GENERALIZATION OF LEVI-OKA'S THEOREM CONCERNING MEROMORPHIC FUNCTIONS

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Dedicated to Professor K. Noshiro on his sixtieth birthday

Introduction

As Fuks [3] stated, every domain of holomorphy or meromorphy over C^n is analytically convex in the sense of Hartogs. Oka [6] proved that every domain over C^n analytically convex in the sense of Hartogs is a domain of holomorphy. Therefore a domain of meromorphy over C^n coincides with a domain of holomorphy over C^n.

In the present paper we shall prove that the envelope of meromorphy of a domain (D, φ) over a Stein manifold S with respect to a family of meromorphic functions on D is p_7-convex in the sense of Docquier-Grauert [2] and, therefore, is a Stein manifold. Especially a domain of meromorphy over S coincides with a domain of holomorphy over S.

A complex manifold M is called of *weak* (or *strong*) *Poincaré type* if for any meromorphic function f on M there exist holomorphic functions g and h on M such that $f=g/h$ on M (and that g and h are coprime at each point of M). From Siegel [8] any complex manifold of Cousin-II type is of strong Poincaré type and from Hitotumatu-Kôta [4] any Stein manifold is of weak Poincaré type.

Let (D, φ) be a domain over a Stein manifold and f be a meromorphic function on D. There exists a meromorphic function $\tilde{f}$ on the domain $(\tilde{\lambda}_f, \tilde{D}_f, \tilde{\varphi}_f)$ of meromorphy of f such that $f=\tilde{f}\circ\tilde{\lambda}_f$. As $\tilde{D}_f$ is a Stein manifold which is of weak Poincaré type, there exist holomorphic functions $\tilde{g}$ and $\tilde{h}$ on $\tilde{D}_f$ such that $\tilde{f}=\tilde{g}/\tilde{h}$ on $\tilde{D}_f$. Then holomorphic functions $g=\tilde{g}\circ\tilde{\lambda}_f$ and $h=\tilde{h}\circ\tilde{\lambda}_f$ on D satisfies $f=g/h$ on D. This means that any domain over a Stein manifold is of weak Poincaré type.

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§ 1. Theorem of continuity

LEMMA 1. *The following three assertions are valid for* $n \geqq 2$.

1) *If* f *is meromorphic in a neighbourhood of* $\bigcap_{p=1}^{\infty}\{z=(z_1, z_2, \ldots, z_n)\,;\, z_1=a_1^p, z_2=a_2^p, \ldots, z_{n-1}=a_{n-1}^p, |z_n| \leqq 1\} \cup \{z\,;\, z_1=a_1^0, z_2=a_2^0, \ldots, z_{n-1}=a_{n-1}^0, |z_n|=1\}$, f *can be meromorphically continued in a neighbourhood of* $\{z\,;\, z_1=a_1^0, z_2=a_2^0, \ldots, z_{n-1}=a_{n-1}^0, |z_n| \leqq 1\}$ *where* $a_j^p \to a_j^0$ *as* $p \to \infty$ *for* $j=1, 2, \ldots, n-1$.

2) *If* f *is meromorphic in a neighbourhood of* $\{z\,;\, |z_1|=1, z_2=0, \ldots, z_{n-1}=0, 0 \leqq z_n \leqq 1\} \cup \{z\,;\, |z_1| \leqq 1, z_2=0, \ldots, z_{n-1}=0, z_n=0\}$, f *can be meromorphically continued in a neighbourhood of* $\{z\,;\, |z_1| \leqq 1, z_2=0, \ldots, z_{n-1}=0, 0 \leqq z_n \leqq 1\}$.

3) *If* f *is meromorphic in a neighbourhood of* $\{z\,;\, |z_1|=1, z_2=0, \ldots, z_{n-1}=0, |z_n| \leqq 1\} \cup \{z\,;\, |z_1| \leqq 1, z_2=0, \ldots, z_{n-1}=0, z_n=0\}$, f *can be meromorphically continued in a neighbourhood of* $\{z\,;\, |z_1| \leqq 1, z_2=0, \ldots, z_{n-1}=0, |z_n| \leqq 1\}$.

Proof. At first we shall prove the equivalence of 1), 2) and 3).

1)$\to$2). Let γ be the supremum of $\delta>0$ such that f can be meromorphically continued in a neighbourhood of

$$C_\delta=\{z\,;\, |z_1| \leqq 1, z_2=0, \ldots, z_{n-1}=0, 0 \leqq z_n \leqq \delta\}.$$

Suppose that $\gamma \leqq 1$. Let $\{\delta^p\,;\, p=1, 2, 3, \ldots\}$ be a sequence of positive numbers $\delta^p<\gamma$ such that $\delta^p \to \gamma$ as $p \to \infty$. Since f is meromorphic in a neighbourhood of

$$\bigcup_{p=1}^{\infty}\{z\,;\, |z_1| \leqq 1, z_2=0, \ldots, z_{n-1}=0, z_n=\delta^p\} \cup \{z\,;\, |z_1|=1, z_2=0, \ldots, z_{n-1}=0, z_n=\gamma\},$$

f can be meromorphically continued in a neighbourhood of C_γ from 1). Therefore we have $\gamma \geqq 1$. Hence f can be meromorphically continued in a neighbourhood of C_1.

2)$\to$3). Let θ be any real number. Since $f(z_1, z_2, \ldots, z_{n-1}, z_n \exp(\sqrt{-1}\,\theta))$ is meromorphic in a neighbourhood of

$$\{z\,;\, |z_1|=1, z_2=0, \ldots, z_{n-1}=0, 0 \leqq z_n \leqq 1\} \cup \{z\,;\, |z_1| \leqq 1, z_2=0, \ldots, z_n=0\},$$

$f(z_1, z_2, \ldots, z_{n-1}, z_n \exp(\sqrt{-1}\,\theta))$ can be meromorphically continued in a neighbourhood of

$$\{z\,;\, |z_1| \leqq 1, z_2=0, \ldots, z_{n-1}=0, 0 \leqq z_n \leqq 1\}$$

from 2). Thus we have proved that $f(z_1, z_2, \ldots, z_n)$ can be meromorphically continued in a neighbourhood of

$$\bigcup_{0\leqq\theta\leqq 2\pi}\{z\ ;\ |z_1|\leqq 1,\ z_2=0,\ \dots,\ z_{n-1}=0,\ 0\leqq z_n\exp(-\sqrt{-1}\,\theta)\leqq 1\}$$
$$=\{z\ ;\ |z_1|\leqq 1,\ z_2=0,\ \dots,\ z_{n-1}=0,\ |z_n|\leqq 1\}.$$

3) $\to$ 1). There exists $\delta>0$ such that f is meromorphic in a neighbourhood of

$$\{z\ ;\ |z_1-a_1^0|<2\delta,\ |z_2-a_2^0|<2\delta,\ \dots,\ |z_{n-1}-a_{n-1}^0|<2\delta,\ |z_n|=1\}.$$

There exists $q>0$ such that $|a_j^p-a_j^0|<\delta$ $(j=1,2,\dots,n-1)$ for $p\geqq q$. Since f is meromorphic in a neighbourhood of

$$\{z\ ;\ z_1=a_1^q,\ z_2=a_2^q,\ \dots,\ z_{n-1}=a_{n-1}^q,\ |z_n|\leqq 1\}\cup\{z\ ;\ |z_1-a_1^q|\leqq\delta,$$
$$z_2=a_2^q,\ \dots,\ z_{n-1}=a_{n-1}^q,\ |z_n|=1\},$$

f can be meromorphically continued in a neighbourhood of

$$\{z\ ;\ |z_1-a_1^q|\leqq\delta,\ z_2=a_2^q,\ \dots,\ z_{n-1}=a_{n-1}^q,\ |z_n|\leqq 1\}.$$

from 3). Of course f can be meromorphically continued in a neighbourhood of

$$\{z\ ;\ z_1=a_1^0,\ z_2=a_2^q,\ \dots,\ z_{n-1}=a_{n-1}^q,\ |z_n|\leqq 1\}.$$

Continuing the same argument we can prove that f can be meromorphically continued in a neighbourhood of

$$\{z\ ;\ z_1=a_1^0,\ z_2=a_2^0,\ \dots,\ z_{n-1}=a_{n-1}^0,\ |z_n|\leqq 1\}.$$

Okuda-Sakai [7] proved the validity of 1). Therefore 1), 2) and 3) are all valid form the above discussion.

LEMMA 2. *If f is meromorphic in* $\{z=(z_1,z_2,\dots,z_n)\ ;\ 1-\varepsilon<|z_1|<1+\varepsilon,\ |z_2|<1+\varepsilon,\ \dots,\ |z_n|<1+\varepsilon\}\cup\{z\ ;\ |z_1|\leqq 1,\ |z_2|<1,\ \dots,\ |z_n|<1\}$, *$f$ can be meromorphically continued in* $\{z\ ;\ |z_1|<1+\varepsilon,\ |z_2|<1+\varepsilon,\ \dots,\ |z_n|<1+\varepsilon\}$.

Proof. We take any a_j with $|a_j|<1$ for $j=1,2,\dots,n-1$. Let δ be any positive number with $\delta<1$. Since f is meromorphic in a neighbourhood of

$$\{z\ ;\ |z_1|=1,\ z_2=a_2,\ \dots,\ z_{n-1}=a_{n-1},\ |z_n|\leqq 1+\varepsilon-\delta\}\cup\{z\ ;\ |z_1|\leqq 1,\ z_2=a_2,$$
$$\dots,\ z_{n-1}=a_{n-1},\ z_n=0\},$$

f can be meromorphically continued in a neighbourhood of

$$\{z\ ;\ |z_1|\leqq 1,\ z_2=a_2,\ \dots,\ z_{n-1}=a_{n-1},\ |z_n|\leqq 1+\varepsilon-\delta\}$$

from 3) of Lemma 1. Therefore f can be meromorphically continued in a neighbourhood of

$$\{z\ ;\ |z_1|\leqq 1,\ |z_2|<1,\ \dots,\ |z_{n-1}|<1,\ |z_n|<1+\varepsilon\}.$$

Continuing the same argument we can prove that f can be meromorphically continued in a neighbourhood of

$$\{z\ ;\ |z_1|\leqq 1,\ |z_2|<1+\varepsilon,\ \dots,\ |z_n|<1+\varepsilon\}.$$

Therefore f can be meromorphically continued in

$$\{z\ ;\ |z_1|<1+\varepsilon,\ |z_2|<1+\varepsilon,\ \dots,\ |z_n|<1+\varepsilon\}.$$

§2. Envelope of meromorphy

In this section we shall define a meromorphic completion, an envelope of meromorphy and a domain of meromorphy. At the same time we can define a holomorphic completion, an envelope of holomorphy and a domain of holomorphy.

Let M be a complex manifold. If there exists a local biholomorphic mapping of a complex manifold V in M, (V, φ) is called an *open set over* M. Moreover, if V is connected, (V, φ) is called a *domain over* M. Let (V, φ) and (V', φ') be open sets over M. If a holomorphic mapping λ of V in V' satisfies $\varphi=\varphi'\circ\lambda$, λ is called a *mapping of* (V, φ) *in* (V', φ'). Consider domains (V, φ) and (V', φ') over M with a mapping λ of (V, φ) in (V', φ'). Let f be a meromorphic (or holomorphic) function on V. A meromorphic (or holomorphic) function f' on V with $f=f'\circ\lambda$ is called a *meromorphic* (or *holomorphic*) *continuation* of f to (λ, V', φ'), or shortly to (V', φ'). Let $\mathfrak{F}$ be a family of meromorphic (or holomorphic) functions on V. If any meromorphic (or holomorphic) function of $\mathfrak{F}$ has a meromorphic (or holomorphic) continuation to (λ, V', φ'), (λ, V', φ'), or shortly (V', φ'), is called a *meromorphic* (or *holomorphic*) *completion of* (V, φ) *with respect to the family* $\mathfrak{F}$. A meromorphic (or holomorphic) completion $(\tilde{\lambda}_{\mathfrak{F}}, \tilde{V}_{\mathfrak{F}}, \tilde{\varphi}_{\mathfrak{F}})$ of (V, φ) with respect to $\mathfrak{F}$, or shortly $(\tilde{V}_{\mathfrak{F}}, \tilde{\varphi}_{\mathfrak{F}})$, is called an *envelope of meromorphy* (or *holomorphy*) *of* (V, φ) *with respect to the family* $\mathfrak{F}$ if the following conditions are satisfied:

Let (λ', V', φ') be another meromorphic (or holomorphic) completion of (V, φ) with respect to $\mathfrak{F}$. Then there exists a mapping ψ of (V', φ') in $(\tilde{V}_{\mathfrak{F}}, \tilde{\varphi}_{\mathfrak{F}})$

with $\tilde{\lambda}_{\mathfrak{F}} = \psi \circ \lambda'$ such that $(\psi, \tilde{V}_{\mathfrak{F}}, \tilde{\varphi}_{\mathfrak{F}})$ is a meromorphic (or holomorphic) completion of (V', φ') with respect to the family $\mathfrak{F}'$ of meromorphic (or holomorphic) continuations of all meromorphic (or holomorphic) functions of $\mathfrak{F}$.

If $\mathfrak{F}$ is the family of all meromorphic (or holomorphic) functions on V, a meromorphic (or holomorphic) completion of (V, φ) with respect to $\mathfrak{F}$ and an envelope of meromorphy (or holomorphy) of (V, φ) with respect to $\mathfrak{F}$ are called shortly a *meromorphic* (or *holomorphic*) *completion* of (V, φ) and an *envelope of meromorphy* (or *holomorphy*) of (V, φ) respectively.

LEMMA 3. *Let (V, φ) be a domain over a complex manifold M and (λ, V', φ') be its meromorphic completion. Then (λ, V', φ') is a holomorphic completion of (V, φ).*

Proof. Let f be a holomorphic function on V. f has a meromorphic continuation f' to (λ, V', φ'). Since $\exp f$ must be meromorphically continued to the function $\exp f'$ on V', f' must be holomorphic in V'.

By the same method as Malgrange [5], who proved the unique existence of the envelope of holomorphy, we shall prove the unique existence of the envelope of meromorphy.

LEMMA 4. *Let (V, φ) be a domain over a complex manifold M and $\mathfrak{F} = \{f_i; i \in I\}$ be a family of meromorphic functions on V. There exists uniquely an envelope of meromorphy of (V, φ) with respect to $\mathfrak{F}$.*

Proof. Consider an open neighbourhood U of a point $a \in M$. Let $(g_i)_{i\in I}$ be a family of meromorphic functions in U indexed by the above I. Let $(g'_i)_{i\in I}$ be another such family defined in a neighbourhood U' of a. If there exists a neighbourhood W of a such that $W \subset U \cap U'$ and $g_i = g'_i$ in W for any $i \in I$, $(g_i)_{i\in I}$ and $(g'_i)_{i\in I}$ are identified. In this manner we shall induce an equivalence relation in the set of all families $(g_i)_{i\in I}$ of meromorphic functions defined in an open neighbourhood of a. An equivalence class is denoted by $(g_i)_a$ and the set of all classes $(g_i)_a$ is denoted by $\mathfrak{R}_{\mathfrak{F},a}$. Let

$$\mathfrak{R}_{\mathfrak{F}} = \bigcup_{a\in M} \mathfrak{R}_{\mathfrak{F},a}.$$

We shall define a mapping p of $\mathfrak{R}_{\mathfrak{F}}$ in M by putting $p(x) = a$ for $x = (g_i)_a \in \mathfrak{R}_{\mathfrak{F}}$. We can induce on $\mathfrak{R}_{\mathfrak{F}}$ a sheaf structure as usual such that $(\mathfrak{R}_{\mathfrak{F}}, p)$ is an open set over M. If we define a mapping ψ of V in $\mathfrak{R}_{\mathfrak{F}}$ by putting

$$\psi(a) = (f_i \circ \varphi^{-1})_{\varphi(a)}$$

for $a \in V$, ψ is a mapping of (V, φ) in $(\mathfrak{R}_{\mathfrak{F}}, p)$. The connected component of the complex manifold $\mathfrak{R}_{\mathfrak{F}}$ containing the connected open set $\psi(V)$ is denoted by $\tilde{V}_{\mathfrak{F}}$. We put

$$\tilde{\varphi}_{\mathfrak{F}} = p \mid \tilde{V}_{\mathfrak{F}}.$$

We shall define a meromorphic function $\tilde{f}_i$ on $\tilde{V}_{\mathfrak{F}}$ for any $i \in I$ so as the germ defined by $\tilde{f}_i$ at $x = (g_i)_a \in \tilde{V}_{\mathfrak{F}}$, defined by a family of meromorphic functions $(g_i)_{i \in I}$ in a neighbourhood of a, coincides with the germ defined by $g_i \circ \tilde{\varphi}_{\mathfrak{F}}$ at x. For $x \in \psi(V)$, we have

$$x = (f_i \circ \varphi^{-1})_{\varphi(a)}$$

for $a \in V$. Therefore we have

$$f_i = \tilde{f}_i \circ \psi$$

for any $i \in I$. Hence $\tilde{f}_i$ is a meromorphic continuation of f_i to $(\psi, \tilde{V}_{\mathfrak{F}}, \tilde{\varphi})$ for any $i \in I$. This means that $(\psi, \tilde{V}_{\mathfrak{F}}, \tilde{\varphi}_{\mathfrak{F}})$ is a meromorphic completion of (V, φ) with respect to $\mathfrak{F}$.

Let (λ, V', φ') be another meromorphic completion of (V, φ) with respect to $\mathfrak{F}$. Let f_i be any meromorphic function of $\mathfrak{F}$. There exists a meromorphic function f'_i on V' with $f_i = f'_i \circ \lambda$. If we define a mapping ψ' of V' in $\mathfrak{R}_{\mathfrak{F}}$ by putting

$$\psi'(a) = (f'_i \circ \varphi'^{-1})_{\varphi'(a)}$$

for any $a \in V'$, ψ' is a mapping of (V', φ') in $(\mathfrak{R}_{\mathfrak{F}}, p)$. Since V' is connected and $\psi(V) \subset \psi'(V')$, we have

$$\psi'(V') \subset \tilde{V}_{\mathfrak{F}}.$$

Therefore ψ' is a mapping of (V', φ') in $(\tilde{V}_{\mathfrak{F}}, \tilde{\varphi}_{\mathfrak{F}})$ too. Moreover we have $\psi = \psi' \circ \lambda$ and

$$f'_i = \tilde{f}_i \circ \psi'$$

for any $i \in I$. Therefore $(\psi, \tilde{V}_{\mathfrak{F}}, \tilde{\varphi}_{\mathfrak{F}})$ is an envelope of meromorphy of (V, φ) with respect to $\mathfrak{F}$.

Now let $(\tilde{\lambda}, \tilde{V}, \varphi)$ and $(\tilde{\lambda}', \tilde{V}', \tilde{\varphi}')$ be envelopes of meromorphy of (V, φ) with respect to $\mathfrak{F}$. There exist, respectively, a mapping ψ of $(\tilde{V}, \tilde{\varphi})$ in $(\tilde{V}', \tilde{\varphi}')$ and a mapping ψ' of $(\tilde{V}', \tilde{\varphi}')$ in $(\tilde{V}, \tilde{\varphi})$ such that

$$\lambda' = \psi \circ \lambda, \ \lambda = \psi' \circ \lambda'.$$

From the theorem of identity $\psi' \circ \psi$ and $\psi \circ \psi'$ are, respectively, identities of $\tilde{V}$ and $\tilde{V}'$. Hence ψ is biholomorphic. In this sense the envelope of meromorphy of (V, φ) with respect to $\mathfrak{F}$ exists uniquely.

§ 3. Pseudoconvexity of an envelope of meromorphy

LEMMA 5. *Let (V, φ) be a domain over an n-dimensional Stein manifold S, $\mathfrak{F} = \{f_i \ ; \ i \in I\}$ be a family of meromorphic functions on V and $(\tilde{\lambda}_{\mathfrak{F}}, \tilde{V}_{\mathfrak{F}}, \tilde{\varphi}_{\mathfrak{F}})$ be the envelope of meromorphy of (V, φ) with respect to $\mathfrak{F}$. Then $(\tilde{V}_{\mathfrak{F}}, \tilde{\varphi}_{\mathfrak{F}})$ is p_7-convex in the sense of Docquier-Grauert* [2] *and, therefore, $\tilde{V}_{\mathfrak{F}}$ is a Stein manifold.*

Proof. We may suppose that $n \geqq 2$. We put

$$D = \{z = (z_1, z_2, \ldots, z_n) \ ; \ |z_1| \leqq 1, \ |z_2| < 1, \ldots, |z_n| < 1\}$$

and

$$\delta D = \{z \ ; \ |z_1| = 1, \ z \in D\}.$$

Consider a continuous mapping ψ of the closure $\overline{D}$ of D in $\tilde{V}_{\mathfrak{F}} \cup \eth \tilde{V}_{\mathfrak{F}}$ with the following properties:

1) $\psi(\delta D) \Subset \tilde{V}_{\mathfrak{F}}$

2) $\psi(\mathring{D}) \subset \tilde{V}_{\mathfrak{F}}$

3) $\tilde{\varphi}_{\mathfrak{F}} \circ \psi$ can be continued to a biholomorphic mapping ξ of a neighbourhood of $\overline{D}$ in S.

From 3) ξ is a biholomorphic mapping of B_ε in S for $0 < \varepsilon \leqq \varepsilon'$ where

$$B_\varepsilon = \{z \ ; \ |z_1| < 1 + \varepsilon, \ |z_2| < 1 + \varepsilon, \ldots, |z_n| < 1 + \varepsilon\}$$

and ε' is a suitable positive number. If we put $G_\varepsilon = \xi(B_\varepsilon)$, $\tilde{\varphi}_{\mathfrak{F}}$ maps $\tilde{\varphi}_{\mathfrak{F}}^{-1}(G_\varepsilon)$ biholomorphically on the subdomain G_ε of S for $0 < \varepsilon \leqq \varepsilon'$. From 1) and 2) ψ can be regarded as a biholomorphic mapping of C_ε in $\tilde{V}_{\mathfrak{F}}$ for $0 < \varepsilon \leqq \varepsilon''$ where

$$C_\varepsilon = D \cup \{z \ ; \ 1 - \varepsilon < |z_1| < 1 + \varepsilon, \ |z_2| < 1 + \varepsilon, \ldots, |z_n| < 1 + \varepsilon\}$$

and ε'' is a suitable positive number.

Now let f_i be any meromorphic function of $\mathfrak{F}$ and $\tilde{f}_i$ be its meromorphic continuation to $(\tilde{\lambda}_{\mathfrak{F}}, \tilde{V}_{\mathfrak{F}}, \tilde{\varphi}_{\mathfrak{F}})$. Then $g_i = \tilde{f}_i \circ \psi$ is meromorphic not only in D but also in C_ε for $0 < \varepsilon \leqq \delta = \min(\varepsilon', \varepsilon'')$. Lemma 4 means that B_ε is a meromorphic completion of C_ε. Therefore there exists a meromorphic contination $\tilde{g}_i$ of g_i

to B_δ. We shall consider the sum space $\tilde{V}_{\mathfrak{F}} \cup B_\delta$. We shall identify a point $x \in \tilde{V}_{\mathfrak{F}}$ and a point $y \in B_\delta$ if

$$\tilde{\varphi}_{\mathfrak{F}}(x) = \xi(y), \ (\tilde{f}_i \circ \tilde{\varphi}_{\mathfrak{F}}^{-1})_{\tilde{\varphi}_{\mathfrak{F}}(x)} = (\tilde{g}_i \circ \xi^{-1})_{\xi(y)}.$$

We can put a complex structure on the quotient space V' of $\tilde{V}_{\mathfrak{F}} \cup B_\delta$ by the equivalence relation induced by the above identification. The holomorphic mappings φ and ξ induce naturally a local biholomorphic mapping φ' of V' in S. The natural injection $\tilde{V}_{\mathfrak{F}} \to \tilde{V}_{\mathfrak{F}} \cup B_\delta$ induces a biholomorphic mapping i of $\tilde{V}_{\mathfrak{F}}$ in V'. i is a mapping of $(\tilde{V}_{\mathfrak{F}}, \tilde{\varphi}_{\mathfrak{F}})$ in (V', φ'). Since $(i \circ \tilde{\lambda}_{\mathfrak{F}}, V', \varphi')$ is a meromorphic completion of (V, φ) with respect to $\mathfrak{F}$ and since $(\tilde{\lambda}_{\mathfrak{F}}, \tilde{V}_{\mathfrak{F}}, \tilde{\varphi}_{\mathfrak{F}})$ is the envelope of meromorphy of (V, φ) with respect to $\mathfrak{F}$, there exists a mapping j of (V', φ') in $(\tilde{V}_{\mathfrak{F}}, \tilde{\varphi}_{\mathfrak{F}})$ such that $\tilde{\lambda}_{\mathfrak{F}} = j \circ i \circ \tilde{\lambda}_{\mathfrak{F}}$. From the theorem of identity $j \circ i$ is the identity of $\tilde{V}_{\mathfrak{F}}$. The natural injection $B_\delta \to V_{\mathfrak{F}} \cup B_\delta$ induces a biholomorphic mapping ψ' of B_δ in V'. It holds that

$$\psi' = i \circ \psi$$

in $D \Subset B_\delta$. Therefore we have

$$\psi = j \circ \psi'$$

in D. Hence we have

$$\psi(D) \Subset V_{\mathfrak{F}}$$

as $\psi'(D) \Subset V'$. Thus we have proved that $(\tilde{V}_{\mathfrak{F}}, \tilde{\varphi}_{\mathfrak{F}})$ is p_7-convex in the sense of Docquier-Grauert [2]. Of course $\tilde{V}_{\mathfrak{F}}$ is a Stein manifold from [2].

§ 4. Main results

Let (V, φ) be a domain over a complex manifold M and f be a meromorphic (or holomorphic) function on V. The envelope $(\tilde{\lambda}_f, \tilde{V}_f, \tilde{\varphi}_f)$ of meromorphy (or holomorphy) of (V, φ) with respect to the family consisting of only f is called the *domain of meromorphy* (or *holomorphy*) *of* f. A domain over M is called a *domain of meromorphy* (or *holomorphy*) if it is a domain of meromorphy of a meromorphic (or holomorphic) function on a domain over M.

PROPOSITION 1. *Let (V, φ) be a domain over a Stein manifold S. Then V is of weak Poincaré type.*

Proof. Let f be a meromorphic function on V and $(\tilde{\lambda}_f, \tilde{V}_f, \tilde{\varphi}_f)$ be the domain of meromorphy of f. There exists a meromorphic continuation $\tilde{f}$ of f

to $(\tilde{\lambda}_f, \tilde{V}_f, \tilde{\varphi}_f)$. From Lemma 5 $\tilde{V}_f$ is a Stein manifold, which is of weak Poincaré type from Hitotumatu-Kôta [3]. There exist holomorphic functions $\tilde{g}$ and $\tilde{h}$ on $\tilde{V}_f$ such that $\tilde{f}=\tilde{g}/\tilde{h}$ on $\tilde{V}_{\mathfrak{F}}$. If we put $g=\tilde{g}\circ\tilde{\lambda}_f$ and $h=\tilde{h}\circ\tilde{\lambda}_f$, g and h are holomorphic functions on V such that $f=g/h$ on V.

If we put

$$S=C^n\times P \ (n\geqq 1),$$

S is a non compact holomorphically convex complex manifold which is not of weak Poincaré type. The authors do not know whether there exists a holomorphically convex complex manifold which is obtained by a proper modification of a Stein space and a domain over which is not of weak Poincaré type. We shall prove the converse of Lemma 3 for domains over a Stein manifold.

PROPOSITION 2. *Let* (V, φ) *be a domain over a Stein manifold* S *and* (λ, V', φ') *be its holomorphic completion. Then* (λ, V', φ') *is a meromorphic completion of* (V, φ).

Proof. Let f be a meromorphic function on V. From Proposition 1 there exist holomorphic functions g and h on V such that $f=g/h$ on V. There exist holomorphic continuations g' and h' of g and h to (λ, V', φ') respectively. Then $f'=g'/h'$ is a meromorphic continuation of f to (λ, V', φ').

COROLLARY. *Let* (V, φ) *be a domain over a Stein manifold. Then* (λ, V', φ') *is a holomorphic completion of* (V, φ) *if and only if it is a meromorphic completion of* (V, φ).

As a special case of the above Corollary we have the following Proposition.

PROPOSITION 3 *Let* (V, φ) *be a domain over a Stein manifold* S. *Then the envelope* $(\tilde{\lambda}_{\mathfrak{R}}, \tilde{V}_{\mathfrak{R}}, \tilde{\varphi}_{\mathfrak{R}})$ *of meromorphy of* (V, φ) *coincides with the envelope* $(\tilde{\lambda}_{\mathfrak{O}}, \tilde{V}_{\mathfrak{O}}, \tilde{\varphi}_{\mathfrak{O}})$ *of holomorphy of* (V, φ).

Proof. From Lemma 3 and Proposition 2 there exist, respectively, a mapping ψ of $(\tilde{V}_{\mathfrak{R}}, \tilde{\varphi}_{\mathfrak{R}})$ in $(\tilde{V}_{\mathfrak{O}}, \tilde{\varphi}_{\mathfrak{O}})$ and a mapping ψ' of $(\tilde{V}_{\mathfrak{O}}, \tilde{\varphi}_{\mathfrak{O}})$ in $(\tilde{V}_{\mathfrak{R}}, \tilde{\varphi}_{\mathfrak{R}})$ such that $\lambda=\psi\circ\lambda'$ and $\lambda'=\psi'\circ\lambda$. From the theorem of identity $\psi'\circ\psi$ and $\psi\circ\psi'$ are, respectively, identities of $\tilde{V}_{\mathfrak{R}}$ and $\tilde{V}_{\mathfrak{O}}$. Hence ψ is a biholomorphic mapping of $\tilde{V}_{\mathfrak{R}}$ in $V_{\mathfrak{O}}$. In this sense $(\tilde{\lambda}_{\mathfrak{R}}, \tilde{V}_{\mathfrak{R}}, \tilde{\varphi}_{\mathfrak{R}})$ coincides with $(\tilde{\lambda}_{\mathfrak{O}}, \tilde{V}_{\mathfrak{O}}, \tilde{\varphi}_{\mathfrak{O}})$.

THEOREM. *Let* (V, φ) *be a domain over a Stein manifold* S. *Then the following* *assertions are equivalent:*

1) (V, φ) *is an envelope of meromorphy with respect to a family of meromorphic functions on a domain over* S.

2) (V, φ) *is a domain of meromorphy.*

3) (V, φ) *is domain of holomorphy.*

4) V *is holomorphically convex.*

Proof. From Lemma 5 1) implies 4). Ouite similarly as in the proof of Lemma 5, 4) follows from 3) by Docquier-Grauert [2]. If V is holomorphically convex, (V, φ) is a domain of meromorphy of a holomorphic function on V from Cartan-Thullen [1]. A domain of meromorphy of a meromorphic function is an envelope of meromorphy with respect to the family consisting of only f.

Roughly speaking, the theory of domains of meromorphy over a Stein manifold coincides almost with the theory of domains of holomorphy over C^n.

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