GENERALIZED SEMIGROUPS OF QUOTIENTS

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In ([6]; pages 36–41), Lambek constructs the maximal ring of quotients $Q(R)$ of a commutative ring R by defining a multiplication on $\varinjlim \mathrm{Hom}_R(D, R)$ where D ranges over all the dense ideals of R, and this generalizes the classical construction of ring of quotients. (cf. [6] for all the references on the subject.)

This programme is carried over, in the first section of this article, to the category of commutative reductive semigroups. Examples show that the maximal semigroup of quotients of a commutative monoid can be different from the classical one.

In [6] again (pages 94–101), the complete ring of quotients of a (not necessarily commutative) ring R is given. It is essentially the bicommutator of the injective envelope of the ring R considered as a right R-module over itself, and it is shown to coincide with the maximal ring of quotients of R. This construction is extended to (not necessarily commutative) monoids in Section 2, and we show that contrary to rings the two constructions do not in general coincide.

In the last section, the notion of rational completion of a ring, as introduced in [5], is also extended to monoids and shown to coincide with the construction of Section 2.

The proofs of the theorems will usually be omitted or only sketched when they are similar to the corresponding ones for rings: the details can then be found in [6].

1. Maximal semigroups of quotients. DEFINITION 1. An ideal D of a commutative semigroup S is *dense* in S whenever for any two distinct elements s_1 and s_2 in S, there exists an element d in D such that $s_1 d \neq s_2 d$.

If a semigroup contains at least one dense ideal, then S is said to be *reductive*. In particular, every monoid is reductive.

From now on in this section, semigroup will always mean commutative reductive semigroup. The letters $S, S', \ldots$ will always designate such semigroups, while the set of (semigroup) homomorphisms between S and S' will be denoted by $\mathrm{Hom}(S, S')$, a typical element of which being $f: S \to S'$. $A_S, B_S, \ldots$ will stand for right S-sets (see [1]), $f: A_S \to B_S$ being an S-homomorphism and $\mathrm{Hom}_S(A, B)$ the set of all such S-homomorphisms between A_S and B_S. If S is a subsemigroup of S', we write $S \subseteq S'$ and similarly $A_S \subseteq B_S$.

PROPOSITION 1. *Let D and D' be ideals of S.*

(1) *If D is dense and $D \subseteq D'$, then D' is dense.*

(2) *If D and D' are dense, then so are $D \cap D'$ and $D \cdot D' = \{dd' : d \in D, d' \in D'\}$.*

(3) *If $f: D_S \to S_S$ and $f': D'_S \to S_S$ both agree on some dense ideal D'' of S, where D and D' are dense in S, then f and f' agree on $D \cap D'$.*

Proof of (3). If $d \in D \cap D'$ then for all $x \in D''$, $(f(d))x = (f(x))d = (f'(x))d = (f'(d))x$ which implies the result.

Now the set $\mathscr{D}(S)$ of all dense ideals of S is directed by the relation $D_i \leqq D_j$ iff $D_i \supseteq D_j$, and if $\kappa_j^i: \mathrm{Hom}_S(D_i, S) \to \mathrm{Hom}_S(D_j, S)$ is defined by $\kappa_j^i(f) = f \mid D_j$ (f restricted to D_j) then $\bar{\mathscr{D}}(S) = \{\mathrm{Hom}_S(D, S): D \in \mathscr{D}(S)\}$ is a directed system. Taking the union of the set $\bar{\mathscr{D}}(S)$ we let $Q(S) = \bigcup \bar{\mathscr{D}}(S)/\equiv$ where for any $f: D_S \to S_S$ and $f': D'_S \to S_S$, $f \equiv f'$ iff f and f' agree on some dense ideal of S. This is an S-congruence relation (cf. [1]) on the right S-set $\cup \bar{\mathscr{D}}(S)$ and $Q(S)$ is obviously a right S-set with $[f]s = [fs]$ where $s \in S$ and $[f]$ is the equivalence class of f modulo $\equiv$, fs being defined by $(fs)(d) = f(ds)$ for all $d \in D$. It is then easy to verify that $Q(S)$ is the direct limit of $\bar{\mathscr{D}}(S)$.

Let us agree to call an S-homomorphism $f: D \to S$, with D a dense ideal of S, a *fraction* on S. We then obtain the following:

THEOREM 2. *If S is a reductive semigroup (resp. monoid), then $Q(S)$ is a reductive semigroup (resp. monoid) with $[f] \cdot [f'] = [f \cdot f']$, where $f \cdot f': D \cdot D' \to S$ is defined by $(f \cdot f')(d \cdot d') = f(d) \cdot f(d')$, $f: D \to S$ and $f': D' \to S$ being two fractions on S. Moreover the mapping $\kappa: S \to Q(S)$ defined by $\kappa(s) = [s/1]$ where $(s/1)(x) = sx$ for all $x \in S$ is an injective homomorphism (resp. unitary).*

To show that this generalizes the classical construction, let $\mathscr{D}'(S)$ be the union of S and the set of principal dense ideals dS of S (i.e., d is cancellable) and $\bar{\mathscr{D}}'(S) = \{\mathrm{Hom}_S(D, S): D \in \mathscr{D}'(S)\}$ the corresponding directed system. Then again one easily verifies that $Q_{cl} = \bigcup \bar{\mathscr{D}}'(S)/\equiv$ is a reductive semigroup (resp. monoid) isomorphic to the direct limit of $\bar{\mathscr{D}}'(S)$, and as in the case of rings, Q_{cl} is also isomorphic over S (i.e. S remains invariant) to the classical semigroup of quotients (resp. monoid of quotients) of S. In fact, every element of $Q_{cl}(S)$ is of the form $[s/1]$ or $[s/d]$ where d is cancellable and $(s/d)(dx) = sx$, and κ factors through $\kappa': S \to Q_{cl}(S)$ with $\kappa'(s) = [s/1]$.

We shall often write more simply $S \subseteq Q_{cl}(S) \subseteq Q(S)$ and equality when the embeddings are isomorphisms. We finally note that when every element of S is cancellable then $Q_{cl}(S) = Q(S)$ and this is also true when every ideal of S is principal.

PROPOSITION 3. *Every equivalence class $[f]$ of $Q(S)$ contains exactly one irreducible fraction (i.e. of which the domain cannot be properly extended) which extends all fractions in the class.*

Proof. If $[f] = \{f_i: D_i \to S: i \in I\}$, define $\bar{f}$ on the union $\bigcup_{i \in I} D_i$ by $\bar{f}(d) = f_i(d)$ if $d \in D_i$.

THEOREM 4. *The following are equivalent.*

(1) $S = Q(S)$.
(2) *Every irreducible fraction has domain S and is thus equal to $s/1$ for some $s \in S$.*
(3) *For every fraction $f: D \to S$ there exists an $s \in S$ such that for all $d \in D$, $f(d) = sd$.*

Proof. (1)$\Rightarrow$(2). For any irreducible fraction $f: D \to S$ there exists an $s \in S$ such that $[f] = [s/1]$ by hypothesis, but f and $s/1$ are both irreducible and thus $f = s/1$.

(2)$\Rightarrow$(3). If $f: D \to S$ is a fraction then there is an $s \in S$ such that $\bar{f} = s/1$ by (2), and thus for all $d \in D$, $f(d) = sd$.

(3)$\Rightarrow$(1). The former says that $[f] = [s/1]$.

PROPOSITION 5. *If $q = [f] \in Q(S)$ then $q^{-1}S = \{s \in S: qs = [sf] \in S\}$ is a dense ideal of S and if $\phi: K \to Q(S)$ is a fraction on $Q(S)$ then $D = \phi^{-1}(S) \cap S \subseteq K$ is a dense ideal of S.*

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Proof. $D \subseteq q^{-1}S$ where $f: D \to S$ since for any $d \in D$, $qd = d[f] = [f(d)/1] \in S$, which implies that $q^{-1}S$ is dense by (2) of Proposition 1. If $k \in K$, then $k^{-1}S$ and $(\phi(k))^{-1}S$ are both dense ideals of S, together with their intersection. If $D' = k^{-1}S \cap (\phi(k))^{-1}S$, then $\phi(kD') = (\phi(k))D' \subseteq S$ and thus $kD' \subseteq D$. Now kD' is dense in S for if $s(kd') = s'(kd')$ for all $d' \in D'$, then $sk = s'k$ (where S and $\kappa(S)$ are identified), and since this is true for all $k \in K$, $s = s'$, the latter being any two elements of S. D is thus dense in S by Proposition 1.

DEFINITION 2. If S is a sub-semigroup of S' and A_S a sub-S-set of S'_S, then A_S is said to be *dense* in S' whenever for any two distinct elements s and s' in S', there exists an $a \in A$ such that $sa \neq s'a$. If D is an ideal of S, then we say that D is *dense in* S'.

PROPOSITION 6. *If $S \subseteq S' \subseteq Q(S)$ then D is a dense ideal of S iff it is dense in S'. In particular, $q^{-1}S$ is dense in S' for all q in $Q(S)$.*

Proof. If for all s in D, $[sf_1] = [sf_2]$, then sf_1 and sf_2 agree on some dense ideal of S which implies that $[f_1] = [f_2]$.

THEOREM 7. $Q(Q(S)) = Q(S)$.

Proof. If $\phi: K \to Q(S)$ is a fraction on $Q(S)$, $D = \phi^{-1}S \cap S$ and f denotes the restriction of ϕ to D, then f is a fraction on S with domain D. Thus for $k \in K$ and $d \in D$, $(\phi(k))d = (\phi(k))[d/1] = k \cdot \phi([d/1]) = k$. $[f(d)/1] = ([f] \cdot k)d$ which means that $\phi(k) = [f] \cdot k$ by the last two propositions, and the result follows from (3) of Theorem 4.

Thus an iteration of the operation yields nothing new.

DEFINITION 3. If $S \subseteq S'$ then S' is said to be a *semigroup of quotients* of S whenever for all $s \in S'$, $s^{-1}S = \{x \in S : sx \in S\}$ is dense in S'.

Or, said otherwise, if s_1 and s_2 are distinct in S' and $s_0 \in S'$, then there is an $s \in S$ such that $s_0 s \in S$ and $s_1 s \neq s_2 s$.

THEOREM 8. *If $\kappa': S \to S'$ is an injective homomorphism, where S' is a semigroup of quotients of S, then there exists a unique injective homomorphism $\bar{\kappa}: S' \to Q(S)$ such that $\bar{\kappa} \cdot \kappa' = \kappa: S \to Q(S)$.*

Proof. For each $s_0 \in S'$, $s_0^*: s_0^{-1}S \to S: s \to s_0 s$ is a fraction on S since $s_0^{-1}S$ is a dense ideal of S by hypothesis, and thus $\bar{\kappa}: S' \to Q(S): s_0 \to [s_0^*]$ is an injective homomorphism since $(s_1 s_2)^*$ and $s_1^* \cdot s_2^*$ agree on the intersection $((s_1 s_2)^{-1}S) \cap (s_1^{-1}S \cdot s_2^{-1}S)$ of their respective domains, and if $[s_1^*] = [s_2^*]$ then for all $d \in s_1^{-1}S \cap s_2^{-1}S$, $s_1 d = s_2 d$, and thus $s_1 = s_2$. κ and $\bar{\kappa} \cdot \kappa'$ are obviously equal and if $\psi \cdot \kappa' = \kappa$ then for all $s \in S'$, $\psi(s)$ and $\bar{\kappa}(s)$ agree on the dense ideal $s^{-1}S$ which proves the uniqueness.

Thus if $Q(S') = S'$ then $\bar{\kappa}$ is an isomorphism and $Q(S)$ is unique up to isomorphism over S with the property that $Q(Q(S)) = Q(S)$. Also by Proposition 6 and Theorem 8, if $S \subseteq S'$ then S' is a semigroup of quotients of S iff for all $s \in S'$, $s^{-1}S$ is a dense ideal of S, or equivalently, iff $S \subseteq S' \subseteq Q(S)$, (more precisely: up to isomorphism over S), and thus the intersection of any family of semigroups of quotients of S is a semigroup of quotients of S.

We now proceed to give some examples. In the rest of this section, R will always denote a commutative ring with identity. By a *semi-ideal* of R, we will understand an ideal of R, the latter being considered as a monoid, while an ideal will be an ordinary ideal, and similarly for semi-fractions and fractions. $Q_{cl}(R)$ and $Q(R)$ will be the classical and maximal monoids of quotients of R regarded as a monoid, with $\bar{Q}_{cl}(R)$ and $\bar{Q}(R)$ the classical and maximal rings of quotients of the ring R as constructed in [6], and density in the sense of Definition 1 as applied to R will be called *semi-density*.

It is clear that an ideal D of R is dense iff it is semi-dense and that $Q_{cl}(R) = \bar{Q}_{cl}(R)$ for the principal semi-ideals of R are those of the form rR and the latter are dense iff r is cancellable in R.

THEOREM 9. $R \subseteq \bar{Q}_{cl}(R) = Q_{cl}(R) \subseteq \bar{Q}(R) \subseteq Q(R)$.

Proof. An ideal D of R is semi-dense iff it is dense, thus every fraction is a semi-fraction and two fractions are equivalent iff they are equivalent as semi-fractions.

The above theorem guarantees the existence of monoids with a maximal monoid of quotients distinct from the classical one, since such examples are known for rings, but before giving an effective example, we review some properties of lower semi-lattices.

PROPOSITION 10. *If M is a lower semi-lattice with largest element* 1 (*i.e., a commutative idempotent monoid*), *then $M = Q_{cl}(M)$ and $Q(M)$ is a semi-lattice with* 1.

Proof. The first statement is true since the only cancellable element of M is 1, and $Q(M)$ is a semi-lattice since for any fraction f on M, $f^2 = f$ (cf. Theorem 16 of [1]).

In [3], Brainerd and Lambek have shown that the maximal ring of quotients $\bar{Q}(R)$ of a Boolean ring R with 1 is its Dedekind completion $D(R)$. If we apply this to the preceding theory, we get:

THEOREM 11. *If R is a Boolean ring with* 1, *then*

$$R = \bar{Q}_{cl}(R) = Q_{cl}(R) \subseteq \bar{Q}(R) = D(R) = Q(R).$$

Proof. By Theorem 9, $R \subseteq D(R) \subseteq Q(R)$, and thus $D(R)$ is a monoid of quotients of R by the remark following Theorem 8. But $D(R) = Q(R)$ since it satisfies (3) of Theorem 4 by Theorem 17 of [1]. (In fact, as pointed out to me by Fred McMorris, Theorem 17 of [1] need not be true if S is a semilattice or even a lattice, but it is true if S is a chain or a Boolean algebra since in the latter two cases infinite distributivity does hold and here $D(R)$ is a Boolean algebra.)

COROLLARY. *If R is a non complete Boolean ring with* 1, *then $Q_{cl}(R) \neq Q(R) = D(R)$.*

2. Complete monoids of quotients. The monoids considered in this section are not necessarily commutative and the letters $M, M', \ldots$ will always denote such monoids. Homomorphism will mean unitary homomorphism (i.e. the identity element is mapped onto the identity element) and $A_M, B_M, \ldots$ will always denote right M-sets.

If A_M is such an M-set, T_A the monoid of all endomaps of the set A and $\phi: M \to T(A)$ the antirepresentation of M in T_A defined for all m in M and a in A by $(\phi(m))(a) = am$, then $\phi(M)$ is a submonoid of T_A and its centralizer (or commutator) H in T_A is by definition the commutator of A_M and is equal to $\mathrm{Hom}_M(A, A)$. Repeating the procedure leads to the bicommutator of A_M which is equal to the monoid $\mathrm{Hom}_H(A, A)$ since A is a left H-set.

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If M is a given monoid, replace in the above introduction A_M by the injective envelope I_M of M_M (cf. [1]) and denote the M-embedding of M_M into I_M by $\imath$. Now once and for all we set $H = \mathrm{Hom}_M(I, I)$ and let $B = B(M)$ be the dual of the bicommutator of I_M, i.e., $B = \mathrm{Hom}_H(I, I)$ with the H-homomorphisms written to the right of their argument. B is then a submonoid of the dual of T_I and ϕ becomes an injective homomorphism with $\phi(M)$ a submonoid of B since for any h in H, i in I and m in M, $h((i)(\phi(m))) = h(im) = (h(i))m = (h(i))(\phi(m))$.

In fact, we shall in the future consider M as a submonoid of B (written $M \subseteq B$) and this makes B into a right M-set. Similarly, ${}_HH$ and ${}_HI$ are both left H-sets, and " evaluation at 1 ", i.e. $\gamma: {}_HH \to {}_HI$ defined by $\gamma(h) = h(1)$, 1 the identity of $M \subseteq I$, is a surjective H-homomorphism since I_M is injective (same proof as for rings again). ϕ introduced above becomes an M-homomorphism $\phi: M_M \to B_M$ since for any i in I, m and m' in M, $(i)\phi(mm') = i(mm') = (im)m' = ((i)(\phi(m)))m'$, and the M-embedding $\imath: M_M \to I_M$ factorizes through $\psi \cdot \phi$, where $\psi: B_M \to I_M$ is defined by $\psi(b) = (1)b$ and is injective (cf. [6]; Lemma 1).

Writing $\psi(B_M) = (1B)_M$ we obtain the following characterization of $1B$ (which corresponds to Proposition 1, page 94 of [6] and of which the proof is essentially the same).

PROPOSITION 12. *If $i \in I$, then $i \in 1B$ if and only if for any h and h' in H, $h(1) = h'(1)$ implies that $h(i) = h'(i)$.*

It is clear that I, $1B$ and B are all right B-sets, that $1B$ becomes a monoid extending M by defining $(1)b \cdot (1)b' = (1)(bb')$ and that $(1B)_B$ is a sub B-set of I_B.

LEMMA. *$g: C_B \to I_B$ is an M-homomorphism if and only if it is a B-homomorphism.*

Proof. For each c in C, define g_c and g'_c by $g_c(b) = g(cb)$ and $g'_c(b) = (g(c))b$ for all b in B. Both are M-homomorphisms of B_M into I_M and can thus be lifted to $\bar{g}_c$ and $\bar{g}'_c$ respectively, both in H, by the injectivity of I_M and the fact that $\psi: B_M \to I_M$ is injective. Now $\bar{g}_c$ and $\bar{g}'_c$ agree on $1B$ by Proposition 12 since they agree on 1, i.e. $g(cb) = (g(c))b$ for all c in C and b in B.

PROPOSITION 13. *I_B is the injective envelope of $(1B)_B$.*

Proof. By the corollary of Theorem 11 of [1] it suffices to show that I_B is injective and that it is an essential extension of $(1B)_B$. Let θ be a B-congruence relation on I_B with restriction to $(1B)_B$ the identity relation. θ is then an M-congruence relation on I_M with restriction to M_M the identity relation, which implies that θ is also the identity on I_M since I_M is an essential extension of M_M (Theorem 7 of [1]) and thus θ is the identity on I_B. The latter is injective for let $f: A_B \to I_B$ and $\delta: A_B \to C_B$, δ injective. There then exists an M-homomorphism $g: C_M \to I_M$ such that $g \cdot \delta = f$ and by the Lemma, g is also a B-homomorphism.

THEOREM 14. *The monoids $B(M)$ and $B(B)$ are isomorphic.*

Proof. The monoids H and $H' = \mathrm{Hom}_B(I_B, I_B)$ are equal since for any i in I, b in B and h in H, $h((i)b) = (h(i))b$.

And so, iterating the construction we have been studying in this section does not lead to anything new, and we call $B(M)$ the *complete monoid of quotients* of M.

PROPOSITION 15. *The following are equivalent.*

(1) *The H-epimorphism* $\gamma: {}_H H \to {}_H I$ *is an isomorphism.*
(2) *The injective M-homomorphism* $\psi: B_M \to I_M$ *is an isomorphism.*
(3) *The two monoids H and B are isomorphic under the correspondence* $h \to b$ *if and only if* $h(1) = (1)b$.
(4) B_M *is injective.*
(5) *The injective B-homomorphism* $\psi: B_B \to I_B$ *is an isomorphism.*
(6) B_B *is injective.*

Proof. Essentially the same as that of Proposition 3, page 95 of [**6**].

PROPOSITION 16. $P = \gamma^{-1}(1B)$ *is a sub-monoid of H and* $\bar{\gamma}: P \to B$ *defined by* $\bar{\gamma}(h) = \psi^{-1} \cdot \gamma(h)$ (*i.e.* $\gamma(h) = b$ *if and only if* $h(1) = (1)b$) *is a surjective homomorphism. Thus* $\gamma': P/\theta \to B$, *with* $\gamma'(\theta h) = \gamma(h)$ *is an isomorphism, where θ defined on P by* $h \,\theta\, h'$ *if and only if* $h(1) = h'(1)$ *is the " kernel " of γ.*

Proof. Let h and h' be in P, i.e., there exist b and b' in B such that $h(1) = (1)b$ and $h'(1) = (1)b'$. Then $h(h'(1)) = h((1)b')\,(h(1))b' = ((1)b)b' = (1)(b \cdot b') \in 1B$, and this implies that P is a monoid. θ is obviously an equivalence relation on P. It is a congruence for let $h_1 \,\theta\, h_2$ and $h'_1 \,\theta\, h'_2$, then $h'_1(1) = h'_2(1)$, which implies that $h_1(h'_1(1)) = h_1(h'_2(1))$, i.e. $h_1 h'_1 \,\theta\, h_1 h'_2$. Similarly $h_1 \,\theta\, h_2$ implies $h_1 h'_2 \,\theta\, h_2 h'_2$ since $h_2 \in P$. Thus by transitivity, $h_1 h'_1 \,\theta\, h_2 h'_2$. The rest is obvious.

Finally as in the case of rings again, $H \cap B$ is the center of H, that is,

$$\{b \in B : (\forall m \in M)\, bm = mb\},$$

while if M is commutative and we write $b(i) = (i)b$ for all $i \in I$ and $b \in B$, then B is the center of H and is thus commutative.

We will now show that $Q(M)$ constructed in the first section can be regarded as a sub-monoid of $B(M)$.

DEFINITION 4. A sub-M-set A_M of B is said to be *M-dense* whenever for any h and h' in H, if $h(1) \neq h'(1)$, there exists an $a \in A$ such that $h((1)a) \neq h'((1)a)$.

PROPOSITION 17. A_M *is M-dense if and only if for any i and i′ in I, if* $i \neq i'$, *then there exists an* $a \in A$ *such that* $(i)a \neq (i')a$.

Proof. This follows from the fact that $\gamma: {}_H H \to {}_H I$ is surjective.

PROPOSITION 18. *If M is commutative, then an ideal D of M is dense in M* (*Definition* 1) *if and only if it is M-dense.*

Proof. If D is dense in M then η defined on I_M by $i \,\eta\, i'$ iff $(\forall d \in D)(i)\,d = (i')\,d$ is an M-congruence relation by the commutativity of M, with restriction to M_M the equality relation by the density of D: thus η is also the equality relation on I_M since I_M is an essential extension of M_M (Theorem 7 of [**1**]) and D is M-dense. The converse is obvious.

PROPOSITION 19. *If A_M is M-dense and if for all $a \in A$, $ba = b'a$, where b and b' are in B, then $b = b'$.*

Proof. $ba = b'a$ implies that $((1)b)a = ((1)b')a$ for all $a \in A$, and the result follows from Proposition 17.

PROPOSITION 20. *If M is commutative then an ideal D of M is dense in M if and only if it is dense in B (Definition 2).*

Proof. By Proposition 18, D is dense in M iff D is M-dense and the result follows from Proposition 19.

PROPOSITION 21. *If D_M and C_M are two sub M-sets of B_M with D_M M-dense, then for any $f \in \mathrm{Hom}_M(D, C)$ there exists a unique $b \in B$ such that $f(d) = bd$ for all $d \in D$.*

Proof. Let again $1D = \psi(D)$ and $1C = \psi(C)$ and define $f^*: (1D)_M \to (1C)_M$ by $f^*((1)d) = (1)(f(d))$. Since f^* is an M-homomorphism and I_M is injective, it can be lifted to $\bar{f}^* \in H$. In fact $\bar{f}^*(1B) \subseteq 1B$ by Proposition 12 for if $h_1(1) = h_2(1)$, h_1 and h_2 in H, then for all $d \in D$,

$$h_1(\bar{f}^*((1)d)) = (h_1(1))(f(d)) = (h_2(1))(f(d)) = h_2(\bar{f}^*((1)d))$$

and thus $(h_1 \cdot \bar{f}^*)(1) = (h_2 \cdot \bar{f}^*)(1)$ by the M-density of D, which implies that for all $b \in B$, $h_1(\bar{f}^*((1)b)) = h_2(f^*((1)b))$. In particular, there exists a $b \in B$ such that $\bar{f}^*(1) = (1)b$, which implies that for all $d \in D$, $(1)(f(d)) = (f^*(1))d = ((1)b)d = (1)bd$, and thus $f(d) = bd$ while the uniqueness of b follows from Proposition 19.

THEOREM 22. *If $f: D_M \to M_M$ is a fraction on M with $[f] \in Q(M)$, then the mapping $[f] \to b$ (b defined in Proposition 21) is an injective homomorphism of $Q(M)$ into B with restriction to M equal to the embedding $\phi: M \to B$.*

Proof. Any two fractions f and f' on M are equivalent iff there is a dense ideal D'' such that for all $d \in D''$, $bd = b'd$, where $f' \leftrightarrow b'$, which means that $b = b'$ by Proposition 19, and thus we get an injection $[f] \to b$. This is a homomorphism for if $f \cdot f' \leftrightarrow b''$, where $f': D'_M \to M_M$, then for all $d \in D \cap D'$,

$$(1)((f \cdot f')(d)) = \bar{f}^*((1)(f'(d))) = \bar{f}^*((1)b'd) = \bar{f}^*(((1)d)b') = (\bar{f}^*((1)d)b'$$

$$= ((1)(f(d)))b' = ((1)bd)b' = (1)((bb')d),$$

where $\bar{f}^*$ was defined in the last proposition, and thus $bb' = b''$ by the uniqueness property. The last assertion is obvious.

As we shall see, the above embedding is not in general surjective, but first we need a new characterization of $Q(M)$.

PROPOSITION 23. *If D_M and C_M are two sub M-sets of B_M with D_M M-dense, then $\mathrm{Hom}_M(D, C)$ and $C:D = \{b \in B \mid bD \subseteq C\}$ are isomorphic sets under the mapping $b \to b^*$, where $b \in B$ and $b^*(d) = bd$ for all $d \in D$. If M is commutative, this extends to an isomorphism of (left and right) M-sets.*

Proof. The mapping is injective by Proposition 19 and surjective by Proposition 21. If M is commutative, then for any $m \in M$, $d \in D$, $b \in B$, $(bm)^*(d) = bmd = (mb^*)d$.

LEMMA. *If $\mathcal{M}$ is a family of left ideals J of M closed under products and containing M, then $\bigcup_{J \in \mathcal{M}} (M:J)$ is a submonoid of B.*

Proof. If $b \in M:J$ and $b' \in M:J'$, then for any $d \in J$ and $d' \in J'$, $(bb')(d'd) = b((b'd')d) \in M$ since $b'd' \in M$ and J is a left ideal, which implies that $bb' \in M:J'J$.

THEOREM 24. *If M is commutative, then $Q(M)$ is isomorphic to $\bigcup_{D \subseteq M} (M:D)$ through the injection $[f] \to b$ of Theorem 22, where D varies over all the dense ideals of M.*

Proof. If $b \in M:D$ for some dense ideal D then using Propositions 23 and 21 we get $[b^*] \to b$, which shows that the above injection is onto.

THEOREM 25. *If M is commutative, then the injection $Q(M) \to B: [f] \mapsto b$ (of Theorem 22) is an isomorphism if and only if for all $b \in B$, $b^{-1}M = \{m \in M \mid bm \in M\}$ is dense in M.*

Proof. The condition says that B is a monoid of quotients of M, by Definition 3, and the result follows from Theorem 8.

We will now give an example of a commutative monoid where $Q(M)$ and $B(M)$ are not isomorphic. In what follows, M will denote a lower semi-lattice $(M, \leqq, 1)$ with largest element 1 (i.e. a commutative idempotent monoid $(M, \cdot, 1)$ with $m \cdot m' = m$ iff $m \leqq m'$), I its Dedekind–MacNeille completion and K an ideal of I (i.e. a semi-filter of $(I, \leqq, 1)$). (cf. **[1]**, Section 3.)

LEMMA. *If $f \in \mathrm{Hom}_M(K, I)$ then for all $k \in K$ and $m \in M$, $k \leqq m$ implies that $f(k) \leqq m$.*

Proof. $k \leqq m$ means that $km = k$ and this implies that $f(k) = f(km) = (f(k))m$, i.e. $f(k) \leqq m$.

PROPOSITION 26. *If $f \in \mathrm{Hom}_M(K, I)$, then for any $k \in K$, $f(k) \leqq k$.*

Proof. If $K_k = \{m \in M \mid k \leqq m\}$ then $k = \inf K_k$ and by the above Lemma $f(k) \leqq m$ for all $m \in K_k$, which implies that $f(k) \leqq k$.

For the rest of this section, we moreover assume that M is a chain (i.e. linearly ordered) with I and K as above (see Theorem 12 of **[1]**).

PROPOSITION 27. $\mathrm{Hom}_M(K, I) = \mathrm{Hom}_I(K, I)$.

Proof. Let $f \in \mathrm{Hom}_M(K, I)$, $k \in K$ and $i \in I$. To show that $f(ki) = (f(k))i$ we must study all the possibilities.

(1) $k \leqq i$: then $f(ki) = f(k) \leqq k \leqq i$ by Proposition 26, and thus $(f(k))i = f(k)$.

(2) $i < k$: then $f(ki) = f(i)$ and we must show that $f(i) = (f(k))i$. By Proposition 26 again, $f(k) \leqq k$ and $f(i) \leqq i$, and this leads to the three sub cases:

(i) $f(i) \leqq f(k) \leqq i \leqq k$: then $(f(k))i = f(k)$ and if $f(i) < f(k)$ then there exists an $m \in M$ with $f(i) < m < f(k)$ which implies that $f(i) = (f(i))m = f(im) = f(m) = f(km) = (f(k))m = m$, a contradiction.

(ii) $f(k) \leqq f(i) \leqq i \leqq k$: then $(f(k))i = f(k)$ and if $f(k) < f(i)$ there is an $m \in M$ with $f(k) < m < f(i)$, which implies that $f(k) = (f(k))m\ f(km) = f(m) = f(im) = (f(i))m = m$, a contradiction.

(iii) $f(i) \leqq i \leqq f(k) \leqq k$: then $(f(k))i = i$ and if $f(i) < i$ there will be another $m \in M$ with $f(i) < m < i$ and thus $f(i) = (f(i))m = f(im) = f(m) = f(km) = (f(k))m = m$, a contradiction.

THEOREM 28. *If $H = \mathrm{Hom}_M(I, I)$ then the epimorphism $\gamma: {}_HH \to {}_HI$ is an H-isomorphism.*

Proof. By the above Proposition, $\mathrm{Hom}_M(I, I) = \mathrm{Hom}_I(I, I)$ and the latter is obviously H-isomorphic to ${}_HI$.

COROLLARY 1. *$\psi: B_M \to I_M$ is an isomorphism.*

Proof. Follows from Proposition 15 and Theorem 28.

COROLLARY 2. *Every $h \in H$ is of the form $h(i) = i$ if $i \leqq k$, and $h(i) = k$ if $k \leqq i$, for all $i \in I$ and some $k \in I$.*

Proof. Set $k = h(1) = \gamma(h)$ in Theorem 28.

COROLLARY 3. *The monoids H and I are isomorphic (through γ) and $B = H$.*

Proof. If h is as in Corollary 2 and $h'(1) = k'$ where $h' \in H$ and $k \leqq k'$, then $\gamma(hh') = h(h'(1)) = h(k') = k$ and $(\gamma(h)) \cdot (\gamma(h')) = h(1) \cdot h'(1) = kk' = k$. Thus H is commutative since I is and $B = H$ for B is then the center of H.

Now from Theorem 18 of [**1**], $M = Q(M)$ and we thus have examples where $Q(M)$ and $B(M)$ are distinct: just take M to be a non-complete chain with largest element 1 (e.g. the rationals with $+\infty$), then its Dedekind–MacNeille completion I will be distinct from M (e.g. the reals with $+\infty$ in the above example).

In fact the fractions in M, M a dense-in-itself chain, can be characterized as follows.

PROPOSITION 29. *The only dense ideals of M are M and $M' = M - \{1\}$.*

Proof. If $m_1 < m_2$ then there exists an $m \in M$ with $m_1 < m < m_2$ since M is dense-in-itself, and $m_1 = m_1 \cdot m \neq m_2 \cdot m = m$, which means that M' is dense. Conversely, let D be a dense ideal distinct from both M and M', i.e., there exists an $m \in M$, not in D, and with $m < 1$. But for all $d \in D$, $d < m$ (for $m \leqq d$ would imply $md = m \in D$) and $md = d = 1d$, which means that D is not dense in M.

PROPOSITION 30. *For every fraction f in M there exists a unique $k \in M$ such that for all m in the domain of f, $f(m) = km$ (i.e. $f(m) = m$ if $m \leqq k$ and $f(m) = k$ if $k \leqq m$).*

Proof. Follows directly from Theorem 18 of [**1**] and Proposition 21.

And thus two fractions *on* M are equivalent if and only if they are defined by the same $k \in M$, which means that $Q(M) = M$.

3. Rational extensions. In this section, M is again a monoid which is not necessarily commutative.

DEFINITION 5. If A_M and B_M are right M-sets, then a *partial M-homomorphism* f of A_M into B_M is an M-homomorphism into B_M with domain a sub-M-set of A_M. f is said to be *irreducible* if it cannot be properly extended.

PROPOSITION 31. *Every partial M-homomorphism of A_M into B_M can be extended to an irreducible one.*

Proof. Define a partial order on the set of partial M-homomorphisms from A_M to B_M by $f \leqq f'$ iff f' extends f. If $\{f_j \mid j \in J\}$ is a chain of such mappings $f_j : (K_j)_M \to B_M$ then $K = \bigcup_{j \in J} K_j$ is a right M-set and one can define $f : K_M \to B_M$ by $f(k) = f_j(k)$ if $k \in K_j$. The rest follows by Zorn's Lemma.

PROPOSITION 32. *If X_M and Y_M are given right M-sets then the relation $\leqq$ defined on the set of sub-M-sets of Y_M by:*

$A_M \leqq B_M(X_M) \Leftrightarrow$

A_M is a sub-M-set of $B_M \subseteq Y_M$ and any two partial M-homomorphisms of B_M into X_M which agree on A_M have a unique common irreducible extension

is a partial order relation.

Proof. Obvious.

PROPOSITION 33. *$A_M \leqq B_M(X_M)$ if and only if A_M is a sub M-set of B_M and any two partial homomorphisms of B_M into X_M which agree on A_M agree on the intersections of their domains.*

Proof. Let $f' : K'_M \to X_M$, $f'' : K''_M \to X_M$, K'_M and K''_M two sub-M-sets of B_M, f' and f'' both agreeing on A_M. It is then possible to define $f : K' \cup K'' \to X_M$ by $f(k) = f'(k)$ if $k \in K'$ and $f(k) = f''(k)$ if $k \in K''$ since by hypothesis f' and f'' agree on $K' \cap K''$. f extends both f' and f'' and it has an irreducible extension $\bar{f}$ by Proposition 31. The converse is trivial.

DEFINITION 6. If M is a submonoid of M', I_M the injective envelope of M_M and M'_M a sub M-set of I_M, then M' is a *rational* extension of M whenever $M_M \leqq M'_M(I_M)$.

For the next theorem we again need the injective homomorphism $\phi : M \to B(M)$ defined by $(i)(\phi(m)) = im$ for all $i \in I$, and we say that an injective homomorphism $\kappa : M \to M'$ is a rational extension of M whenever M' is a rational extension of $\kappa(M)$.

THEOREM 34. *$\phi : M \to B(M)$ is a rational extension. Moreover, an injective homomorphism $\kappa : M \to M'$ is a rational extension of M if and only if there exists a unique injective homomorphism $\bar{\kappa} : M' \to B(M)$ such that $\bar{\kappa} \cdot \kappa = \bar{\phi}$.*

Proof. Two partial M-homomorphisms from $(1B)_M$ into I_M which agree on M can be extended to two M-endomorphisms of I_M which agree on $(1B)_M$ by Proposition 12 and the original ones thus agree on the intersection of their domains, i.e. ϕ is a rational extension. If

κ is a rational extension and h and h' in H agree on M, then they agree on M', which implies that M' is a submonoid of $1B$ by Proposition 12 again: we thus let $\bar{\kappa}((1)b) = b$. Conversely any two partial M-homomorphisms of M'_M into I_M which agree on M agree on the intersection of their domains since they can be regarded as partial homomorphisms of $(1B)_M$ into I_M.

DEFINITION 7. If $\kappa: M \to \overline{M}$ is a rational extension of M such that for every other rational extension $\kappa': M \to M'$ there exists a unique injective homomorphism $\bar{\kappa}: M' \to \overline{M}$ with $\bar{\kappa} \cdot \kappa' = \kappa$, then κ is called a *rational completion* of M.

Since it is unique up to isomorphism " over M ", we may talk about *the rational completion* of M.

COROLLARY 1. *$\phi: M \to B(M)$ is the rational completion of M.*

COROLLARY 2. *Every monoid of quotients of a commutative monoid is a rational extension of M.*

COROLLARY 3. *The intersection of any family of rational extensions of a monoid M is a rational extension of M.*

We will end with a new characterization of commutative monoids of quotients, but first we prove:

PROPOSITION 35. *An ideal D of a monoid M is M-dense if and only if $D_M \leqq M_M(I_M)$.*

Proof. If h and h' in H agree on D_M the same is true for their restriction to M_M, and the condition implies that they must agree on M_M, i.e. $h(1) = h'(1)$, and thus D is M-dense. Conversely if f and f' are partial M-homomorphisms agreeing on D, then their respective extensions to $\bar{f}$ and $\bar{f}'$ in H also agree on D and thus on M.

In the rest of this section, M is a submonoid of the commutative monoid $M' \subseteq B(M)$ and D is an ideal of M.

PROPOSITION 36. *D is dense in M' (Definition 2) if and only if $D_M \leqq M'_M(I_M)$.*

Proof. Let D be dense in M' and f and f' be two partial M-homomorphisms of M'_M into I_M which agree on D: their respective extensions to $\bar{f}$ and $\bar{f}'$ in H then also agree on D. Now define θ on I_M by $i\theta i'$ iff $(\forall d \in D)(id = i'd)$: this is an M-congruence relation on I_M since M' is commutative, and with restriction to M'_M equal to the identity relation. Since I_M is an essential extension of M'_M ([1], Theorem 7 and its Corollary) θ is also the identity relation on I_M. If now $\bar{f}(1) = i$ and $\bar{f}'(1) = i'$, then for all $d \in D$, $(\bar{f}(1))d = \bar{f}(d) = \bar{f}'(d) = (\bar{f}'(1))d$, which implies that $\bar{f}(1) = \bar{f}'(1)$. Thus $\bar{f}$ and $\bar{f}'$ agree on $1B$ and in particular f and f' agree on the intersection of their domains. Conversely if $D_M \leqq M'_M(I_M)$ and m_0 and m_1 are in M' with $m_0 d = m_1 d$ for all $d \in D$, then m_j^* defined by $m_j^*(m) = m_j m$, $(j = 0, 1)$, for all $m \in M'$, gives two M-homomorphisms of M'_M into I_M which agree on D_M and thus on M'_M. In particular, $m_0^*(1) = m_0 = m_1 = m_1^*(1)$.

THEOREM 37. *The following are equivalent.*

(1) *D is dense in M.*
(2) *D is dense in every monoid of quotients of M.*

(3) *D is dense in* M' (*in particular in* $B(M)$).
(4) *D is M-dense.*
(5) $D_M \leqq M'_M(I_M)$.
(6) $D_M \leqq M_M(I_M)$.
(7) $D_M \leqq M_M(M_M)$.

Proof. The equivalence of the first six has already been proved in Propositions 6, 20, 18, 36 and 35. (7) implies (1) for let m and m' in M' be such that for all d in D, $md = m'd$. m and m' then generate two M-endomorphisms of M_M which agree on D and thus on M by condition (7), and so $m = m'$. That (6) implies (7) is obvious.

Applying this to monoids of quotients, we obtain the fact that M' is a monoid of quotients of M if and only if for all m in M', $m^{-1}M \leqq M_M(M_M)$.

Since the results of Section 1 were first written (see **[2]**), the notion of maximal monoid of quotients has been generalised in **[8]** to noncommutative monoids by using the analogue of Proposition 4, page 96 of **[6]** as a definition of dense ideal. It can be shown that Theorem 22 of the present paper remains valid in the noncommutative case.

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