_Q}$ and define $\mathcal{O}_{\mathcal{R}^{trun}}$ to be $\mathcal{R}^{trun} \cap \mathcal{O}_{\mathcal{R}}$. Note that $\mathcal{O}^{con}_{\mathcal{R}}$ is contained in $\mathcal{O}_{\mathcal{R}}$. There is a projection map $pr : \mathcal{R} \to \mathcal{R}^{trun}$, which is the direct sum of the projection maps described in Section 2.3. By the definition of each summand of $\mathcal{O}^{con}_{\mathcal{R}}$ we see that

$$\ker(pr) \cap \mathcal{O}_{\mathcal{R}} \subset \mathcal{O}^{con}_{\mathcal{R}}. \tag{21}$$

We may view $\bigoplus_{j=0}^{a-1} \widehat{\mathcal{B}}$ $\left(\text{resp., } \bigoplus_{j=0}^{a-1} \mathcal{B}^{\dagger}\right)$ as a subspace of $\mathcal{R}$ $\left(\text{resp., } \mathcal{R}^{\dagger}\right)$ using the maps (6). Let C (resp., M and b) denote the endomorphism of $\mathcal{R}^{\dagger}$ that acts on the Q-coordinate by C_Q $\left(\text{resp., } M_Q \text{ and } \mathbf{diag}\left(b_Q, \dots, b_Q\right)\right)$. This gives an operator $U_p \circ C : \mathcal{R}^{\dagger} \to \mathcal{R}^{\dagger}$. From formulas (17), (19) and (20), we have

$$U_p \circ C \left(\mathcal{O}^{con}_{\mathcal{R}}\right) \subset \mathcal{O}^{con}_{\mathcal{R}}. \tag{22}$$

Also, by equation (16) know

$$\begin{aligned}(bM)^{\nu}\alpha_0 N_0 (bM)^{-1} &= C,\\ (bM)^{\sigma}\alpha N (bM)^{-1} &= C^{\nu^{a-1}+\nu^{a-2}+\cdots+1}.\end{aligned} \tag{23}$$

For each Q, we have

$$\begin{aligned} b_Q &\equiv 1 \bmod \mathfrak{m},\\ M_Q &\equiv \mathbf{diag}\left(u_{Q,0}^{n_{Q,0}} g_0, \dots, u_{Q,a-1}^{n_{Q,a-1}} g_{a-1}\right) \bmod \mathfrak{m},\end{aligned} \tag{24}$$

with $g_j \in \mathbb{F}_q\left[\left[u_{Q,j}\right]\right]^{\times}$.

6. Normed vector spaces and Newton polygons

For the convenience of the reader, we recall some definitions and facts about Newton polygons and normed p-adic vector spaces. Most of what follows is well known (see, e.g., [25] or [22] for many standard facts on p-adic functional analysis). However, we do find it necessary to introduce some notation and definitions that are not standard. In particular, we introduce the notion of a *formal basis*, which allows us to compute Fredholm determinants by estimating columns (in contrast to estimating rows, which is the approach taken in [1]).

6.1. Normed vector spaces and Banach spaces

Let V be a vector space over L with a norm $|\cdot|$ compatible with the p-adic norm $|\cdot|_p$ on L. We will assume that for every $x \in V\backslash\{0\}$, the norm $|x|$ lies in $|L^{\times}|_p$, the norm group of $L^{\times}$. We say V is a *Banach space* if it is also complete. Let $V_0 \subset V$ denote the subset consisting of $x \in V$ satisfying $|x| \leq 1$ and let $\overline{V} = V_0 / \mathfrak{m} V_0$. If W is a subspace of V, we automatically give W the subspace norm unless otherwise specified.

Definition 6.1. Let I be a set. We let $\mathbf{s}(I)$ denote the set of families $x = (x_i)_{i\in I}$, with $x_i \in L$ such that $|x| = \sup_{i\in I} |x_i|_p < \infty$. Then $\mathbf{s}(I)$ is a Banach space with the norm $|\cdot|$. We let $\mathbf{c}(I) \subset \mathbf{s}(I)$ be the subspace of families with $\lim_{i\in I} x_i = 0$ (note that this agrees with $\mathbf{c}(I)$ defined in [25, Section I]).

Definition 6.2. A *formal basis* of V is a subset $G = \{e_i\}_{i\in I} \subset V$ with a norm-preserving embedding $V \to \mathbf{s}(I)$, where e_i gets mapped to the element in $\mathbf{s}(I)$ with 1 in the i-coordinate and 0 otherwise.[2] We regard V a subspace of $\mathbf{s}(I)$.

[2]In [15] we use the term *integral basis*.

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Definition 6.3. An *orthonormal basis* of V is a formal basis $G = \{e_i\}_{i \in I} \subset V$ such that $V \subset \mathbf{c}(I)$. This inclusion is an equality if V is a Banach space. By [25, Proposition I], every Banach space over L has an orthonormal basis. Thus, every Banach space is of the form $\mathbf{c}(I)$.

Example 6.4. Let V be the Banach space $\mathcal{O}_L[[t]] \otimes \mathbb{Q}_p$. Then $\{t^n\}_{n \in \mathbb{Z}_{\geq 0}}$ is a formal basis of V and there is an isomorphism $V \cong \mathbf{s}(\mathbb{Z}_{\geq 0})$. By [25, Lemme I], any orthonormal basis of V reduces to an $\mathbb{F}_q$-basis of $\overline{V} = \mathbb{F}_q[[t]]$ and thus must be uncountable. The Tate algebra $L\langle t \rangle \subset V$ is a Banach space, which we may identify with $\mathbf{c}(\mathbb{Z}_{\geq 0})$.

6.1.1. Restriction of scalars to E

Let I be a set. Assume that $V \subset \mathbf{s}(I)$ has G as a formal basis. We may regard V as a vector space over E. Let $\zeta_1 = 1, \zeta_2, \ldots, \zeta_a \in \mathcal{O}_L$ be elements that reduce to a basis of $\mathbb{F}_q$ over $\mathbb{F}_p$ mod $\pi_\circ$ and set $I_E = I \times \{1, \ldots, a\}$. We define

$$G_E = \{\zeta_j e_i\}_{(i,j) \in I_E}.$$

Note that G_E is a formal basis of V over E.

6.2. Completely continuous operators and Fredholm determinants

6.2.1. Completely continuous operators

Let V be a vector space over L with norm $|\cdot|$. Let $G = \{e_i\}_{i \in I}$ be a formal basis of V. We assume I is a countable set. Let $u : V \to V$ (resp., $v : V \to V$) be an L-linear (resp., E-linear) operator. Let $(n_{i,j})$ be the matrix of u with respect to the basis G.

Definition 6.5. For $i \in I$, we define $\mathbf{row}_i(u, G) = \inf_{j \in I} v_p(n_{i,j})$ and $\mathbf{col}_i(u, G) = \inf_{j \in I} v_p(n_{j,i})$. That is, $\mathbf{row}_i(u, G)$ (resp., $\mathbf{col}_i(u, G)$) is the smallest p-adic valuation that occurs in the ith row (resp., column) of the matrix of u. Note that $\mathbf{col}_i(u, G) = \log_p |u(e_i)|$.

Definition 6.6. Assume that $V = \mathbf{c}(I)$. We say that u is *completely continuous* if it is the p-adic limit of L-linear operators with finite-dimensional image. This is equivalent to $\lim_{i \in I} \mathbf{row}_i(u, G) = \infty$ [21, Theorem 6.2]. We make the analogous definition for v.

6.2.2. Fredholm determinants

We continue with the notation from Section 6.2.1. We define the *Fredholm determinant* of u with respect to G to be the formal sum

$$\begin{aligned} \det(1 - su \mid G) &= \sum_{n=0}^{\infty} c_n s^n, \\ c_n &= (-1)^n \sum_{\substack{S \subset I \\ |S| = n}} \sum_{\sigma \in \mathrm{Sym}(S)} \mathrm{sgn}(\sigma) \prod_{i \in S} n_{i,\sigma(i)}. \end{aligned} \tag{25}$$

We define the Fredholm determinant $\det_E(1 - sv \mid G_E)$ in an analogous manner using the matrix of v with respect to G_E. Note that there is no reason a priori for the sums c_n to converge. We will say that $\det(1 - su \mid G)$ is *well defined* if each c_n converges.

Lemma 6.7. *Assume that V is a Banach space with orthonormal basis G and that u is completely continuous. Then* $\det(1 - su \mid G)$ *is well defined and is an entire function in s. Furthermore, if G' is another orthonormal basis of V, we have* $\det(1 - su \mid G) = \det(1 - su \mid G')$. *The analogous result holds for v.*

Proof. See [25, Proposition 7]. □

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Definition 6.8. Continue with the notation from Lemma 6.7. We let $\det(1-su \mid V)$ denote the Fredholm determinant $\det(1-su \mid G)$. By Lemma 6.7, this determinant does not depend on our choice of orthonormal basis. We define $\det_E(1-sv \mid V)$ similarly.

6.2.3. Newton polygons of operators

Definition 6.9. Let $*$ be either p or q. Let $f(t)=\sum a_n t^n \in L\langle t\rangle^\times$ be an entire function. We define the $*$-adic Newton polygon $NP_*(f)$ to be the lower convex hull of the points $(n, v_*(a_n))$. For $r>0$, we let $NP_*(f)_{<r}$ denote the 'subpolygon' of $NP_*(f)$ consisting of all segments whose slope is less than r.

Definition 6.10. Adopt the notation from Section 6.2.2. Assume that $\det(1-su \mid G)$ $\left(\text{resp., } \det_E(1-sv \mid G_E)\right)$ is well defined and an entire function in s. Then we define $NP_*(u \mid G)$ (resp., $NP_*(v \mid G_E)$) to be $NP_*(\det(1-su \mid G))$ $\left(\text{resp., } NP_*\left(\det_E(1-sv \mid G_E)\right)\right)$. Further assume that V is a Banach space and u (resp., v) is completely continuous. Then by Lemma 6.7, the Fredholm determinant does not depend on the choice of orthonormal basis, so we define $NP_*(u \mid V)$ (resp., $NP_*(v \mid V)$) to be $NP_*(u \mid G)$ (resp., $NP_*(v \mid G_E)$).

Definition 6.11. Set $d \in \mathbb{Z}_{\geq 1} \cup \infty$ and let $A=\{c_n\}_{n\geq 1}^d$ be a nondecreasing sequence of real numbers. If $d=\infty$ we will make the assumption that $\lim_{n\to\infty} c_n = \infty$. Let P_A be the 'polygon' of length d consisting of vertices $(0,0), (1,c_1), (2, c_1+c_2), \dots$. We write

$$NP_*(f) \succeq A$$

if the polygon $NP_*(f)$ lies above P_A at every x-coordinate where both are defined.

The following lemma allows us to bound $NP_p(v \mid G_E)$ by estimating the columns of the matrix representing v.

Lemma 6.12. *Assume that* $\lim_{i\in I} \mathbf{col}_{(i,1)}(v, G_E) = \infty$. *If v is ν^{-1}-semilinear, then the Fredholm determinant* $\det(1-sv \mid G_E)$ *is well defined and we have*

$$NP_p(v \mid G_E) \succeq \left\{\mathbf{col}_{(i,1)}(v, G_E)\right\}_{i\in I}^{\times a},$$

where the superscript '$\times a$' means each slope is repeated a times.

Proof. Note that $v\left(\zeta_j e_i\right) = \zeta_j^{\nu^{-1}} v(e_i)$, which implies $\mathbf{col}_{(i,j)}(v, G_E) = \mathbf{col}_{(i,1)}(v, G_E)$ for each j. In particular, $\lim_{(i,j)\in I_E} \mathbf{col}_{(i,j)}(v, G_E) = \infty$, so $\det_E(1-sv \mid G_E)$ is well defined. By the definition of c_n in equation (25), we see that

$$\begin{aligned} NP_p(v \mid G_E) &\succeq \left\{\mathbf{col}_{(i,j)}(v, G_E)\right\}_{(i,j)\in I_E} \\ &= \left\{\mathbf{col}_{(i,1)}(v, G_E)\right\}_{i\in I}^{\times a}. \end{aligned}$$

□

6.2.4. Computing Newton polygons using *a*th roots

When estimating the Newton polygon of an L-linear completely continuous operator u on V, it is convenient to work with an E-linear operator v that is an ath root of u. The reason is that we can translate p-adic bounds on $\det_E(1-sv \mid V)$ to q-adic bounds on $\det(1-su \mid V)$.

Lemma 6.13. *Let V be a Banach space. Let v be a completely continuous E-linear operator on V and let $u=v^a$. Assume that u is L-linear. We further assume that* $\det(1-su \mid V)$ *has coefficients in E (a priori, its coefficients could lie in L). Let $\frac{1}{a}NP_p(v \mid V)$ denote the polygon where both the x-coordinates and y-coordinates of the points in $NP_p(v \mid V)$ are scaled by a factor of $\frac{1}{a}$. Then $NP_q(u \mid V) = \frac{1}{a}NP_p(v \mid V)$.*

Proof. Some version of this lemma is present in most papers proving 'Hodge bounds' for exponential sums (see, e.g., [4] or [1]). The proof of [15, Lemma 6.25] is easily adapted to our situation. □

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7. Finishing the proof of Theorem 1.1

7.1. The Monsky trace formula

Let us recall the Monsky trace formula in the case of curves. For a complete treatment, see [22] or [29, Section 10]. Let $\Omega^i_{\mathcal{B}^\dagger}$ denote the space of i-forms of $\mathcal{B}^\dagger$ [22, Section 4]. The map σ induces a map $\sigma_i : \Omega^i_{\mathcal{B}^\dagger} \to \Omega^i_{\mathcal{B}^\dagger}$ sending xdy to $x^\sigma d(y^\sigma)$. As in [28, Section 3], there exist trace maps $\mathrm{Tr}_i : \Omega^i_{\mathcal{B}^\dagger} \to \sigma\left(\Omega^i_{\mathcal{B}^\dagger}\right)$. Let Θ_i denote the map $\sigma_i^{-1} \circ \mathrm{Tr}_i$. For $\omega \in \Omega^1_{\mathcal{B}^\dagger}$ and $x \in \mathcal{B}^\dagger$, we have

$$\Theta_1(x\omega^\sigma) = \Theta_0(x)\omega. \tag{26}$$

Consider the L-function

$$L\left(\rho^{wild} \otimes \chi^{\otimes p^j}, V, s\right) = \prod_{x \in V} \frac{1}{1 - \rho^{wild} \otimes \chi^{\otimes p^j}(Frob_x)s^{\deg(x)}}, \tag{27}$$

which is a slight modification of equation (1). Fix a tuple $\mathbf{r} = (r_Q)_{Q \in W}$ of positive rational numbers. Monsky shows that if the r_Q are sufficiently small (so $\mathcal{B}(0,\mathbf{r}]$ consists of functions with sufficiently small radius of overconvergence), the operator $\Theta_i \circ \alpha f^{-\Gamma_j}$ is completely continuous on $\Omega^i_{\mathcal{B}(0,\mathbf{r}]}$. The Monsky trace formula states

$$L\left(\rho^{wild} \otimes \chi^{\otimes p^j}, V, s\right) = \frac{\det\left(1 - s\Theta_1 \circ \alpha f^{-\Gamma_j} \,\middle|\, \Omega^1_{\mathcal{B}(0,\mathbf{r}]}\right)}{\det\left(1 - s\Theta_0 \circ \alpha f^{-\Gamma_j} \middle| \mathcal{B}(0,\mathbf{r}]\right)}, \tag{28}$$

where $\alpha, f,$ and Γ_j are as in Sections 5.3.2 and 5.3.3. Thus, we may estimate $L\left(\rho^{wild} \otimes \chi^{\otimes p^j}, V, s\right)$ by estimating operators on the space of 1-forms and 0-forms.

In our situation we may simplify equation (28). The map $\mathcal{A}^\dagger \to \mathcal{B}^\dagger$ is étale, which implies $\Omega_{\mathcal{B}^\dagger} = \pi^*\Omega_{\mathcal{A}^\dagger}$. Since $\Omega_{\mathcal{A}^\dagger} = \mathcal{A}^\dagger \frac{dt}{t}$, we see that $\Omega_{\mathcal{B}^\dagger} = \mathcal{B}^\dagger \frac{dt}{t}$. In particular, we have $\Omega_{\mathcal{B}(0,\mathbf{r}]} = \mathcal{B}(0,\mathbf{r}] \frac{dt}{t}$. Also, since $\frac{dt}{t} = \frac{1}{q}\left(\frac{dt}{t}\right)^\sigma$, we know by equation (26) that $\Theta_1\left(x\frac{dt}{t}\right) = \frac{1}{q}\Theta_0(x)\frac{dt}{t}$. Thus, we have $\Theta_1 = U_q$ and $\Theta_0 = qU_q$. Then equation (28) becomes

$$L\left(\rho^{wild} \otimes \chi^{\otimes p^j}, V, s\right) = \frac{\det\left(1 - sU_q \circ \alpha f^{-\Gamma_j} \middle| \mathcal{B}(0,\mathbf{r}]\right)}{\det\left(1 - sqU_q \circ \alpha f^{-\Gamma_j} \middle| \mathcal{B}(0,\mathbf{r}]\right)}. \tag{29}$$

As $\det\left(1 - sU_q \circ \alpha f^{-\Gamma_j} \,\middle|\, \mathcal{B}(0,\mathbf{r}]\right) \in 1 + s\mathcal{O}_L[[s]]$, we know that $\frac{1}{\det\left(1 - sqU_q \circ \alpha f^{-\Gamma_j} \,\middle|\, \mathcal{B}(0,\mathbf{r}]\right)}$ lies in $1 + qs\mathcal{O}_L[[qs]]$. This means each slope of $NP_q\left(\frac{1}{\det\left(1 - sqU_q \circ \alpha f^{-\Gamma_j} \,\middle|\, \mathcal{B}(0,\mathbf{r}]\right)}\right)$ is at least 1. In particular, we have

$$NP_q\left(L\left(\rho^{wild} \otimes \chi^{\otimes p^j}, V, s\right)\right)_{<1} = NP_q\left(U_q \circ \alpha f^{-\Gamma_j} \middle| \mathcal{B}(0,\mathbf{r}]\right)_{<1}.$$

Note that ρ and $\rho^{wild} \otimes \chi^{\otimes p^j}$ are Galois conjugates. Thus, $L(\rho, V, s)$ and $L\left(\rho^{wild} \otimes \chi^{\otimes p^j}, V, s\right)$ are Galois conjugates. This gives

$$\begin{aligned} NP_q(L(\rho, V, s))_{<1} &= \frac{1}{a} NP_q\left(L\left(\rho^{wild} \otimes \bigoplus_{j=0}^{a-1} \chi^{\otimes p^j}, V, s\right)\right)_{<1} \\ &= \frac{1}{a} NP_q\left(U_q \circ \alpha N \,\middle|\, \bigoplus_{j=0}^{a-1} \mathcal{B}(0,\mathbf{r}]\right)_{<1}, \end{aligned} \tag{30}$$

where N is the dual Frobenius structure from Proposition 5.9.

7.2. Estimating $NP_q\left(U_q \circ \alpha N \middle| \bigoplus_{j=0}^{a-1} \mathcal{B}(0,\mathbf{r}]\right)$

In this subsection we estimate the q-adic Newton polygon of $U_q \circ \alpha N$ acting on $\bigoplus_{j=0}^{a-1} \mathcal{B}(0,\mathbf{r}]$.

Proposition 7.1. *We have*

$$\frac{1}{a} NP_q\left(U_q \circ \alpha N \,\middle|\, \bigoplus_{j=0}^{a-1} \mathcal{B}(0,\mathbf{r}]\right)_{<1} \geq \{ \underbrace{0,\dots,0}_{g-1+r_0+r_1+r_\infty-\Omega_\rho} \} \bigsqcup \left(\bigsqcup_{i=1}^{\mathbf{m}} S_{\tau_i} \right),$$

where S_{τ_i} is the slope set defined in Section 1.1 and r_* is the cardinality of $\eta^{-1}(*)$ defined in Section 3.2.

We break the proof up into several steps.

7.2.1. The twisted space and the ath root

We view $\bigoplus_{j=0}^{a-1} \widehat{\mathcal{B}}$ and $\bigoplus_{j=0}^{a-1} \mathcal{B}(0,\mathbf{r}]$ as subspaces of $\mathcal{R}$, as described in Section 5.3.4. Unfortunately, the global Frobenius structure αN will not have nice growth properties like the local Frobenius structures studied in Section 5.1. Instead, we have to 'twist' this subspace using the matrices bM defined in Section 5.3.4. Define the spaces

$$\widehat{V} = bM\left(\bigoplus_{j=0}^{a-1} \widehat{\mathcal{B}}\right),$$
$$V = bM\left(\bigoplus_{j=0}^{a-1} \mathcal{B}(0,\mathbf{r}]\right),$$

which we regard as subspaces of $\mathcal{R}$. After decreasing $\mathbf{r}$ we may assume that

$$V = \widehat{V} \bigcap \bigoplus_{Q\in W} \bigoplus_{j=0}^{a-1} \mathcal{E}_Q\left(0, r_Q\right]. \tag{31}$$

In fact, equation (31) holds as long as bM, viewed as a matrix with elements in $\bigoplus_{Q\in W} \mathcal{E}_Q^\dagger$, has entries contained in $\bigoplus_{Q\in W} \mathcal{E}_Q\left(0, r_Q\right]$. From equation (23) we know that $U_q \circ C^{\nu^{a-1}+\cdots+1}$ and $U_p \circ C$ act on V. Since $U_q \circ C^{\nu^{a-1}+\cdots+1} = \left(U_p \circ C\right)^a$, Lemma 6.13 tells us that

$$\begin{aligned} NP_q\left(U_q \circ \alpha N \,\middle|\, \bigoplus_{j=0}^{a-1} \mathcal{B}(0,\mathbf{r}]\right) &= NP_q\left(U_q \circ C^{\nu^{a-1}+\cdots+1} \,\middle|\, V\right) \\ &= \frac{1}{a} NP_p\left(U_p \circ C \mid V\right). \end{aligned} \tag{32}$$

Proposition 7.2. *The following hold:*

1. *We have $pr\left(\widehat{V}_0\right) = \mathcal{O}_{\mathcal{R}^{trun}}$, where pr is the projection map defined in Section 5.3.4.*
2. *Both $\ker\left(pr : V \to \mathcal{R}^{trun}\right)$ and $\ker\left(pr : \widehat{V} \to \mathcal{R}^{trun}\right)$ have dimension $a\left(g-1+r_0+r_1+r_\infty-\Omega_\rho\right)$ as vector spaces over L.*

To prove Proposition 7.2 we need the following lemma:

Lemma 7.3. *Let $f : R \to S$ be a continuous map of Banach spaces such that $f(R_0) \subset S_0$. If $\overline{f} : \overline{R} \to \overline{S}$ is surjective, then f is surjective and $f(R_0) = S_0$. Furthermore,*

$$\overline{\ker(f)} = \ker\left(\overline{f}\right).$$

Proof. This is proven by approximating the image and kernel of f. For more details see [15, Lemma 7.3]. □

Proof of Proposition 7.2. Let us first consider $\widehat{V}$. Define a function $\mu : W \to \mathbb{N}$ by

$$\mu(Q) = \begin{cases} 1, & \eta(Q) \in \{0, \infty\}, \\ p, & \eta(Q) = 1. \end{cases} \tag{33}$$

Let $\overline{M}$ be the reduction of M mod $\mathfrak{m}$. By Lemma 7.3 and expression (24), we may prove the corresponding result for the map

$$\overline{pr} : \overline{M}\left(\bigoplus_{j=0}^{a-1} \overline{B}\right) \to \overline{\mathcal{R}^{trun}} = \bigoplus_{Q\in W} \bigoplus_{j=0}^{a-1} u_{Q,j}^{-\mu(Q)} \mathbb{F}_q \left[\left[u_{Q,j}^{-1}\right]\right]. \tag{34}$$

Define the divisor

$$D_j = \sum_{i=1}^{r_1} (p-1)\left[P_{1,i}\right] - \sum_{Q\in W} n_{Q,j}[Q].$$

By expression (24), we know the kernel of equation (34) is

$$\bigoplus_{j=0}^{a-1} H^0\left(X, \mathcal{O}_X\left(D_j\right)\right).$$

Since $(p-1)r_1 = \deg(\eta)$, we know from formula (14) that $\deg(D_i) \geq \deg(\eta) - r_0 - r_\infty$. By equation (4) and the Riemann–Roch theorem, we see that $H^0\left(X, \mathcal{O}_X\left(D_j\right)\right)$ has dimension $g - 1 + r_0 + r_1 + r_\infty - \sum n_{Q,j}$. Then from equation (15) we know that the kernel of equation (34) has dimension $a\left(g - 1 + r_0 + r_1 + r_\infty - \Omega_\rho\right)$ as an $\mathbb{F}_q$-vector space. To prove the result for V, first note that

$$\ker\left(pr : \mathcal{R} \to \mathcal{R}^{trun}\right) \subset \bigoplus_{Q\in W} \bigoplus_{j=0}^{a-1} \mathcal{E}_Q\left(0, r_Q\right],$$

as the kernel consists of functions with finite-order poles. The proposition follows from equation (31). □

7.2.2. Choosing a basis

For the remainder of this section, we let $v = U_p \circ C$, which we view as an operator on V. Then define $J \subset \mathbb{N} \times W \times \{0, \ldots, a-1\}$ by

$$J = \{(n, Q, j) \mid n \geq \mu(Q),\ j \in \{0, \ldots, a-1\}\}, \tag{35}$$

where μ is the function defined in equation (33). The set $\left\{u_{Q,j}^{-n}\right\}_{(n,Q,j)\in J}$ is an orthonormal basis for $\mathcal{R}^{trun}$ over L (recall that $u_{Q,j}$ is the element of $\mathcal{R}$ with u_Q in the (Q, j)-coordinate and zeros in the other coordinates). Let K be a set with $\dim_L(\ker_L(pr|_V))$ elements and set $I = J \sqcup K$. For $i = (n, Q, j) \in J$, choose an element $e_i \in V_0$ with $pr(e_i) = u_{Q,j}^{-n}$. By the first part of Proposition 7.2 we know that such an e_i exists. We also choose an orthonormal basis $\{e_i\}_{i\in K} \subset V_0$ of $\ker_L(pr|_V)$ indexed by K. Then $G = \{e_i\}_{i\in I}$ is an orthonormal basis of $\widehat{V}$ over L. By formula (21) there exists $c_i \in \mathcal{O}_{\mathcal{R}}^{con}$ for each $i \in I$ with

$$e_i = \begin{cases} u_{Q,j}^{-n} + c_i, & i = (n, Q, j) \in J, \\ c_i, & i \in K. \end{cases} \tag{36}$$

Define the space

$$V^{con} = \left(\mathcal{O}_{\mathcal{R}}^{con} \cap \widehat{V}\right) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$

Endow V^{con} with a norm so that $V_0^{con} = \mathcal{O}_{\mathcal{R}}^{con} \cap \widehat{V}$ (recall that the subscript 0 denotes the subset of elements of norm ≤ 1). We now scale each element $e_i \in G$ by an element x_i in $\mathcal{O}_L$ to obtain a formal basis G^{con} of V^{con}. We break up the definition of x_i into four cases: The first case is when $i \in K$, and the other three cases correspond to the three types of points $Q \in W$ described in Section 5.3.3. We define

$$x_i = \begin{cases} 1, & i \in K, \\ \pi_{as_Q}^{q(\mathbf{e}_Q,j)} \pi_{s_Q}^{pn}, & i = (n, Q, j),\ \eta(Q) \in \{0, \infty\} \text{ and } \rho_Q^{wild} \text{ is unramified}, \\ \pi_{as_Q}^{q(\mathbf{e}_Q,j)} \pi_{s_Q}^{pn}, & i = (n, Q, j),\ \eta(Q) \in \{0, \infty\} \text{ and } \rho_Q^{wild} \text{ is ramified}, \\ p^{b(n)}, & i = (n, Q, j) \text{ and } \eta(Q) = 1, \end{cases} \tag{37}$$

where $b(n)$ is the function defined in Section 4.2. From the definition of $\mathcal{O}_{\mathcal{R}}^{con}$, we see that $G^{con} = \{x_i e_i\}$ is a formal basis of V^{con}. Indeed, we just selected the x_i appropriately for each summand in the definition of $\mathcal{O}_{\mathcal{R}}^{con}$.

Proposition 7.4. *We have*

$$\det_E \left(1 - sU_p \circ C \middle| V\right) = \det_E \left(1 - sU_p \circ C \middle| G_E^{con}\right).$$

Proof. For $Q \in W$, define a sequence $b_{Q,1}, b_{Q,2}, \ldots \in \mathcal{O}_L$ such that $\left\{\ldots, u_Q^2, u_Q^1, 1, b_{Q,1}u_Q^{-1}, b_{Q,2}u_Q^{-2}, \ldots\right\}$ is a formal basis of $\mathcal{E}_Q\left(0, r_Q\right]$. For $i \in K$ set $y_i = 1$, and for $i = (n, Q, j)$ set $y_i = b_{Q,n}$. Then $G^{\mathbf{r}} = \{y_i e_i\}$ is an orthonormal basis of V. In particular, we have

$$\begin{aligned} \det_E \left(1 - sU_p \circ C \middle| V\right) &= \det_E \left(1 - sU_p \circ C \middle| G_E^{\mathbf{r}}\right) \\ &= \det_E \left(1 - sU_p \circ C \middle| G_E^{con}\right). \end{aligned}$$

The second equality follows from observing that the matrices of $U_p \circ C$ for the bases $G_E^{\mathbf{r}}$ and G_E^{con} are similar. □

7.2.3. Estimating the column vectors

To estimate the column vectors we will need the following lemma:

Lemma 7.5. *For any $n \geq 0$, we have $\pi_\circ^n \mathcal{O}_{\mathcal{R}}^{con} \cap \widehat{V} = \pi_\circ^n V_0^{con}$.*

Proof. Set $z \in \pi_\circ^n \mathcal{O}_{\mathcal{R}}^{con} \cap \widehat{V}$. Then $\pi_\circ^{-n} z \in \mathcal{O}_{\mathcal{R}}^{con}$, and since $\widehat{V}$ is a vector space we have $\pi_\circ^{-n} z \in \widehat{V}$. It follows that $z \in \pi_\circ^n \left(\mathcal{O}_{\mathcal{R}}^{con} \cap \widehat{V}\right)$. The other direction is similar. □

We now estimate $\mathbf{col}_{(i,1)}\left(v, G_E^{con}\right)$ for each $i \in I$. We break this up into the four cases used for defining x_i.

(I) For $i \in K$, we have $x_i e_i = e_i$. We know from equation (36) that $e_i \in \mathcal{O}^{con}_{\mathcal{R}}$. By formula (22) we know $v\left(\mathcal{O}^{con}_{\mathcal{R}}\right) \subset \mathcal{O}^{con}_{\mathcal{R}}$, which means $v(e_i) \in V_0^{con}$. Thus, $\mathbf{col}_{(i,1)}\left(v, G_E^{con}\right) \geq 0$ and

$$\left\{\mathbf{col}_{(i,1)}\left(v, G_E^{con}\right)\right\}_{i\in K} \geq \{\underbrace{0,0,\dots,0}_{a\left(g-1+r_0+r_1+r_\infty-\Omega_\rho\right)}\}.$$

The multiplicity of the zeros follows from Proposition 7.2.

(II) Fix Q with $\eta(Q) = 1$ and let $i = (n, Q, j) \in J$. By equation (35), we consider only tuples (n, Q, j) with $n \geq p$. Recall from equation (37) that $x_i = p^{b(n)}$ and from equation (36) that $e_i = u_{Q,j}^{-n} + c_i$ with $c_i \in \mathcal{O}^{con}_{\mathcal{R}}$. Write $n = k + pm$, where $0 \leq k < p$. By formula (22) we have $v(x_i c_i) \in p^{b(n)}\mathcal{O}^{con}_{\mathcal{R}}$, and by formula (17) we have $v\left(x_i u_{Q,j}^{-n}\right) \subset p^m \mathcal{O}^{con}_{\mathcal{R}}$. From the definition of $b(n)$ in Section 4.2, we know $b(n) \geq m$, which implies $v(x_i e_i) \in p^m \mathcal{O}^{con}_{\mathcal{R}}$. Lemma 7.5 tells us that $v(x_i e_i) \in p^m\left(V_0^{con}\right)$. Thus, we have $\mathbf{col}_{(i,1)}\left(v, G_E^{con}\right) \geq m$. This gives

$$P_Q = \left\{\mathbf{col}_{((n,Q,j),1)}\left(v, G_E^{con}\right)\right\}_{\substack{n\geq p\\ 0\leq j<a}} \geq \{1,2,3,\dots\}^{\times ap}.$$

(III) Fix $Q \in W$ such that $\eta(Q) \in \{0, \infty\}$ and ρ_Q^{wild} is unramified. Consider $i = (n, Q, j) \in J$. By equation (35) we consider only tuples (n, Q, j) where $n \geq 1$. Recall from equation (37) that $x_i = \pi_{a\mathfrak{s}_Q}^{q(\mathbf{e}_Q,j)} \pi_{\mathfrak{s}_Q}^{pn}$ and from equation (36) that $e_i = u_{Q,j}^{-n} + c_i$ with $c_i \in \mathcal{O}^{con}_{\mathcal{R}}$. Then by formulas (19) and (22), we see that $v(x_i e_i) \in \pi_{\mathfrak{s}_Q}^{n(p-1)} \pi_{a\mathfrak{s}_Q}^{-\omega_Q} \mathcal{O}^{con}_{\mathcal{R}}$. Again, by Lemma 7.5 we see that $v(x_i e_i) \in \pi_{\mathfrak{s}_Q}^{n(p-1)} \pi_{a\mathfrak{s}_Q}^{-\omega_Q}\left(V_0^{con}\right)$. This gives

$$P_Q = \left\{\mathbf{col}_{((n,Q,j),1)}\left(v, G_E^{con}\right)\right\}_{\substack{n\geq 1\\ 0\leq j<a}} \geq \left\{\frac{1}{\mathfrak{s}_Q} - \frac{\omega_Q}{a\mathfrak{s}_Q(p-1)}, \frac{2}{\mathfrak{s}_Q} - \frac{\omega_Q}{a\mathfrak{s}_Q(p-1)}, \dots\right\}^{\times a}.$$

(IV) Finally, fix $Q \in W$ such that $\eta(Q) \in \{0, \infty\}$ and ρ_Q^{wild} is ramified. Repeating the argument from case III but replacing $\mathfrak{s}_Q$ with s_Q gives

$$P_Q = \left\{\mathbf{col}_{((n,Q,j),1)}\left(v, G_E^{con}\right)\right\}_{\substack{n\geq 1\\ 0\leq j<a}} \geq \left\{\frac{1}{s_Q} - \frac{\omega_Q}{as_Q(p-1)}, \frac{2}{s_Q} - \frac{\omega_Q}{as_Q(p-1)}, \dots\right\}^{\times a}.$$

We put everything together to get

$$\left\{\mathbf{col}_{(i,1)}\left(v, G_E^{con}\right)\right\}_{i\in I} \geq \{\underbrace{0,0,\dots,0}_{g-1+r_0+r_1+r_\infty-\Omega_\rho}\}^{\times a} \bigsqcup \left(\bigsqcup_{Q\in W} P_Q\right).$$

Then by Lemma 6.12 we see that $\det\left(1 - sv, G_E^{con}\right)$ converges and that

$$NP_p\left(v \mid G_E^{con}\right) \geq \{\underbrace{0,0,\dots,0}_{g-1+r_0+r_1+r_\infty-\Omega_\rho}\}^{\times a^2} \bigsqcup \left(\bigsqcup_{Q\in W} P_Q^{\times a}\right).$$

Then from Proposition 7.4 we have

$$\frac{1}{a} NP_p\left(U_p \circ C \mid V\right) \geq \{\underbrace{0,0,\dots,0}_{g-1+r_0+r_1+r_\infty-\Omega_\rho}\}^{\times a} \bigsqcup \left(\bigsqcup_{Q\in W} P_Q\right).$$

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When Q is from case II, each slope in P_Q is at least 1. Also, when Q is from case III, we know from formula (18) that each slope in P_Q is at least 1. This gives

$$\frac{1}{a}NP_p\left(U_p \circ C \mid V\right)_{<1} \geq \{\underbrace{0,\ldots,0}_{g-1+r_0+r_1+r_\infty-\Omega_\rho}\}^{\times a} \bigsqcup \left(\bigsqcup_{i=1}^{\mathbf{m}} S_{\tau_i}^{\times a}\right).$$

Proposition 7.1 follows from (32).

7.3. Finishing the proof

We now finish the proof of Theorem 1.1. From equation (30) and Proposition 7.1, we know

$$NP_q(L(\rho, V, s))_{<1} \geq \{\underbrace{0,\ldots,0}_{g-1+r_0+r_1+r_\infty-\Omega_\rho}\} \bigsqcup \left(\bigsqcup_{i=1}^{\mathbf{m}} S_{\tau_i}\right).$$

Comparing equation (1) with equation (27) gives

$$L(\rho, V, s) = L(\rho, s) \cdot \prod_{\substack{Q \in W \\ Q \neq \tau_i}} \left(1 - \rho\left(Frob_Q\right) s\right).$$

This product has $r_0 + r_1 + r_\infty - \mathbf{m}$ terms, each accounting for a slope 0 segment. Thus,

$$NP_q(L(\rho, s))_{<1} \geq \{\underbrace{0,\ldots,0}_{g-1+\mathbf{m}-\Omega_\rho}\} \bigsqcup \left(\bigsqcup_{i=1}^{\mathbf{m}} S_{\tau_i}\right).$$

From the Euler–Poincaré formula (see, e.g., [24]) we know that $L(\rho, s)$ has degree $2(g-1+\mathbf{m}) + \sum\left(s_{\tau_i} - 1\right)$. This accounts for the remaining slope 1 segments. The proof is complete.

List of symbols

X	A smooth proper curve over $\mathbb{F}_q$.
ρ	An Artin character on X.
s_Q	The Swan conductor of ρ at Q.
$\mathbf{e}_Q$	The exponent of ρ at Q.
ϵ_Q	The natural number between 0 and $q-2$ such that $\frac{\epsilon_Q}{q-1}$ represents $\mathbf{e}_Q$.
ω_Q	The sum of the p-adic digits of ϵ_Q.
$\tau_1, \ldots, \tau_{\mathbf{m}}$	The points of X where ρ is ramified.
$\mathbf{m}$	The number of points in X where ρ is ramified.
Ω_ρ	The sum $\sum\limits_{Q \in W} \frac{1}{a(p-1)} \omega_Q$.
$HP(\rho)$	The Hodge polygon associated to ρ.
$\mathcal{E}_t$	The Amice ring with parameter t.
$\mathcal{E}_t^\dagger$	The bounded Robba ring with parameter t.
$\mathcal{E}(0, r]$	The ring of bounded functions that converge in the annulus $0 < v_p(t) \leq r$.
η	A morphism from X to $\mathbb{P}^1$ with specific ramification properties.
W	The points of X lying above $\{0, 1, \infty\}$ – that is, $\eta^{-1}(\{0, 1, \infty\})$.
e_Q	The ramification index of η at $Q \in X$.
V	The affine curve $\eta^{-1}\left(\mathbb{P}^1 - \{0, 1, \infty\}\right)$ contained in X.
$\mathbf{V}$	A formal lift of V.
$\mathcal{V}^{rig}$	The rigid fibre of $\mathbf{V}$.

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$B^\dagger$	The ring of 'overconvergent' functions on $\mathbf{V}$.
$\mathcal{B}^\dagger$	The ring of 'overconvergent' functions on $\mathcal{V}^{rig}$.
$\mathbf{r}$	A tuple $\mathbf{r} = (r_Q)_{Q\in W}$ of positive rational numbers, where each r_Q represents a radius of convergence around the point Q.
$\mathcal{B}(0,\mathbf{r}]$	The ring of functions on $\mathcal{V}$ converging on the annulus $0 < v_p(u_Q) \leq r_Q$ for each $Q \in W$.
$\mathcal{D}_{\mathbf{e},s}$	A subspace of $\bigoplus_{j=0}^{a-1} \mathcal{O}_{\mathcal{E}^\dagger}$ depending on a tame exponent $\mathbf{e}$ and a Swan conductor s .
$\mathcal{D}$	A subspace of $\mathcal{O}_{\mathcal{E}^\dagger}$ used for studying U_p for the second type of Frobenius endomorphism.
α_0	The p -Frobenius structure associated to ρ^{wild}.
α	The q-Frobenius structure associated to ρ^{wild}.
N	The q-Frobenius structure associated to $\bigoplus_{j=0}^{a-1} \chi^{\otimes p^j}$.
N_0	The p -Frobenius structure associated to $\bigoplus_{j=0}^{a-1} \chi^{\otimes p^j}$.
C_Q	The well-behaved p -Frobenius structure associated to ρ_Q.
b_Q	The 'change of basis' element between the wild global and local p -Frobenius structures at Q.
M_Q	The 'change of basis' matrix between the tame global and local p -Frobenius structures at Q.
$\mathcal{R}_Q^\dagger$	a copies of the bounded Robba ring with parameter u_Q: $\bigoplus_{j=0}^{a-1} \mathcal{E}_Q^\dagger$.
$\mathcal{O}_{\mathcal{R}_Q^\dagger}$	a copies of the integral Robba ring with parameter u_Q: $\bigoplus_{j=0}^{a-1} \mathcal{O}_{\mathcal{E}_Q^\dagger}$.
$\mathcal{O}_{\mathcal{R}_Q}^{con}$	The subspace of functions at Q with precise growth conditions determined by $\eta(Q)$ and the ramification datum at Q.
$\mathcal{R}$	The direct sum over $Q \in W$ of each $\mathcal{R}_Q$.
$\mathcal{R}^\dagger$	The direct sum over $Q \in W$ of each $\mathcal{R}_Q^\dagger$.
$\mathcal{R}_Q^{trun}$	Truncated elements of $\mathcal{R}_Q$ depending on $\eta(Q)$.
$\mathcal{R}^{trun}$	The direct sum over $Q \in W$ of each $\mathcal{R}_Q^{trun}$.
$\mathcal{O}_{\mathcal{R}}^{con}$	The direct sum of $\mathcal{O}_{\mathcal{R}_Q}^{con}$ over $Q \in W$.
C	The endomorphism of $\mathcal{R}$ acting on the Q coordinate by C_Q.
M	The endomorphism of $\mathcal{R}$ acting on the Q coordinate by M_Q.
b	The endomorphism of $\mathcal{R}$ acting on the Q coordinate by $\mathbf{diag}(b_Q)$.

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