Adv. Appl. Probab. **55,** 321–355 (2023)
*doi:*10.1017/apr.2022.20

FULL CLASSIFICATION OF DYNAMICS FOR ONE-DIMENSIONAL CONTINUOUS-TIME MARKOV CHAINS WITH POLYNOMIAL TRANSITION RATES

CHUANG XU,* *Technical University of Munich*
MADS CHRISTIAN HANSEN** *** AND
CARSTEN WIUF,** **** *University of Copenhagen*

Abstract

This paper provides a full classification of the dynamics for continuous-time Markov chains (CTMCs) on the nonnegative integers with polynomial transition rate functions and without arbitrary large backward jumps. Such stochastic processes are abundant in applications, in particular in biology. More precisely, for CTMCs of bounded jumps, we provide necessary and sufficient conditions in terms of calculable parameters for explosivity, recurrence versus transience, positive recurrence versus null recurrence, certain absorption, and implosivity. Simple sufficient conditions for exponential ergodicity of stationary distributions and quasi-stationary distributions as well as existence and nonexistence of moments of hitting times are also obtained. Similar simple sufficient conditions for the aforementioned dynamics together with their opposite dynamics are established for CTMCs with unbounded forward jumps. Finally, we apply our results to stochastic reaction networks, an extended class of branching processes, a general bursty single-cell stochastic gene expression model, and population processes, none of which are birth–death processes. The approach is based on a mixture of Lyapunov–Foster-type results, the classical semimartingale approach, and estimates of stationary measures.

Keywords: Density-dependent continuous-time Markov chains; stochastic reaction networks; explosivity; recurrence; transience; certain absorption; positive and null recurrence; stationary and quasi-stationary distributions

2020 Mathematics Subject Classification: Primary 60J28; 37A25; 05C92
Secondary 92C42; 92C40; 92D25

1. Introduction

Continuous-time Markov chains (CTMCs) on a countable state space are widely used in applications, for example, in genetics [20], epidemiology [37], ecology [24], biochemistry and systems biology [45], sociophysics [44], and queueing theory [26]. For a CTMC on a countable state space, criteria for dynamical properties (explosivity, recurrence, certain absorption, positive recurrence, etc.) are among the fundamental topics and areas of interest.

Received 18 June 2021; revision received 27 March 2022.
* Postal address: Faculty of Mathematics, Technical University of Munich, 85748 Garching bei München, Germany. Email address: Chuang.Xu@ma.tum.de
** Postal address: Department of Mathematical Sciences, University of Copenhagen, Copenhagen, 2100, Denmark.
*** Email address: madschansen@gmail.com
**** Email address: wiuf@math.ku.dk

A primary source of inspiration for our work comes from stochastic reaction network (SRN) theory, where examples are abundant. In the present context, SRNs are CTMC models of *(chemical) reaction networks* with polynomial transition rates [4] (see Section 4.1 for a precise definition). In particular, we are interested in one-species reaction networks, where the reactions take the form $n\mathrm{S} \xrightarrow{\kappa} m\mathrm{S}$ for two nonnegative integers, n, m, and $\kappa > 0$, a positive reaction rate constant. Here S represents a (chemical) species common to all reactions in the network, and the reaction represents the conversion of n molecules of the species S into m molecules of the same species. Each reaction has a transition rate, a propensity to 'fire'. The transition rate of $n\mathrm{S} \xrightarrow{\kappa} m\mathrm{S}$ is $\eta(x) = \kappa x(x-1)\ldots(x-n+1)$, $x \in \mathbb{N}_0$. Whenever the reaction fires, the corresponding Markov chain on the state space $\mathbb{N}_0$ jumps from the current state x to the state $x+m-n$, the number of S molecules after the firing of the reaction. Different reactions may contribute to the same transition in the state space.

While the chemical terminology may suggest that the usage of such models is restricted, this is by far not so. In fact, SRNs have widespread use in the sciences with species interpreted as agents, individuals, and similar entities, and reactions as interactions among these [4]. One might emphasize the susceptible–infectious–recovered (SIR) model in epidemiology as a particular example [37].

Consider the following two examples of one-species SRNs from the recent literature, consisting of seven and five reactions, respectively [1]:

$$\mathrm{S} \underset{2}{\overset{1}{\rightleftharpoons}} 2\mathrm{S} \underset{4}{\overset{4}{\rightleftharpoons}} 3\mathrm{S} \underset{1}{\overset{6}{\rightleftharpoons}} 4\mathrm{S} \xrightarrow{1} 5\mathrm{S}, \qquad \mathrm{S} \underset{2}{\overset{1}{\rightleftharpoons}} 2\mathrm{S} \underset{1}{\overset{3}{\rightleftharpoons}} 3\mathrm{S} \xrightarrow{1} 4\mathrm{S}. \tag{1}$$

A key issue is to understand whether the graphical representation of the reaction networks determines the dynamics of the corresponding CTMCs, irrespective of their initial values. The first network is *explosive* (except if the initial state is 0, which forms a singleton communicating class), while the second is *positive recurrent* on the positive integers (again 0 forms a singleton class) [1], which might be inferred from known birth–death process (BDP) criteria [5]. However, these criteria are *not* computationally simple and blind to the graphical structure of the networks. A simple explanation for the drastic difference in the dynamics of these two random walks on $\mathbb{N}_0$ is desirable but remains unknown [1].

Motivated by the above concern, we provide criteria for dynamical properties of CTMCs on $\mathbb{N}_0$ with polynomial-like transition rates and without the possibility of arbitrary large negative jumps, as in the examples above. These CTMCs are ubiquitous in applications [7]. Specifically, we provide simple threshold criteria for the existence and nonexistence of moments of hitting times, positive recurrence and null recurrence, and exponential ergodicity of stationary distributions and quasi-stationary distributions (QSDs) in terms of *four* easily computable parameters, derived from the transition rates. Additionally, we provide necessary and sufficient conditions for explosivity, recurrence versus transience, certain absorption, and implosivity. These conditions provide simple explanations for the dynamical discrepancies between the two SRNs in (1).

Our approach is to apply the classical semimartingale approach used in Lamperti's problem [29], as well as Lyapunov–Foster theory [12, 32, 33] with delicately constructed Lyapunov functions (in particular, we make use of the techniques in [32]). The problem of finding neat and desirable necessary and sufficient conditions for dynamical properties of CTMCs has existed for a long time [33, 34, 12]. However, the fact that this has not been accomplished yet indicates that it might be a nontrivial task. A main contribution of this paper is to *identify* a large class of CTMCs (without a built-in detailed balanced structure) for which computationally

TABLE 1. Parameter regions with different dynamical properties. Implosive (E), positive recurrent (A+B+E), null recurrent (C), transient and non-explosive (D), explosive (F), and impossible parameter combinations (gray); exponential ergodicity (B+E), uniform exponential convergence of QSD (E), no QSD or ergodic stationary distribution (C+D+F). The parameter regions below $\alpha = 0$ assume Ω is finite.

	$\alpha < 0$	$\alpha = 0$					$\alpha > 0$
		$\gamma < 0$	$\gamma = 0$	$\beta < 0 < \gamma$	$\beta = 0$	$\beta > 0$	
$R = 0$	A		C			D	D
$R = 1$	B	A	C	C	C	D	D
$R = 2$	E	A	A	A	C	D	F
$R > 2$	E	E	E	E	E	F	F

simple sufficient and necessary criteria can be established for dynamical properties of interest. Our criteria save the effort of constructing Lyapunov functions and applying Lyapunov–Foster theory case by case. Also, a case-by-case approach is ignorant of the underlying graphical structure of the Markov chain.

The simple necessary and sufficient conditions for the dynamical properties are determined by calculating up to four parameters, R, α, β, and γ, that are expressed in terms of the coefficients of the first two terms of the polynomial-like transition rate functions (the specific assumptions are given in Section 2). For illustration, let Ω be the set of jump sizes, and let

$$\lambda_\omega(x) = a_\omega x^{d_\omega} + b_\omega x^{d_\omega - 1} + \mathrm{O}\big(x^{d_\omega - 2}\big), \qquad \omega \in \Omega, \tag{2}$$

be the transition rate functions, where d_ω is the degree of λ_ω and O is Landau's symbol. Define $R = \max_{\omega\in\Omega} d_\omega$ and

$$\alpha = \sum_{\omega\,:\,d_\omega = R} a_\omega \omega, \qquad \gamma = \sum_{\omega\,:\,d_\omega = R} b_\omega \omega + \sum_{\omega\,:\,d_\omega = R-1} a_\omega \omega, \qquad \beta = \gamma - \frac{1}{2} \sum_{\omega\,:\,d_\omega = R} a_\omega \omega^2. \tag{3}$$

Based on these four parameters, a *full* classification of the dynamical properties can be achieved (see Theorems 1, 3, 7, and 9) and is summarized in Table 1 below. The parameters α, β, γ depend only on the coefficients of the monomials of degree R and $R - 1$. Furthermore, the parameter α might be interpreted as a sum over the jump sizes, weighted by the coefficients of the monomials of degree R. Similarly, γ might be interpreted as a sum over the jump sizes, weighted by the coefficients of the monomials of degree $R - 1$.

To see the power of our results, consider the following SRN, which is not a BDP:

$$0 \underset{\kappa_2}{\overset{\kappa_1}{\rightleftharpoons}} m\mathrm{S} \underset{\kappa_4}{\overset{\kappa_3}{\rightleftharpoons}} (m+1)\mathrm{S} \xrightarrow{\kappa_5} (m+3)\mathrm{S}, \tag{4}$$

where m is a positive integer, and κ_i, $i = 1, \ldots$, is a positive rate constant. Then $R = m + 1$, $\alpha = 2\kappa_5 - \kappa_4$,

$$\beta = \kappa_3 - m\kappa_2 + \tfrac{m^2+m-1}{2}\kappa_4 - (m^2 + m + 2)\kappa_5,$$

https://doi.org/10.1017/apr.2022.20 Published online by Cambridge University Press

and

$$\gamma = \kappa_3 - m\kappa_2 + \frac{m(m+1)}{2}\kappa_4 - m(m+1)\kappa_5.$$

The criteria established in Section 3 (and collected in Table 1) imply that the SRN is (in the sense of the underlying irreducible CTMC on $\mathbb{N}_0$)

(a) explosive almost surely (a.s.) if and only if (i) $\alpha > 0$ or (ii) $\alpha = 0$, $\beta > 0$, $R > 2$, and non-explosive if either (i) or (ii) fails;

(b) recurrent if and only if (iii) $\alpha < 0$ or (iv) $\alpha = 0$, $\beta \le 0$, and transient if and only if both (iii) and (iv) fail;

(c) positive recurrent if and only if (iii), (v) $\alpha = 0$, $\beta < 0$, or (vi) $\alpha = 0$, $\beta = 0$, $R > 2$ holds, and null recurrent if and only if (vii) $\alpha = 0$, $\beta = 0$, $R = 2$;

(d) implosive if and only if (iii) or (viii) $\alpha = 0$, $\beta \le 0$, $R > 2$ holds, and non-implosive if and only if both (ii) and (viii) fail.

Here implosive means positive recurrent with uniformly bounded expected first return time. (See Subsection 3.5 for the precise definition.) The above example shows the applicability and simplicity of our results; in fact, the computations could easily be implemented in a software program that takes a reaction network as input and outputs the network dynamical properties. Furthermore, the example illustrates the richness of the dynamical properties that might reside within a single example by varying the parameters of the model. All possibilities for α, β and R are covered (the parameter γ is irrelevant for SRNs [46]). The stability of the chain depends only on α (which is independent of m), unless $\alpha = 0$, in which case the sign of β determines the stability. If so, then $\beta = \kappa_3 - m\kappa_2 - 3\kappa_5$, which depends on m. Thus, if $\kappa_3 - \kappa_2 - 3\kappa_5 > 0$, then if $m > 1$ is chosen large enough, the stability of the chain flips. The parameter α plays a role similar to that of the largest Lyapunov exponent for $\alpha \neq 0$. Analogously, when $\alpha = 0$, the parameter β determines the stochastic stability and hence plays a role similar to that of the second-largest Lyapunov exponent in the critical case.

Brief description of our approach

Although our approach is essentially based on Lyapunov–Foster-type results, the sharp criteria for diverse dynamical properties of CTMCs are established by combining a mixture of results [6, 12, 16, 32, 33]; in particular [32] provides useful criteria.

The most prominent difficulties in deriving necessary and sufficient conditions for dynamical properties of general CTMCs, with multiple jump sizes, lie in the non-calculability of stationary distributions/measures, as well as the nonexistence of orthogonal polynomials [28]. This also explains why, in general, a partial result in terms of a sufficient but not a necessary condition, by construction of a Lyapunov function, is likely. Here we discover that Lyapunov–Foster theory and the semimartingale approach are indeed enough to derive necessary and sufficient conditions. To obtain conditions that are not only sufficient but also necessary, we check whether the negation of a condition is also sufficient for the reverse dynamical property. Moreover, to show null recurrence, we also rely on estimates for stationary measures in [6]. Finally, we would like to point out that some of the Lyapunov functions we use appear to be rarely used in the literature.

Comparison with results in the literature

Complete classification of dynamical properties seems quite rare in the literature. Here, we summarize relevant results together with the methods applied.

Reuter provided necessary and sufficient conditions for explosivity of CTMCs (known as Reuter's criterion) [38], but these conditions are difficult to check except in special cases, e.g. for BDPs [28] and competition processes [39]. This is due to the fact that the conditions involve infinitely many algebraic equations.

Karlin and McGregor established threshold results for explosivity and recurrence, as well as certain absorption of BDPs with and without absorbing states, by means of the so-called Karlin–McGregor integral representation formula [28]. The existence of such a formula is essentially due to the tridiagonal structure of the Q-matrix. For the same reason, it is delicate to extend such an approach to *generalized BDPs*: pure birth processes [14], one-sided skip-free CTMCs [13, 15, 16], and recently (higher-dimensional) quasi-birth–death processes (QBDPs) with tridiagonal block structure of the Q-matrix [22]. There are *no* restrictions on the types of transition rates for the results in [28] to hold, except the BDP assumption. In contrast, for our results to hold, we require the polynomial transition rates to have uniformly bounded degrees, but we do not require the BDP assumption.

In the context of QSDs, there are few threshold results for certain absorption, existence and uniqueness, and quasi-ergocidity of QSDs. Van Doorn [42, 43] obtained ergodicity, existence and nonexistence, and uniqueness of QSDs for absorbed BDPs, also building on the Karlin–McGregor integral representation formula. Later, Ferrari *et al.* [23] generalized the results in [43]. They derived a necessary and sufficient condition for the existence of a QSD on the positive integers for which zero is an absorbing state, using the so-called renewal dynamical approach, assuming the CTMC is non-explosive, and that the absorption time is finite and unbounded with probability one. Then the existence of a QSD is equivalent to finiteness of the exponential moment of the absorption time, for one initial transient state (and hence all of them). But such a moment condition is again not straightforward to verify, pending the assumptions.

To sum up, general checkable threshold criteria for dynamical properties of CTMCs (absorbed and non-absorbed), other than generalized BDPs, are few. We identify a class of CTMCs with polynomial-like transition rates and without arbitrary large backward jumps for which simple, checkable criteria for absorbed and non-absorbed CTMCs are found, based on the coefficients of the polynomials. The price for this is to impose some further mild regularity conditions, in addition to the two requirements mentioned above.

Impact of our work and further extensions

The following are some possible extensions of our work from the theoretical perspective:

- The sufficient condition for the existence of ergodic stationary distributions and QSDs allows us to further investigate the tail asymptotics of these distributions [47] and the computation of these distributions (in a forthcoming paper).
- The novel combination of the approaches presented here can further be extended to establish criteria for the dynamics of one-dimensional CTMCs with asymptotic polynomial transition rates, and higher-dimensional CTMCs with Q-matrix of a certain block structure, in analogy with QBDPs and BDPs.
- A deeper understanding of the threshold parameters may provide insight into the dynamics of higher-dimensional CTMCs on lattices.

https://doi.org/10.1017/apr.2022.20 Published online by Cambridge University Press

From the perspective of applications, we have the following:

- The criteria can be applied to completely classify the dynamics of one-dimensional mass-action SRNs, and in particular, we can prove the so-called positive recurrence conjecture [2] for weakly reversible reaction networks in one dimension [46].
- The criteria can be used to establish bifurcations of one-dimensional SRNs (in a forthcoming paper).

Outline

In Section 2, the notation and standing assumptions are introduced. Section 3 develops threshold criteria for dynamical properties of CTMCs. Applications to SRNs, a class of branching processes, a general bursty single-cell stochastic gene expression model, and population processes of non-BDP type are provided in Section 4. Proofs of the main results are provided in Section 5. Additional tools used in the proofs as well as proofs of some elementary propositions are in the appendix.

2. Preliminaries and assumptions

Let $\mathbb{R}$, $\mathbb{R}_{\geq 0}$, $\mathbb{R}_{>0}$ be the sets of real, nonnegative real, and positive real numbers, respectively. Let $\mathbb{Z}$ be the set of integers, $\mathbb{N}=\mathbb{Z}\cap\mathbb{R}_{>0}$, and $\mathbb{N}_0=\mathbb{N}\cup\{0\}$. For $x, y\in\mathbb{N}$, let $x^{\underline{y}}=x(x-1)\cdots(x-y+1)$ be the descending factorial of x.

Let $(Y_t:t\geq 0)$ (or Y_t for short) be a CTMC on a closed, infinite state space $\mathcal{Y}\subseteq\mathbb{N}_0$ with conservative transition rate matrix $Q=(q_{x,y})_{x,y\in\mathcal{Y}}$; that is, every row sums to zero. A set $A\subseteq\mathcal{Y}$ is *closed* if $q_{x,y}=0$ for all $x\in A$ and $y\in\mathcal{Y}\setminus A$ [36]. Assume the *absorbing set* $\partial\subsetneq\mathcal{Y}$ is finite (potentially empty) and closed. Hence, $\mathcal{Y}\setminus\partial$ is unbounded.

Let $\Omega=\{y-x: q_{x,y}>0,\ \text{for some } x, y\in\mathcal{Y}\}$ be the set of jump sizes. For $\omega\in\Omega$, define the transition rate function by

$$\lambda_\omega(x)=q_{x,x+\omega},\qquad x\in\mathcal{Y}.$$

Let $\Omega_\pm=\{\omega\in\Omega:\operatorname{sgn}(\omega)=\pm 1\}$ be the sets of forward and backward jump sizes, respectively. Throughout, we assume the following regularity conditions:

(A1) $\Omega_+\neq\varnothing$, $\Omega_-\neq\varnothing$.

(A2) $\#\Omega_-<\infty$.

(A3) $\sum_{\omega\in\Omega}\lambda_\omega(x)|\omega|<\infty$, for all $x\in\mathcal{Y}$.

(A4) There exist $u, M\in\mathbb{N}$ such that λ_ω is a strictly positive polynomial of degree $\leq M$ on the set $\mathcal{Y}\setminus\{0,\ldots,u-1\}$, for all $\omega\in\Omega$.

(A5) $\mathcal{Y}\setminus\partial$ is irreducible.

If either $\Omega_+=\varnothing$ or $\Omega_-=\varnothing$, then Y_t is a pure birth or death process (possibly with multiple jump sizes). The classification of states and the dynamics of such processes are simpler than under (**A1**). Indeed, one can derive parallel results from the corresponding results under (**A1**). Assumption (**A2**) implies Y_t cannot make arbitrary large negative jumps. Assumption (**A3**) is a regularity condition that ensures that functions like x, $\log x$, and $\log\log x$ are in the domain of the infinitesimal generator of the CTMC, in order to serve as Lyapunov functions. If Ω is finite, then (**A2**) and (**A3**) are automatically fulfilled. In that case, the sums above are trivially polynomials for large x.

https://doi.org/10.1017/apr.2022.20 Published online by Cambridge University Press

Assumption (**A4**) implies that the Markov chain can make all jumps in Ω with positive probability from any 'large' state $x \in \mathcal{Y}$, $x \geq u$. Moreover, (**A3**) and (**A4**) together imply that $\sum_{\omega\in\Omega} \lambda_\omega(x)$ and $\sum_{\omega\in\Omega} \lambda_\omega(x)\omega$ are polynomials of degree $\leq M$ for $x \in \mathcal{Y} \setminus \{0, \ldots, u-1\}$ (Proposition 1). If (**A4**) fails, simple examples show that $\sum_{\omega\in\Omega} \lambda_\omega(x)$ and $\sum_{\omega\in\Omega} \lambda_\omega(x)\omega$ may not be polynomials. That these sums are polynomials for large states is an essential property that we rely on in proofs.

Assumption (**A4**) is common in applications, especially in the context of chemical reaction networks and population processes [3, 19]. Assumption (**A5**) is also standard and is generally satisfied in applications [12, 32], potentially by restricting the state space. One can show that (**A4**) and (**A5**) together imply that $\mathcal{Y} \setminus \partial$ is infinite; thus the assumptions are not compatible with a finite state space.

With the above assumptions, the following three parameters are well-defined and finite:

$$R = \max\{\deg(\lambda_\omega) : \omega \in \Omega\}, \quad \alpha = \lim_{x\to\infty} \frac{\sum_{\omega\in\Omega} \lambda_\omega(x)\omega}{x^R}, \quad \gamma = \lim_{x\to\infty} \frac{\sum_{\omega\in\Omega} \lambda_\omega(x)\omega - \alpha x^R}{x^{R-1}}.$$

If $R = 0$, then trivially $\gamma = 0$. In particular, if Ω is finite, the following additional parameter is also well-defined and finite:

$$\beta = \gamma - \frac{1}{2} \lim_{x\to\infty} \frac{\sum_{\omega\in\Omega} \lambda_\omega(x)\omega^2}{x^R},$$

with $\beta < \gamma$. The parameter α encodes the sign of the average jump size of the chain. It is straightforward to verify that (3) is a consequence of the above parameter definitions, owing to the asymptotic expansions (2) of the transition rate functions.

Example 1. Recall the example (4) in the introduction,

$$0 \underset{\kappa_2}{\overset{\kappa_1}{\rightleftharpoons}} m\mathrm{S} \underset{\kappa_4}{\overset{\kappa_3}{\rightleftharpoons}} (m+1)\mathrm{S} \xrightarrow{\kappa_5} (m+3)\mathrm{S}.$$

Then $\Omega = \{1, 2, -1, m, -m\}$, $m \geq 1$, and

$$\lambda_1(x) = \left\{ \begin{array}{ll} \kappa_3 x^{\underline{m}} & \text{if } m > 1, \\ \kappa_3 x^{\underline{m}} + \kappa_1 & \text{if } m = 1, \end{array} \right\} = \kappa_3 x^m + \mathrm{O}\big(x^{m-1}\big),$$

$$\lambda_2(x) = \left\{ \begin{array}{ll} \kappa_5 x^{\underline{m+1}} & \text{if } m \neq 2, \\ \kappa_5 x^{\underline{m+1}} + \kappa_1 & \text{if } m = 2, \end{array} \right\} = \kappa_5 x^{m+1} - \kappa_5 \tfrac{m(m+1)}{2} x^m + \mathrm{O}\big(x^{m-2}\big),$$

$$\lambda_{-1}(x) = \begin{cases} \kappa_4 x^{\underline{m+1}} = \kappa_4 x^{m+1} - \kappa_4 \tfrac{m(m+1)}{2} x^m + \mathrm{O}\big(x^{m-1}\big) & \text{if } m > 1, \\ \kappa_4 x^{\underline{m+1}} + \kappa_2 x^{\underline{m}} = \kappa_4 x^{m+1} + \big(\kappa_2 - \kappa_4 \tfrac{m(m+1)}{2}\big) x^m + \mathrm{O}\big(x^{m-1}\big) & \text{if } m = 1, \end{cases}$$

$$\lambda_m(x) = \kappa_1 \quad \text{if } m > 2,$$

$$\lambda_{-m}(x) = \kappa_2 x^{\underline{m}} = \kappa_2 x^m + \mathrm{O}\big(x^{m-1}\big) \quad \text{if } m > 1.$$

Hence, $R = m + 1$, and

$$\alpha = 2\kappa_5 - \kappa_4, \quad \gamma = \kappa_3 - m\kappa_2 + \tfrac{m(m+1)}{2}\kappa_4 - m(m+1)\kappa_5,$$

$$\beta = \kappa_3 - m\kappa_2 + \tfrac{m^2+m-1}{2}\kappa_4 - (m^2 + m + 2)\kappa_5,$$

https://doi.org/10.1017/apr.2022.20 Published online by Cambridge University Press

TABLE 2. Labeling of the conditions in the main theorems in Section 3.

(C1)	$\alpha > 0, R > 1,$	(C2)	$\alpha = 0, \beta > 0, R > 2,$
(C3)	$\alpha < 0,$	(C4)	$R \leq 1,$
(C5)	$\alpha = 0, R = 2,$	(C6)	$\alpha = 0, \beta \leq 0,$
(C7)	$\alpha > 0,$	(C8)	$\alpha = 0, \beta > 0,$
(C9)	$\alpha = 0, \beta < 0, R > 1,$	(C10)	$\alpha = \beta = 0, R > 2,$
(C11)	$\alpha = 0, \gamma < 0, R = 1,$	(C12)	$\alpha = 0, \beta \leq 0, \gamma > 0, R = 1,$
(C13)	$\alpha = 0, R = 0,$	(C14)	$\alpha = 0, \beta < 0, R = 1,$
(C15)	$\alpha = 0, \beta \leq 0, R = 1,$	(C16)	$\alpha = \beta = 0, R = 2,$
(C17)	$\alpha < 0, R \geq 1,$	(C18)	$\alpha = 0, \beta \leq 0, R > 2,$
(C19)	$\alpha < 0, R > 1,$	(C20)	$\alpha < 0, R \leq 1,$
(C21)	$\alpha = 0, \beta < 0, R = 2.$		

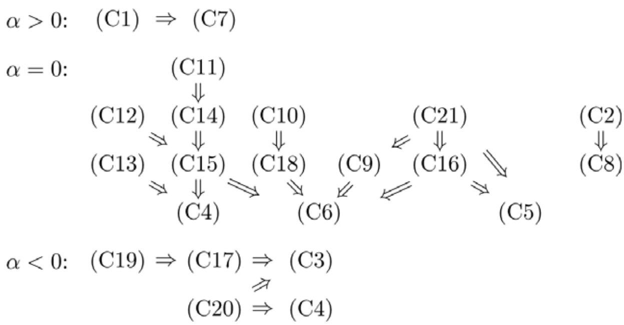


FIGURE 1. Flow diagram of implications among the 21 conditions.

by (3). As mentioned in the introduction, the sign of α determines the stochastic stability of the CTMC [33]. Hence, α plays a similar role as the largest Lyapunov exponent. Analogously, when $\alpha = 0$, the parameter β determines the stochastic stability and hence plays a role similar to that of the second-largest Lyapunov exponent in the critical case.

3. Criteria for dynamical properties

In this section, we provide threshold criteria for various dynamical properties in terms of R, α, β, γ. Proofs are relegated to Section 5. For ease of comparison, we collect all parameter conditions used in the main theorems below. These are listed in the order in which they appear in the main theorems; see Table 2. Figure 1 shows implications among the 21 conditions.

3.1. Explosivity and non-explosivity

The sequence $J = (J_n)_{n\in\mathbb{N}_0}$ of *jump times* of a CTMC Y_t are defined by $J_0 = 0$ and $J_n = \inf\{t \geq J_{n-1} : Y_t \neq Y_{J_{n-1}}\}$, $n \geq 1$, where $\inf \varnothing = \infty$ by convention. The *lifetime* is denoted by $\zeta = \sup_n J_n$. The process Y_t is said to *explode* (with positive probability) at $y \in \mathcal{Y}$ if $\mathbb{P}_y(\{\zeta < \infty\}) > 0$. In particular, Y_t explodes a.s. at $y \in \mathcal{Y}$ if $\mathbb{P}_y(\{\zeta < \infty\}) = 1$, and does not explode at $y \in \mathcal{Y}$ if $\mathbb{P}_y(\{\zeta < \infty\}) = 0$ [33]. Hence, Y_t does not explode if $Y_0 \in \partial$ (since ∂ is closed and finite), and $\mathbb{E}_y(\zeta) < \infty$ implies that Y_t explodes at y a.s. Recall that non-explosivity and explosivity

https://doi.org/10.1017/apr.2022.20 Published online by Cambridge University Press

are class properties. They hold for either all or no states in $\mathcal{Y}\setminus\partial$. Hence, we simply say Y_t is *explosive* (respectively, explosive a.s.) if it explodes with positive probability (respectively, explodes a.s.) at some state in $\mathcal{Y}\setminus\partial$, and Y_t is *non-explosive* if it does not explode at some state in $\mathcal{Y}\setminus\partial$.

We present necessary and sufficient conditions for explosivity and non-explosivity.

Theorem 1. *Assume* (**A1**)–(**A5**)*, and that* Ω *is finite. Then* Y_t *is explosive with positive probability if and only if either (C1) or (C2) holds. Moreover,* Y_t *is explosive a.s. whenever it is explosive, provided* $\partial=\varnothing$*.*

Theorem 2. *Assume* (**A1**)–(**A5**) *and that* Ω *is infinite. Then* Y_t *is explosive if (C1) holds, and it is non-explosive if one of the three conditions (C3), (C4), (C5) holds. Moreover,* Y_t *is explosive a.s. whenever it is explosive, provided* $\partial=\varnothing$*.*

Explosion might occur with probability less than one for CTMCs with *non*-polynomial transition rates and $\partial=\varnothing$ [32]. Reuter's criterion and generalizations of it provide necessary and sufficient conditions for explosivity (with positive probability) for general CTMCs in terms of convergence or divergence of a series [13, 30, 38]. However, these conditions are *not* easy to check. In comparison, for CTMCs with polynomial transition rates, Theorem 1 provides an *explicit* and checkable necessary and sufficient condition.

3.2. Recurrence versus transience, and certain absorption

For a nonempty subset $A\subseteq\mathcal{Y}$, let $\tau_A=\inf\{t\geq 0: Y_t\in A\}$ be the *hitting time* of A, with the convention that $\inf\varnothing=\infty$. If $Y_0\in A$, then $\tau_A=0$. Let $\tau_A^+=\inf\{t\geq J_1: Y_t\in A\}$ be the *first return time* to A. Obviously, $\tau_A=\tau_A^+$ if and only if $Y_0\notin A$. The process Y_t has *certain absorption* if the hitting time of ∂ is finite a.s. for all $Y_0\in\mathcal{Y}$.

Theorem 3. *Assume* (**A1**)–(**A5**) *and that* Ω *is finite.*

(i) Assume $\partial=\varnothing$*. Then* Y_t *is recurrent if either (C3) or (C6) holds, while it is transient if neither of them holds.*

(ii) Assume $\partial\neq\varnothing$*. Then* Y_t *has certain absorption if and only if either (C3) or (C6) holds.*

The results show that CTMCs with polynomial transition rates cannot have an infinite series of critical transitions from recurrence to transience, for varying parameter values. This stands in contrast to the case of CTMCs with non-polynomial transition rates, as discovered in [32]. One might hope that this phenomenon carries over to CTMCs with polynomial transition rates in dimensions higher than one.

Theorem 4. *Assume* (**A1**)–(**A5**) *and that* Ω *is infinite.*

(i) Assume $\partial=\varnothing$*. Then* Y_t *is recurrent if (C3) holds, and it is transient if (C7) holds.*

(ii) Assume $\partial\neq\varnothing$*. Then* Y_t *has certain absorption if (C3) holds, while it does not have certain absorption if (C7) holds.*

3.3. Moments of hitting times

Below we present threshold results on the existence of moments of hitting times for recurrent states only, as transient states have infinite return time. Therefore, in light of Theorem 3, we investigate the existence and nonexistence of moments of hitting times only for $\alpha<0$ and for $\alpha=0,\ \beta\leq 0$. Moreover, limited by the tools we apply, we do not discuss existence and nonexistence of moments of absorption times for $\partial\neq\varnothing$. Hence, we assume Y_t is irreducible on

$\mathcal{Y}$ (equivalently, $\partial=\varnothing$) and provide existence and nonexistence of moments of hitting times for states in $\mathcal{Y}$.

Theorem 5. *Assume* (**A1**)–(**A5**), $\partial=\varnothing$, *and that* Ω *is finite. Then the following hold:*

(i) There exists a finite nonempty subset $B\subseteq\mathcal{Y}$ *such that*

$$\mathbb{E}_x\left(\tau_B^{\epsilon}\right)<+\infty,\quad \forall x\in\mathcal{Y},\quad \forall 0<\epsilon<\delta, \tag{5}$$

for $\delta>0$, *provided one of the conditions (C3), (C9), (C10) holds; for* $0<\delta<1/2$, *provided (C13) holds; for* $0<\delta<\frac{\beta}{\beta-\gamma}$, *provided (C14) holds; and for* $0<\delta<1$, *provided (C16) holds. In particular,* $\mathbb{E}_x(\tau_B)<+\infty$, *provided one of the conditions (C3), (C9), (C10), (C11) holds.*

(ii) There exists a finite nonempty subset $B\subseteq\mathcal{Y}$ *such that*

$$\mathbb{E}_x\left(\tau_B^{\epsilon}\right)=+\infty,\quad \forall x\in\mathcal{Y}\setminus B,\quad \forall\epsilon>\delta,$$

for $\delta>1$, *provided (C13) holds; for* $\delta>\frac{\beta}{\beta-\gamma}$, *provided (C15) holds; and for* $\delta>1$, *provided (C16) holds. In particular,* $\mathbb{E}_x(\tau_B)=+\infty$ *provided (C12) holds.*

Theorem 6. *Assume* (**A1**)–(**A5**), $\partial=\varnothing$, *and that* Ω *is infinite. If (C3) is fulfilled and* $0<\delta\le 1$, *then* (5) *holds.*

3.4. Positive recurrence and null recurrence

We provide sharp criteria for positive and null recurrence, as well as exponential ergodicity of stationary distributions and QSDs.

If $\partial=\varnothing$, then $\tau_\partial=\infty$ a.s., and the conditional process $(Y_t:\tau_\partial>t)$ reduces to $(Y_t:t\ge 0)$. If $\partial\ne\varnothing$, and $\tau_\partial<\infty$ a.s. (that is, Y_t has certain absorption), then the process conditioned to never be absorbed, $(Y_t:\tau_\partial>t)$, is referred to as the *Q-process* [11, 18].

The process $(Y_t:\tau_\partial>t)$ on $\mathcal{Y}\setminus\partial$ is said to be *exponentially ergodic* if there exist a probability measure μ_* and $0<\delta<1$ such that for all probability measures μ on $\mathcal{Y}\setminus\partial$, there exists a constant $C_\mu>0$ such that

$$|\mathbb{P}_\mu(Y_t\in B|\tau_\partial>t)-\mu_*(B)|\le C_\mu\delta^t,\quad \forall t>0,\ B\subseteq\mathcal{Y}\setminus\partial$$

(see [25]). The measure μ^* is also said to be exponentially ergodic. In particular, if C_μ can be chosen independently of μ, then $(Y_t:\tau_\partial>t)$ and μ^* is said to be *uniformly exponentially ergodic*. Moreover, if $\partial=\varnothing$, then μ_* is the unique ergodic stationary distribution; if $\partial\ne\varnothing$, then μ_* is a *quasi-limiting distribution* (QLD) [18].

If $\partial\ne\varnothing$, a probability measure ν on $\mathcal{Y}\setminus\partial$ is a QSD for Y_t if for all $t\ge 0$ and all sets $B\subseteq\mathcal{Y}\setminus\partial$,

$$\mathbb{P}_\nu(Y_t\in B|\tau_\partial>t)=\nu(B).$$

Any QLD is a QSD [18]. The existence of a QSD implies certain absorption, and exponential ergodicity of the Q-process implies existence of a unique QSD [18]. A probability measure ν on $\mathcal{Y}\setminus\partial$ is a *quasi-ergodic distribution* if, for any $x\in\mathcal{Y}\setminus\partial$ and any bounded function f on $\mathcal{Y}\setminus\partial$ [10, 27], the following limit holds:

$$\lim_{t\to\infty}\mathbb{E}_x\left(\frac{1}{t}\int_0^t f(Y_s)\mathrm{d}s\middle|\tau_\partial>t\right)=\int_{\mathcal{Y}\setminus\partial}f\mathrm{d}\nu.$$

A quasi-ergodic distribution is in general different from a QSD [27].

https://doi.org/10.1017/apr.2022.20 Published online by Cambridge University Press

Theorem 7. *Assume* (**A1**)–(**A5**) *and that* Ω *is finite.*

(i) Assume $\partial = \varnothing$ *and that* Y_t *is recurrent. Then* Y_t *is positive recurrent and there exists a unique stationary distribution* π *on* $\mathcal{Y}$*, if and only if one of the conditions (C3), (C9), (C10), (C11) holds, while* Y_t *is null recurrent if and only if none of the conditions (C3), (C9), (C10), (C11) hold. Moreover,* Y_t *is exponentially ergodic if either (C17) or (C18) holds.*

(ii) Assume $\partial \neq \varnothing$ *and that* Y_t *has certain absorption. Then there exist no QSDs if none of the conditions (C3), (C9), (C10), (C11) hold. In contrast, there exists a unique uniformly exponentially ergodic QLD, supported on* $\mathcal{Y} \setminus \partial$*, if either (C18) or (C19) holds. Morevover, it is also a unique quasi-ergodic distribution and the unique stationary distribution of the Q-process.*

Theorem 8. *Assume* (**A1**)–(**A5**) *and that* Ω *is infinite.*

- *(i) Assume* $\partial = \varnothing$*. Then* Y_t *is positive recurrent and there exists a unique stationary distribution* π *on* $\mathcal{Y}$ *if (C3) holds. Moreover,* π *is exponentially ergodic if (C17) holds.*
- *(ii) Assume* $\partial \neq \varnothing$*. Then there exist no QSDs if (C7) holds, while there exists a unique uniformly exponentially ergodic QLD supported on* $\mathcal{Y} \setminus \partial$ *if (C19) holds.*

We provide some perspectives:

- The convergence (or ergodicity) in Theorem 8(ii) is uniform with respect to the initial distribution, while in contrast, the convergence in Theorem 8(i) is not uniform. Indeed, for the subcritical linear BDP, the stationary distribution is exponentially ergodic but not uniformly so [5].
- Indeed, one can obtain *uniform* exponential ergodicity in Theorem 7(i) with (C18) or (C19) by choosing a non-reachable absorption set (potentially empty), hence imposing that the time to extinction is infinite. In this case, the QSD is in fact a stationary distribution [12].
- The subtle difference between the conditions for positive recurrence and for exponential ergodicity of QSDs lies in the fact that we have no a priori estimate of the *decay parameter*

$$\psi_0 = \inf\left\{\psi > 0 \colon \liminf_{t\to\infty} e^{\psi t}\mathbb{P}_x(X_t = x) > 0\right\}$$

(which is independent of x) [12]. We cannot compare ψ_0 with $-\alpha$ when $R > 1$, or with $-\beta$ when $R > 2$ and $\alpha = 0$. Refer to the constructive proofs (using Lyapunov functions) in Appendix A for details. Hence, one may believe that the condition we provide for quasi-ergodicity generically is stronger than that for ergodicity.
The only gap cases that remain for QSDs are (C11), (C20), (C21), where neither existence of a QSD nor exponential ergodicity of the Q-process are known to occur, provided one of the three conditions hold.

3.5. Implosivity

Assume $\partial = \varnothing$. Then Y_t is irreducible on $\mathcal{Y}$. Let $B \subsetneq \mathcal{Y}$ be a nonempty proper subset. Then Y_t *implodes towards* B [32] if there exists $t_* > 0$ such that

$$\mathbb{E}_y(\tau_B) \le t_*, \quad \forall y \in \mathcal{Y} \setminus B.$$

https://doi.org/10.1017/apr.2022.20 Published online by Cambridge University Press

Exponential ergodicity of the Q-process	$\Rightarrow^*$	Existence of a QSD	$\Rightarrow^*$	Certain absorption
$\Uparrow$		$\Downarrow$		$\Updownarrow$
Implosivity	$\Rightarrow^*$	Ergodicity	$\Rightarrow^*$	Recurrence

FIGURE 2. Flow diagram of implications among the 21 conditions.

Implosion towards a single state $x \in \mathcal{Y}$ implies finite expected first return time to the state, and thus positive recurrence of x. Indeed,

$$\mathbb{E}_x\left(\tau_x^+\right) \leq \mathbb{E}_x(J_1) + \sup_{\{y\,:\,y \neq x\}} \mathbb{E}_y(\tau_x) < \infty,$$

where $\tau_x^+ = \tau_{\{x\}}^+$, $\tau_x = \tau_{\{x\}}$, and J_1 has finite expectation since x is not absorbing. Hence, Y_t does not implode towards any transient or null recurrent state.

The process Y_t is *implosive* if Y_t implodes towards any state of $\mathcal{Y}$, and otherwise, Y_t is *non-implosive*. Hence, implosivity implies positive recurrence. If Y_t implodes towards a finite nonempty subset of $\mathcal{Y}$, then Y_t is implosive (see Proposition 17).

Theorem 9. *Assume* (**A1**)–(**A5**)*,* $\partial = \varnothing$*, and that* Ω *is finite. Then* Y_t *is implosive, and there exists* $\epsilon > 0$ *such that for every nonempty finite subset* $B \subseteq \mathcal{Y}$ *and every* $x \in \mathcal{Y} \setminus B$*,*

$$\mathbb{E}_x\left(\exp\left(\tau_B^{\epsilon}\right)\right) < \infty,$$

if either (C18) or (C19) holds, while Y_t *is non-implosive otherwise.*

Theorem 10. *Assume* (**A1**)–(**A5**)*,* $\partial = \varnothing$*, and that* Ω *is infinite. Then* Y_t *is implosive if (C19) holds.*

3.6. Relations between absorbed CTMCs and non-absorbed CTMCs

It is worth comparing the properties of absorbed CTMCs to those of non-absorbed CTMCs, as first discussed by Karlin and McGregor [28]. Indeed, in the case of a finite set of absorbing states $\partial \neq \varnothing$ and an irreducible state space $\mathcal{Y} \setminus \partial$, one can add positive transition rates to the transition matrix from the states in ∂ to a finite subset of states in $\mathcal{Y} \setminus \partial$, such that $\mathcal{Y}$ is irreducible for the new chain. Conversely, if $\partial = \varnothing$, then one can prescribe a finite set $\partial' \subseteq \mathcal{Y}$, and delete all transitions from ∂' to $\mathcal{Y}' = \mathcal{Y} \setminus \partial$, so that $\mathcal{Y}'$ is irreducible and ∂' an absorbing set for the new chain. These operations can be viewed as simple extensions to those of [28], proposed in the context of BDPs. As the dynamical properties we discuss generically are determined by transitions among states of large values, the operations provide a way to link the dynamics of an absorbed CTMC with that of a corresponding non-absorbed CTMC, and vice versa.

Figure 2 shows the implications among the properties, in agreement with the parameter conditions derived in the main theorems. In Examples 2 and 3 below, counterexamples are given. It remains unknown whether exponential ergodicity of the Q-process implies implosivity, and whether ergodicity implies existence of a QSD; see Figure 2.

Example 2. (i) Consider the sublinear BDP on $\mathbb{N}_0$ with birth rates $\lambda_j = a$ and death rates $\mu_j = b$ for $j \in \mathbb{N}$. We have $\partial = \{0\}$, $R = 0$, and $\alpha = a - b$. Hence, the process is non-explosive for any initial state by Theorem 1. By [43], the process has certain absorption with decay parameter $\psi_0 = \left(\sqrt{a} - \sqrt{b}\right)^2$ when $a \leq b$, and it admits a continuum family of QSDs when $\alpha < 0$. This shows that existence of a QSD does not imply exponential ergodicity of the Q-process in general.

https://doi.org/10.1017/apr.2022.20 Published online by Cambridge University Press

(ii) Consider the linear BDP on $\mathbb{N}_0$ with birth rates $\lambda_j = aj$ and death rates $\mu_j = bj$ for $j \in \mathbb{N}$. Assume $a \le b$. We have $\partial = \{0\}$, $R = 1$, and $\alpha = a - b$. Hence, the process is non-explosive for any initial state by Theorem 1. By [43], the process has certain absorption with decay parameter $\psi_0 = \left(\sqrt{a} - \sqrt{b}\right)^2$ when $a \le b$, and it admits no QSDs for $\alpha = 0$ (hence, $\beta = -a < 0 = \gamma$), while it admits a continuum family of QSDs for $\alpha < 0$. This shows that the process has certain absorption, but no QSDs for $\alpha = 0$, which is also justified by Theorem 7(ii). Moreover, it also shows that ergodicity of the non-absorbed process does not imply uniqueness of a QSD or exponential ergodicity of the Q-process.

(iii) Consider the superlinear BDP on $\mathbb{N}_0$ with birth rates $\lambda_j = j^2$ and death rates $\mu_j = j^2$ for $j \in \mathbb{N}$. We have $\partial = \{0\}$, $R = 2$, $\alpha = 0$, and $\beta = -1 < 0$. Hence, the process is non-explosive and has certain absorption by Theorem 1 and Theorem 3(ii). By [43], the process admits either no QSDs or a continuum family of QSDs. This shows that certain absorption does not imply exponential ergodicity of the Q-process.

Implosivity is indeed a stronger property than positive recurrence (e.g., when $R \le 1$, $\alpha < 0$), as shown in the following example (see also Table 1).

Example 3. Let Y_t be an irreducible BDP on $\mathbb{N}_0$ with $\Omega = \{1, -1\}$ and

$$\lambda_{-1}(x) = x, \quad \lambda_1(x) = 1, \quad x \in \mathbb{N}_0.$$

In this case, $R = 1$ and $\alpha = -1$. By Theorems 7 and 9, Y_t is positive recurrent and admits an ergodic stationary distribution, but Y_t is non-implosive.

4. Applications

4.1. Stochastic reaction networks

A *reaction network* $(\mathcal{C}, \mathcal{R})$ on a finite set $\mathcal{S} = \{S_1, \ldots, S_n\}$ (with elements called *species*) is an edge-labeled finite digraph with node set $\mathcal{C}$ (with elements called *complexes*) and edge set $\mathcal{R}$ (with elements called *reactions*), such that the elements of $\mathcal{C}$ are nonnegative linear combinations of species, $y = \sum_{i=1}^n y^i S_i$, identified with vectors $y = \left(y^1, \ldots, y^n\right)$ in $\mathbb{N}_0^n$. Reactions are directed edges between complexes, written as $y \to y'$. We assume that every species has a positive coefficient in some complex, and that every complex is in some reaction. Hence, the reaction network can be deduced from the reactions alone, and it is customary simply to list (or draw) the reactions. If $n = 1$, the reaction network is a one-species reaction network.

A *stochastic reaction network* (SRN) is a reaction network together with a CTMC $X(t)$, $t \ge 0$, on $\mathbb{N}_0^n$, modeling the number of molecules of each species over time. A reaction $y \to y'$ *fires* with transition rate $\eta_{y\to y'}(x)$, in which case the chain jumps from $X(t) = x$ to $x + y' - y$ [4]. The Markov process with transition rates $\eta_{y\to y'} : \mathbb{N}_0^n \to \mathbb{R}_{\ge 0}$, $y \to y' \in \mathcal{R}$, has Q-matrix

$$q_{x,x+\omega} = \sum_{y\to y'\in\mathcal{R}\,:\,y'-y=\omega} \eta_{y\to y'}(x).$$

Hence, the transition rate from the state x to $x + \omega$ is

$$\lambda_\omega(x) = \sum_{y\to y'\in\mathcal{R}\,:\,y'-y=\omega} \eta_r(x).$$

For (stochastic) mass-action kinetics, the transition rate for $y \to y'$ is

$$\eta_{y\to y'}(x) = \kappa_{y\to y'} \tfrac{(x)!}{(x-y)!} 1_{\{x'\,:\,x'\ge y\}}(x), \quad x \in \mathbb{N}_0^n$$

https://doi.org/10.1017/apr.2022.20 Published online by Cambridge University Press

(in accordance with the transition rate introduced in the introduction for one-species reaction networks), where $x! := \prod_{i=1}^{n} x_i!$, and $\kappa_{y\to y'}$ is a positive reaction rate constant [3, 4]. Generally, we number the reactions and write $\kappa_1, \kappa_2, \ldots$ for convenience.

In this section, we apply the results developed in Section 3 to some examples of SRNs.

Example 4. Consider the following two reaction networks:

$$(A)\quad \varnothing \underset{\kappa_2^{\mathrm{A}}}{\overset{\kappa_1^{\mathrm{A}}}{\rightleftharpoons}} \mathrm{S}, \qquad \text{and} \qquad (B)\quad \varnothing \underset{\kappa_2^{\mathrm{B}}}{\overset{\kappa_1^{\mathrm{B}}}{\rightleftharpoons}} \mathrm{S}, \quad 2\mathrm{S} \xrightarrow{\kappa_3^{\mathrm{B}}} 3\mathrm{S},$$

with $\Omega = \{-1, 1\}$ in both cases and with transition rates

$$\lambda_{-1}^{A}(x) = \kappa_2^{A} x, \qquad \lambda_1^{A}(x) = \kappa_1^{A},$$
$$\lambda_{-1}^{B}(x) = \kappa_2^{B} x, \qquad \lambda_1^{B}(x) = \kappa_1^{B} + \kappa_3^{B} x(x-1),$$

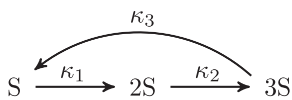
respectively. By Theorem 7, the first is positive recurrent and admits an exponentially ergodic stationary distribution on $\mathbb{N}_0$, since $\alpha = -\kappa_2^{A}$ and $R = 1$, while by Theorem 1, the second reaction network is explosive for any initial state, since $\alpha = \kappa_3^{B} > 0$ and $R = 2$. Indeed, these two reaction networks are *structurally equivalent* in the sense that there is only one irreducible component $\mathbb{N}_0$ [48].

Example 5. Consider the following pair of SRNs from the introduction:

$$\mathrm{S} \underset{2}{\overset{1}{\rightleftharpoons}} 2\mathrm{S} \underset{4}{\overset{4}{\rightleftharpoons}} 3\mathrm{S} \underset{1}{\overset{6}{\rightleftharpoons}} 4\mathrm{S} \xrightarrow{1} 5\mathrm{S}, \qquad \mathrm{S} \underset{2}{\overset{1}{\rightleftharpoons}} 2\mathrm{S} \underset{1}{\overset{3}{\rightleftharpoons}} 3\mathrm{S} \xrightarrow{1} 4\mathrm{S}. \tag{6}$$

For the first reaction network, $R = 4$, $\alpha = 0$, and $\beta = 1$, and for the second, $R = 3$, $\alpha = 0$, and $\beta = 0$. By Theorem 1, the first is explosive for any initial state and the second does not explode for any initial state.

Example 6. (i) Consider a strongly connected reaction network:

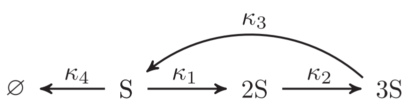


For the underlying CTMC Y_t, $\Omega = \{1, -2\}$, and

$$\lambda_1(x) = \kappa_1 x + \kappa_2 x(x-1), \qquad \lambda_{-2}(x) = \kappa_3 x(x-1)(x-2).$$

Hence Y_t is irreducible on $\mathbb{N}$ with 0 a neutral state. Moreover, $R = 3$ and $\alpha = -2\kappa_3$. By Theorem 7, there exists a unique exponentially ergodic stationary distribution on $\mathbb{N}$.

(ii) Consider a similar reaction network including direct degradation of S:



The threshold parameters are the same as in (i). Let $\partial = \{0\}$ with $\mathbb{N}_0 \setminus \partial = \mathbb{N}$ an irreducible component. By Theorem 7, the network has a uniformly exponentially ergodic QSD on $\mathbb{N}$.

https://doi.org/10.1017/apr.2022.20 Published online by Cambridge University Press

4.2. An extended class of branching processes

Consider an extended class of branching processes [16] with transition rate matrix $Q = (q_{x,y})_{x,y\in\mathbb{N}_0}$:

$$q_{x,y} = \begin{cases} r(x)\mu(y-x+1) & \text{if} \quad y \geq x-1 \geq 0 \quad \text{and} \quad y \neq x, \\ -r(x)(1-\mu(1)) & \text{if} \quad y = x \geq 1, \\ q_{0,y} & \text{if} \quad y > x = 0, \\ -q_0 & \text{if} \quad y = x = 0, \\ 0 & \text{otherwise,} \end{cases} \tag{7}$$

where μ is a probability measure on $\mathbb{N}_0$, $q_0 = \sum_{y\in\mathbb{N}} q_{0,y}$, and $r(x)$ is a positive finite function on $\mathbb{N}_0$. Assume the following:

(**H1**) $\mu(0) > 0$, $\mu(0) + \mu(1) < 1$;

(**H2**) $\sum_{y\in\mathbb{N}} q_{0,y} y < \infty$, $E = \sum_{k\in\mathbb{N}_0} k\mu(k) < \infty$;

(**H3**) $r(x)$ is a polynomial of degree $R \geq 1$ for large x.

The next theorem follows from the results in Section 3. We would like to mention that the results below provide conditions for different dynamical regimes in terms of only R and M. In contrast, the condition for positive recurrence in [16] also depends on the integrability of a definite integral as well as summability of a series, which nonetheless never appear.

Theorem 11. *Assume* (**H1**)–(**H3**). *Let Y_t be a process generated by the Q-matrix given in* (7) *and $Y_0 \neq 0$. Then Y_t is non-explosive if one of the following conditions holds: (1) $R \leq 1$, (2) $E < 1$, (3) $R = 2$, $E = 1$, while it is explosive with positive probability if (4) $E > 1$, $R > 1$. Furthermore, the following hold:*

(i) If $q_0 > 0$, then Y_t is irreducible on $\mathbb{N}_0$ and is

(i-1) recurrent if $E < 1$, and transient if $E > 1$;

(i-2) positive recurrent and exponentially ergodic if $E < 1$;

(i-3) implosive if $R > 1$ and $E < 1$.

(ii) If $q_0 = 0$, then $\partial = \{0\}$, and Y_t has certain absorption if $E < 1$, while it does not if $E > 1$. Moreover, the process admits no QSDs if $E > 1$, while it admits a uniformly exponentially ergodic QSD on $\mathbb{N}$ if $R > 1$ and $E < 1$.

Proof. For all $k \in \mathbb{N} \cup \{-1\}$, let

$$\lambda_k(x) = \begin{cases} r(x)\mu(k+1) & \text{if } x \in \mathbb{N}, \\ q_{0k} & \text{if } x = 0. \end{cases}$$

By (**H1**), $\mu(k) > 0$ for some $k \in \mathbb{N}$. Note that $\mathcal{Y} \setminus \partial$ is irreducible, with $\partial = \varnothing$ if $q_0 > 0$ and $\partial = \{0\}$ if $q_0 = 0$. Hence, regardless of q_0, by positivity of r, (**A1**)–(**A2**) are satisfied with $\Omega_- = \{-1\}$ and $\Omega_+ = \{y : q_{0y} > 0\} \cup (\operatorname{supp} \mu \setminus \{0, 1\} - 1)$. Moreover, (**H2**)–(**H3**) imply (**A3**)–(**A4**). Let $r(x) = ax^R + bx^{R-1} + \mathrm{O}\big(x^{R-2}\big)$ with $a > 0$. Since $R \geq 1$, it coincides with $\max\{\deg(\lambda_\omega) : \omega \in \Omega\}$. It is straightforward to verify that

$$\alpha = a(E-1), \quad \beta = \big(\tfrac{1}{2}a + b\big)(E-1) - \tfrac{1}{2}aE', \quad \text{and} \quad \gamma = b(E-1),$$

https://doi.org/10.1017/apr.2022.20 Published online by Cambridge University Press

where $E' = \sum_{k\in\mathbb{N}} k(k-1)\mu(k) > 0$. Hence α has the same sign as $E-1$, and $\beta < 0$ whenever $E = 1$ (or equivalently, $\alpha = 0$). Furthermore, $\alpha = 0$ implies $\gamma = 0$. In addition, in light of the fact that $R \geq 1$, the condition $E \geq 1$ decomposes into three possibilities:

$$E > 1, \quad \text{or} \quad E = 1,\ R > 1, \quad \text{or} \quad E = R = 1.$$

Then the conclusions follow directly from Theorems 1, 3, 7 and 9. □

Corollary 1. *Assume* (**H1**)–(**H3**)*, that μ has finite support, and that $\{y : q_{0y} > 0\}$ is finite. Then Y_t is non-explosive if and only if either $R = 1$ or $E \leq 1$. Furthermore, the following hold:*

(i) If $q_0 > 0$, then Y_t is irreducible and is

(i-1) recurrent if $E \leq 1$, and transient otherwise;

(i-2) positive recurrent if and only if $E < 1$, or $E = 1$, $R > 1$, while it is null recurrent if and only if $E = R = 1$; furthermore, Y_t is exponentially ergodic if $E < 1$, or if $E = 1$, $R > 2$;

(i-3) implosive if and only if either of the two conditions $R > 1$, $E < 1$, or $R > 2$, $E = 1$ holds.

(ii) If $q_0 = 0$, then $\partial = \{0\}$, and Y_t has certain absorption if and only if $E \leq 1$. Moreover, the process admits no QSDs if $E > 1$, or $E = R = 1$, while it admits a uniformly exponentially ergodic QSD on $\mathbb{N}$ if either $R > 1$, $E < 1$, or $R > 2$, $E = 1$.

Proof. Based on the proof of Theorem 11, the conclusions follow from Corollaries 2, 4, 8, and 10. □

The extended branching process under more general assumptions (allowing more general forms of r) is addressed in [16]. In that reference, the conditions given for the dynamic behavior of the process seem more involved than here and even become void in some situations (e.g., in [16, Corollary 1.5(iii)], where the definite integral indeed is always infinite under (**H1**)–(**H3**).)

4.3. A general single-cell stochastic gene expression model

To model single-cell stochastic gene expression with bursty production, we propose the following one-species *generalized* reaction network (consisting potentially of infinitely many reactions) with mass-action kinetics:

$$m\mathrm{S} \xrightarrow{c_m\mu_m(\mathrm{k})} (m+k)\mathrm{S}, \quad m = 0, \ldots, J_1, \qquad m\mathrm{S} \xrightarrow{r_m} (m-1)\mathrm{S}, \quad m = 1, \ldots, J_2, \tag{8}$$

where $c_m \geq 0$ for $m = 0, \ldots, J_1$, $r_m \geq 0$ for $m = 1, \ldots, J_2$, $J_1 \in \mathbb{N}_0$, $J_2 \in \mathbb{N}$, and μ_m, for $m = 0, \ldots, J_1$, are probability distributions on $\mathbb{N}$. Assume the following:

(**H4**) $J_1 \leq J_2$, $c_0 > 0$, $c_{J_1} > 0$, $r_1 > 0$, and $r_{J_2} > 0$.

(**H5**) $E_m = \sum_{k=1}^{\infty} k\mu_m(k) < \infty$, for $m = 0, \ldots, J_1$.

This network encompasses several single-cell stochastic gene expression models in the presence of bursting; see e.g. [8, 17, 31, 41]. Ergodicity and an exact formula for the ergodic stationary distribution (when it exists) are the main concerns of these references. The first set of J_1 reactions accounts for bursty production of mRNA copies with transcription rate c_m and

https://doi.org/10.1017/apr.2022.20 Published online by Cambridge University Press

burst size distribution μ_m. The second set of J_2 reactions accounts for degradation of mRNA with degradation rate r_m [17, 41].

The network (8) reduces to the specific model studied in the following references:

- [17, Section 4] (see also [31, Section 3.2]), when $J_1 = 0$, $J_2 = 1$, and μ_0 is a geometric distribution.
- [40], when $J_1 = 0$, $J_2 = 1$, and μ_0 is a negative binomial distribution.
- [31, Example 3.6], when $J_1 = J_2 = 1$, and $\mu_0 = \mu_1$ are geometric distributions.
- [21] when $J_1 = 2$, $J_2 = 3$, $\mu_0 = \delta_1$, $\mu_2 = \delta_k$ for some $k \in \mathbb{N}$, and $c_1 = r_2 = 0$. Here δ_i is the Dirac delta measure at i.

Theorem 12. *Assume* (**H4**)–(**H5**)*, and that μ_m has finite support whenever $c_m > 0$ for $m = 0, \ldots, J_1$. Then the process Y_t associated with the network* (8) *is irreducible on $\mathbb{N}_0$, and it is positive recurrent and there exists an ergodic stationary distribution on $\mathbb{N}_0$ if and only if one of the following conditions holds:*

(i) $J_1 < J_2$,

(ii) $J_1 = J_2$ *and* $c_{J_2}E_{J_2} < r_{J_2}$,

(iii) $J_1 = J_2 > 2$, $c_{J_2}E_{J_2} = r_{J_2}$, *and* $c_{J_2-1}E_{J_2-1} \le r_{J_2-1} + \frac{1}{2}c_{J_2}\big(E_{J_2} + E'_{J_2}\big)$,

(iv) $J_1 = J_2 = 2$, $c_{J_2}E_{J_2} = r_{J_2}$, *and* $c_{J_2-1}E_{J_2-1} < r_{J_2-1} + \frac{1}{2}c_{J_2}\big(E_{J_2} + E'_{J_2}\big)$,

(v) $J_1 = J_2 = 1$, $c_{J_2}E_{J_2} = r_{J_2}$, *and* $c_{J_2-1}E_{J_2-1} < r_{J_2-1}$,

where $E'_m = \sum_{k=1}^{\infty} k^2 \mu_m(k)$. Moreover, the stationary distribution is exponentially ergodic if one of (i), (ii), and (iii) holds. Furthermore, the process Y_t is implosive if and only if (iii) or $J_2 > 1$ with (i) or (ii).

Proof. We have $\Omega = \{-1\} \cup \left(\cup_{j=0}^{J_1} \operatorname{supp} \mu_j\right)$, and

$$\lambda_{-1}(x) = \sum_{j=1}^{J_2} r_j x^{\underline{j}}, \qquad \lambda_k(x) = \sum_{j=0}^{J_1} c_j \mu_j(k) x^{\underline{j}},$$

for $k \in \mathbb{N}$ and $x \in \mathbb{N}_0$, where $x^{\underline{j}} = \prod_{i=0}^{j-1}(x-i)$ is the descending factorial. By (**H4**), (**A1**)–(**A2**) are satisfied; moreover, the irreducibility of Y_t also follows from [48]. Under (**H5**), the mass-action kinetics yield (**A3**)–(**A4**). Since $J_1 \le J_2$ by (**H4**), we have $R = J_2 \ge 1$. Since

$$\sum_{\omega\in\Omega} \lambda_\omega(x)\omega = -\sum_{j=J_1+1}^{J_2} r_j x^{\underline{j}} + \sum_{j=1}^{J_1} \big(c_j E_j - r_j\big) x^{\underline{j}} + c_0 E_0,$$
$$\sum_{\omega\in\Omega} \lambda_\omega(x)\omega^2 = \sum_{j=J_1+1}^{J_2} r_j x^{\underline{j}} + \sum_{j=1}^{J_1} \big(c_j E'_j + r_j\big) x^{\underline{j}} + c_0 E'_0,$$

we have $\alpha = c_{J_1}E_{J_1}\delta_{J_1,J_2} - r_{J_2}$, where $\delta_{i,j}$ is the Kronecker delta. When $\alpha = 0$, we have

$$J_1 = J_2, \quad c_{J_2}E_{J_2} = r_{J_2}, \quad \gamma = c_{J_2-1}E_{J_2-1} - r_{J_2-1},$$
$$\beta = c_{J_2-1}E_{J_2-1} - r_{J_2-1} - \frac{1}{2}c_{J_2}\big(E_{J_2} + E'_{J_2}\big).$$

https://doi.org/10.1017/apr.2022.20 Published online by Cambridge University Press

The combination of the conditions (i) and (ii) is equivalent to $\alpha < 0$; the condition (iii) is equivalent to $R > 2$, $\alpha = 0$, $\beta \leq 0$; the condition (iv) is equivalent to $R = 2$, $\alpha = 0$, $\beta < 0$; the condition (v) is equivalent to $R = 1$, $\alpha = 0$, $\gamma < 0$. The conclusions then follow from Theorems 7 and 9. □

Corollary 2. *Assume* **(H4)**–**(H5)**, *and that μ_m has infinite support and $c_m > 0$ for some $m = 0, \ldots, J_1$. Then Y_t is irreducible on $\mathbb{N}_0$, and is positive recurrent with an ergodic stationary distribution if one of Theorem 12(i), Theorem 12(ii), and Theorem 12(v) holds. Moreover, the stationary distribution is exponentially ergodic if either Theorem 12(i) or Theorem 12(ii) holds. In addition, the process Y_t is implosive if $J_2 > 1$.*

Proof. Based on the proof of Theorem 12, the conclusions follow directly from Corollaries 8 and 10. □

4.4. Stochastic populations under bursty reproduction

Two stochastic population models with bursty reproduction are investigated in [8].

The first model is a Verhulst logistic population process with bursty reproduction. The process Y_t is a CTMC on $\mathbb{N}_0$ with transition rate matrix $Q = (q_{x,y})_{x,y\in\mathbb{N}_0}$ satisfying

$$q_{x,y} = \begin{cases} c\mu(j)x & \text{if } y = x + j,\ j \in \mathbb{N}, \\ \frac{c}{K}x^2 + x & \text{if } y = x - 1 \in \mathbb{N}_0, \\ 0 & \text{otherwise}, \end{cases}$$

where $c > 0$ is the reproduction rate, $K \in \mathbb{N}$ is the typical population size in the long-lived metastable state prior to extinction [8], and μ is the burst size distribution. Assume the following:

(H6) $E_b = \sum_{k=1}^{\infty} k\mu(k) < \infty$.

Approximations of the mean time to extinction and QSD are discussed in [8] against various burst size distributions of finite mean (e.g., Dirac measure, Poisson distribution, geometric distribution, negative-binomial distribution). Nevertheless, the existence of a QSD is not proved there. Here we prove the certain absorption and ergodicity of the QSD for this population model.

Theorem 13. *Assume* **(H6)**. *The Verhulst logistic model Y_t with bursty reproduction has certain absorption. Moreover, there exists a uniformly exponentially ergodic QSD on $\mathbb{N}$ trapped to zero.*

Proof. We have $\Omega = \operatorname{supp} \mu \cup \{-1\}$, $\lambda_{-1}(x) = \frac{c}{K}x^2 + x$, $\lambda_k(x) = c\mu(k)x$, for $k \in \mathbb{N}$ and $x \in \mathbb{N}$. Let $\partial = \{0\}$; then $\mathbb{N}_0 \setminus \partial = \mathbb{N}$ is irreducible [48]. Hence **(A1)**–**(A5)** are satisfied. Moreover, $R = 2$, $\alpha = -\frac{c}{K} < 0$, and thus the conclusions follow from Theorems 3 and 7, together with Corollaries 4 and 8 for finite supp μ and infinite supp μ, respectively. □

The second model is a runaway model of a stochastic population including bursty pair reproduction [8]. This model can be described as a generalized reaction network, where c, K, and μ are defined as in the first model. The survival probability of this population model is addressed in [8]. Nevertheless, it turns out that this model is explosive for any initial state.

Theorem 14. *Assume* **(H6)**. *The runaway model is explosive.*

https://doi.org/10.1017/apr.2022.20 Published online by Cambridge University Press

Proof. We have $\Omega = \text{supp}\,\mu \cup \{-1\}$, $\lambda_k(x) = \frac{c}{K}\mu(k)x(x-1)$, $\lambda_{-1}(x) = x$, for $k \in \mathbb{N}$ and $x \in \mathbb{N}$. Let $\partial = \{0, 1\}$. Then $\mathbb{N}_0 \setminus \partial = \mathbb{N} \setminus \{1\}$ is irreducible [48]. Hence (**A5**) is valid. Moreover, it is easy to verify that (**A1**)–(**A4**) are also satisfied. In addition, $R = 2$, $\alpha = \frac{c}{K}E_b > 0$, and thus the conclusions follow from Theorem 1 and Corollary 2 for finite supp μ and infinite supp μ, respectively. □

5. Proofs

5.1. Proof of Theorem 1

Hereafter, we use the notation $[m, n]_1$ ($[m, n[_1$, etc.) for the set of consecutive integers from m to n, with $m, n \in \mathbb{N}_0 \cup \{+\infty\}$. The notation is adopted from [48].

Moreover, throughout the proofs, we assume without loss of generality that $\mathcal{Y} = \mathbb{N}_0$, and $\partial \subseteq \{0\}$ for ease of exposition. Indeed, the dynamical properties of the CTMCs discussed in this paper depend only on the transition structure of the states $x \in \mathcal{Y}$ with large values of x. By Assumptions (**A4**)–(**A5**), all jumps in Ω are possible. When $\partial \neq \varnothing$, it is standard to 'glue' all states in ∂ to be a single state 0, since ∂ is finite and the set of states in $\mathcal{Y} \setminus \partial$ one jump away from ∂ is also finite, by (**A2**).

We prove the conclusions case by case.
(a) Assume $\partial = \varnothing$. Then Y_t is irreducible on $\mathbb{N}_0$, and one can directly apply Propositions 2 and 3 with appropriate Lyapunov functions to be determined.
(b) Assume $\partial \neq \varnothing$. Let Z_t be the irreducible CTMC on the state space $\mathcal{Y} \setminus \partial$ with $Z_0 = Y_0$ and transition operator $\widetilde{Q}$ being Q restricted to $\mathcal{Y} \setminus \partial$:

$$\widetilde{q}_{x,y} = q_{x,y}, \quad \text{for all } x,\ y \in \mathcal{Y} \setminus \partial \text{ and } x \neq y.$$

In the following, we show that Z_t is explosive if and only if Y_t is explosive, and hence the case (b) reduces to the case (a). This equivalence is not quite trivial. There is a positive probability that, starting from any non-absorbing state, the chain will jump to an absorbing state in a finite number of steps. So we need to show that this will not happen with probability one. Otherwise, explosivity is not possible. Assume first that Z_t is explosive. Then $\widetilde{Q}v = v$ for some bounded nonnegative nonzero v. Let $u_x = v_x \mathbb{1}_{\mathcal{Y}\setminus\partial}(x)$, $\forall x \in \mathcal{Y}$. It is straightforward to verify that $Qu = u$. By Proposition 4, Y_t is also explosive. Conversely, assume that Y_t is explosive; then $Qu = u$ for some bounded nonnegative nonzero u. Let $w = u|_\partial$, i.e., $w_x = u_x$ for all $x \in \partial$. Since $Q|_\partial$ is a lower-triangular matrix with nonpositive diagonal entries, and $w \geq 0$, it is readily deduced that $w = 0$ by Gaussian elimination. This implies from $Qu = u$ that $\widetilde{Q}v = v$ with $v = u|_{\mathcal{Y}\setminus\partial}$. Hence Z_t is explosive. To sum up, Z_t is explosive if and only if Y_t is explosive.

Based on the above analysis, it remains to prove the conclusions for the case (a) using Propositions 2 and 3. We first prove the conclusions assuming Ω is finite.

(i) We prove explosivity by Proposition 2. Let the lattice interval $A = [0, x_0 - \min\Omega_-[_1$ for some $x_0 > 1$ to be determined. Since $\#\Omega_- < \infty$, $A \subseteq \mathbb{N}_0$ is finite. Let f be decreasing and bounded such that $f(x) = \mathbb{1}_{[0,x_0[_1}(x) + x^{-\delta}\mathbb{1}_{[x_0,\infty[_1}(x)$ for all $x \geq x_0$, with $\delta > 0$ to be determined. Obviously, Proposition 2(i) is satisfied for the set A. Next we verify the conditions in Proposition 2(ii). It is easy to verify by straightforward calculation that

$$Qf(x) < -\epsilon \quad \text{for all } x \in \mathbb{N}_0 \setminus A,$$

where $\epsilon = \delta\alpha/2$ provided (C1) holds with $\delta < R - 1$, or $\epsilon = \delta(\beta - \delta\vartheta)/2$ provided (C2) holds with $\delta < \min\{\beta/\vartheta, R - 2\}$, and x_0 is chosen large enough. Since $\delta > 0$ can be arbitrarily small,

https://doi.org/10.1017/apr.2022.20 Published online by Cambridge University Press

in either case, there exist δ and ϵ such that the conditions in Proposition 2 are fulfilled, and thus $\mathbb{E}_x\zeta < +\infty$ for all $x \in \mathcal{Y}$. In particular, Y_t is explosive a.s.

(ii) Now we prove non-explosivity using Proposition 3. Let $f(x) = \log\log(x+1)$ and $g(x) = (|\alpha| + |\beta| + 1)(x+M)$ for all $x \in \mathbb{N}_0$, with some $M > 0$ to be determined. One can show that all the conditions in Proposition 3 are satisfied with some large constant $M > 0$, provided neither (C1) nor (C2) holds. Hence Y_t is non-explosive.

5.2. Proof of Theorem 2

We first prove explosivity under the condition (C1). Let f be as in the proof of Theorem 1(i), and let

$$\alpha_- = \lim_{x\to\infty} \frac{\sum_{\omega\in\Omega_-} \lambda_\omega(x)\omega}{x^R}, \qquad \alpha_+ = \lim_{x\to\infty} \frac{\sum_{\omega\in\Omega_+} \lambda_\omega(x)\omega}{x^R}.$$

Then $\alpha = \alpha_+ + \alpha_-$. Since $\alpha > 0$, we have $R_+ = R$ and there exists $\epsilon_0 \in]0,1[$ such that $\alpha_- + (1-\epsilon_0)\alpha_+ > 0$. By Proposition 1, there exist $N_0,\ u' \in \mathbb{N}$ such that

$$\frac{\sum_{\omega\in\Omega_+\cap[1,N_0]_1} \lambda_\omega(x)\omega}{\sum_{\omega\in\Omega_+} \lambda_\omega(x)\omega} \geq 1 - \epsilon_0 \quad \text{for all } x \geq u'.$$

By (**A3**), $\Omega \setminus]N_0, \infty[_1$ is finite. Hence, choosing $x_0 \geq u'$ large, we have for all $x \in \mathcal{Y} \setminus A$

$$\begin{aligned}
Qf(x) &= \sum_{\omega\in\Omega_-} \lambda_\omega(x)\big((x+\omega)^{-\delta} - x^{-\delta}\big) + \sum_{\omega\in\Omega_+} \lambda_\omega(x)\big((x+\omega)^{-\delta} - x^{-\delta}\big) \\
&\leq \sum_{\omega\in\Omega_-} \lambda_\omega(x)\big((x+\omega)^{-\delta} - x^{-\delta}\big) + \sum_{\omega\in\Omega_+\cap[1,N_0]_1} \lambda_\omega(x)\big((x+\omega)^{-\delta} - x^{-\delta}\big) \\
&\leq x^{-\delta} \sum_{\omega\in\Omega_-} \lambda_\omega(x)\big(-\omega\delta x^{-1} + \mathrm{O}(x^{-2})\big) + x^{-\delta} \sum_{\omega\in\Omega_+\cap[1,N_0]_1} \lambda_\omega(x)\big(-\omega\delta x^{-1} + \mathrm{O}(x^{-2})\big) \\
&= -\delta\alpha_- x^{R-1-\delta} + \mathrm{O}\big(x^{R-2-\delta}\big) - \delta x^{-1-\delta} \sum_{\omega\in\Omega_+\cap[1,N_0]_1} \lambda_\omega(x)\omega \\
&\leq -\delta\alpha_- x^{R-1-\delta} + \mathrm{O}\big(x^{R-2-\delta}\big) - \delta(1-\epsilon_0)x^{-1-\delta} \sum_{\omega\in\Omega_+} \lambda_\omega(x)\omega \\
&= -\delta\big(\alpha_- + (1-\epsilon_0)\alpha_+\big)x^{R-1-\delta} + \mathrm{O}\big(x^{R-2-\delta}\big) < -\epsilon,
\end{aligned}$$

where

$$\epsilon = \frac{\delta\big(\alpha_- + (1-\epsilon_0)\alpha_+\big)}{2},$$

and $\delta < R - 1$. The rest of the argument is the same as that of the proof of Theorem 1(i).

Next, we prove non-explosivity under (C3), (C4), or (C5). Let f and g be as in (ii) in the proof of Theorem 1. By (**A3**), for some large $M > 0$ to be determined, for all $x \in \mathbb{N}_0$,

$$\begin{aligned}
Qf(x) &= \sum_{\omega\in\Omega_-} \lambda_\omega(x)\big(\log\log(x+1+\omega) - \log\log(x+1)\big) \\
&\quad + \sum_{\omega\in\Omega_+} \lambda_\omega(x)\big(\log\log(x+1+\omega) - \log\log(x+1)\big) \\
&= \sum_{\omega\in\Omega_-} \lambda_\omega(x)\log\left(1+\frac{\log\big(1+\frac{\omega}{x+1}\big)}{\log(x+1)}\right) + \sum_{\omega\in\Omega_+} \lambda_\omega(x)\log\left(1+\frac{\log\big(1+\frac{\omega}{x+1}\big)}{\log(x+1)}\right) \\
&= \sum_{\omega\in\Omega_-} \lambda_\omega(x)\left(\frac{\omega}{(x+1)\log(x+1)} + \mathrm{O}\big((x+1)^{-2}(\log(x+1))^{-1}\big)\right) \\
&\quad + \sum_{\omega\in\Omega_+} \lambda_\omega(x)\log\left(1+\frac{\log\big(1+\frac{\omega}{x+1}\big)}{\log(x+1)}\right) \\
&\le \sum_{\omega\in\Omega_-} \lambda_\omega(x)\left(\frac{\omega}{(x+1)\log(x+1)} + \mathrm{O}\Big((x+1)^{-2}(\log(x+1))^{-1}\Big)\right) \\
&\quad + \sum_{\omega\in\Omega_+} \lambda_\omega(x)\frac{\omega}{(x+1)\log(x+1)} \\
&= \frac{1}{(x+1)\log(x+1)}\sum_{\omega\in\Omega} \lambda_\omega(x)\omega + \mathrm{O}\Big((x+1)^{R-2}(\log(x+1))^{-1}\Big) \\
&= \alpha\frac{(x+1)^{R-1}}{\log(x+1)} + \mathrm{O}\Big((x+1)^{R-2}(\log(x+1))^{-1}\Big) \le g(f(x)),
\end{aligned}$$

provided (C3), (C4), or (C5) holds. The rest of the proof is the same as that of Theorem 1(ii).

5.3. Proof of Theorem 3

Let $h_A = \mathbb{P}_{Y_0}(\tau_A < \infty)$ be the *hitting probability* [36]. In particular, h_A is called the *absorption probability* if A is a closed communicating class. To verify conditions for certain absorption requires the following property for hitting probabilities. For any set $A \subseteq \mathcal{Y}$, we write $h_A(i)$ for h_A, to emphasize the dependence of the hitting probability on the initial state $i \in \mathcal{Y}$. In particular, if $A = \{x\}$ is a singleton, we simply write h_x for h_A.

Assume without loss of generality that $\mathcal{Y} = \mathbb{N}_0$, and $\partial \subseteq \{0\}$.

(i) We first show recurrence and transience. The idea of applying the classical semimartingale approach originates from [29]. It suffices to show recurrence and transience for the embedded discrete-time Markov chain $\widetilde{Y}_n$ of Y_t.

To show recurrence, let $Z_n = \log\log(\widetilde{Y}_n + 1)$. Since a one-to-one bicontinuous transformation of the state space preserves the Markov property and recurrence, it suffices to show recurrence for Z_n. In light of the expression for the transition probability of $\widetilde{Y}_n$, we have

$$\mathbb{E}(Z_{n+1} - Z_n | Z_n = \log\log(x+1)) = \frac{1}{\sum_{\omega\in\Omega}\lambda_\omega(x)}\sum_{\omega\in\Omega}\lambda_\omega(x)(\log\log(x+\omega) - \log\log x).$$

https://doi.org/10.1017/apr.2022.20 Published online by Cambridge University Press

By tedious but straightforward computation, we have the following asymptotic expansion:

$$\mathbb{E}\big(Z_{n+1}-Z_n|Z_n=\log\log(x+1)\big)=\frac{\alpha x^R+\beta x^{R-1}-\vartheta x^{R-1}(\log x)^{-1}+\mathrm{O}\big(x^{R-2}\big)}{(1+x)\log(1+x)\sum_{\omega\in\Omega}\lambda_\omega(x)}.$$

From this asymptotic expansion, and noting that $\vartheta>0$, we have

$$\mathbb{E}\big(Z_{n+1}-Z_n|Z_n=\log\log(x+1)\big)\le 0,\qquad \forall n\in\mathbb{N}_0,\ \text{ for all large } x,$$

provided either (C3) or (C6) holds. From Proposition 6 follows the recurrence of Z_n, and thus the recurrence of $\widetilde{Y}_n$ as well.

Next, we prove the transience of $\widetilde{Y}_n$ under the reverse conditions (that is, neither (C3) nor (C6) holds). Let $Z'_n=1-\big(1+\widetilde{Y}_n\big)^{-\delta}$, with $\delta>0$ to be determined. Again, Z'_n is a Markov chain, and $\widetilde{Y}_n\to\infty$ if and only if $Z'_n\to 1$, which implies that (14) is fulfilled for Z'_n with $M=1$, since $\widetilde{Y}_n$ on a subset of $\mathbb{N}_0$ is irreducible. Similarly to the above computation, we have the asymptotic expansion

$$\mathbb{E}\Big(Z'_{n+1}-Z'_n|Z'_n=1-(1+x)^{-\delta}\Big)=\frac{\delta}{(1+x)^{\delta+1}\sum_{\omega\in\Omega}\lambda_\omega(x)}\big(\alpha x^R+(\beta-\delta\vartheta)x^{R-1}+\mathrm{O}\big(x^{R-2}\big)\big).$$

Hence

$$\mathbb{E}\Big(Z'_{n+1}-Z'_n|Z'_n=1-(1+x)^{-\delta}\Big)\ge 0\qquad \forall n\in\mathbb{N}_0$$

for all large x (and so for all values of $z=Z'_n$ in some interval $C\le z<1$), provided $\alpha>0$ or $\alpha=0,\ \beta>0$ with $\delta<\frac{\beta}{\vartheta}$. By Proposition 7,

$$\mathbb{P}\Big(\lim_{n\to\infty}Z'_n=1\Big)=1,$$

that is,

$$\mathbb{P}\Big(\lim_{n\to\infty}\widetilde{Y}_n=\infty\Big)=1,$$

meaning $\widetilde{Y}_n$ is transient.

(ii) Let $\widetilde{\omega}\in\Omega_+$. Let $k_0=\min\{l\in\mathbb{N}: l\widetilde{\omega}\in\mathbb{N}\}$. Define Z_t to be a CTMC on $\mathbb{N}_0$ with transition matrix $\widetilde{Q}=(\widetilde{q}_{xy})$ satisfying, for all $x\ne y,\ x,y\in\mathbb{N}_0$,

$$\widetilde{q}_{xy}=\begin{cases} q_{xy} & \text{if } x\in\mathbb{N},\\ 1 & \text{if } x=0 \text{ and } y=j\widetilde{\omega},\ j=1,\ldots,k_0,\\ 0 & \text{otherwise.}\end{cases}$$

It is easy to verify that Z_t is irreducible on $\mathbb{N}_0$. In the following, we show that the recurrence of Z_t is equivalent to the absorption of Y_t, which yields the conclusion.

On the one hand, applying (i) to Z_t, we have that Z_t is recurrent if and only if (C3) or (C6) holds. On the other hand, from Proposition 9, Z_t is recurrent if and only if $h_0^Z(i)=1$ for all $i\in\mathcal{Y}$, where $h_0^Z(i)$ is the hitting probability for Z_t. By Proposition 8, $h_0^Z(i)=1$ for all $i\in\mathcal{Y}$ if and only if $(1,\ldots,1)$ is the minimal nonnegative solution to the linear equations

$$\begin{cases} x_i=1, & i=0,\\ \sum_{j\in\partial\setminus\{i\}}\widetilde{q}_{ij}(x_i-x_j)=0, & i\in\mathbb{N},\end{cases}$$

https://doi.org/10.1017/apr.2022.20 Published online by Cambridge University Press

which, by the definition of $\widetilde{Q}$, are identical to

$$\begin{cases} x_i = 1, & i = 0, \\ \sum_{j\in\mathcal{Y}\setminus\{i\}} q_{ij}\big(x_i - x_j\big) = 0, & i \in \mathbb{N}. \end{cases} \tag{9}$$

By Proposition 8, Y_t has certain absorption if and only if $(1, \ldots, 1)$ is the minimal nonnegative solution to (9). Hence the recurrence of Z_t is equivalent to the certain absorption of Y_t.

5.4. Proof of Theorem 4

If Ω is infinite, then by (**A3**), Ω_+ is also infinite, while Ω_- is finite. Hence, the asymptotic expansions of the sum over all negative jumps in Ω_- in the proof of Theorem 3 remain valid, while the asymptotic expansions of the sum over all positive jumps in Ω_+ might fail, in that the sum is infinite. Nevertheless, an upper estimate of $Qf(x)$ for certain Lyapunov functions is still possible, as demonstrated in the proof of Theorem 2. A careful examination of the arguments of Theorem 3 shows that the desired upper estimates of $Qf(x)$ hold under the respective conditions listed in Theorem 4, by replacing the asymptotic expansions by one-sided inequalities.

5.5. Proof of Theorem 5

Assume without loss of generality that $\mathcal{Y} = \mathbb{N}_0$.

We first prove the existence of moments of hitting times assuming Ω is finite, by applying Proposition 10(i) case by case. We now prove the existence of moments under (C13) for $0 < \delta < 1/2$. Let $f(x) = \sqrt{x+1}$ for $x \in \mathbb{N}_0$. One can directly verify that for every $0 < \sigma = 2\delta < 2$, there exists $0 < c < +\infty$ such that $Qf^{\sigma}(x) \le -cf^{\sigma-2}(x)$ for all large x. By Proposition 10(i), there exists $a > 0$ such that

$$\mathbb{E}_x\Big(\tau^{\epsilon}_{\{f\le a\}}\Big) < +\infty, \quad \forall x \in \mathcal{Y},\ \forall 0 < \epsilon < \sigma/2.$$

Moreover, $\{f \le a\}$ is finite since $\lim_{x\to\infty} f(x) = +\infty$.

Analogous arguments apply to the cases (C16) for $0 < \delta < 1$ and (C10) for $\delta > 0$ with $f(x) = \log(x+1)$; the condition (C14) for $0 < \delta < \frac{\beta}{\beta-\gamma}$ with $f(x) = \sqrt{x+1}$; the condition (C9) with $f(x) = \log\log(x+1)$, and the condition (C3) with $f(x) = x+1$.

Next we prove the nonexistence of hitting times by Proposition 10(ii) assuming Ω is finite. For all cases, let $f(x) = g(x)$, and specifically, let $f(x) = x+1$ in the cases (C13) for $\delta > 1$ and (C15) for $\delta > \frac{\beta}{\beta-\gamma}$, and $f(x) = \log(x+1)$ in the case (C16) for $\delta > 1$. Note that in the case (C15) for $\delta > \frac{\beta}{\beta-\gamma}$, we have that $-\frac{\beta}{\vartheta} < 1$ is equivalent to $\gamma > 0$. The tedious but straightforward verification of the conditions (ii-1)–(ii-4) in Proposition 10 is left to the interested reader.

5.6. Proof of Theorem 6

As alluded to in the proof of Theorem 4, the desired upper estimate of $Qf(x)$ with the same f under (C3) still holds as in the proof of Theorem 5, by replacing the asymptotic expansions by one-sided inequalities.

5.7. Proof of Theorem 7

Assume without loss of generality that $\mathcal{Y} = \mathbb{N}_0$ and $\partial \subseteq \{0\}$. We prove this theorem by Propositions 12–13, assuming Ω is finite. We emphasize that the nonexistence of a QSD rests only on the failure of certain absorption.

(i) Since Y_t is recurrent, we have that $\widetilde{Y}_n$ is recurrent by Proposition 5. Let $\widetilde{\pi}$ be its unique (up to a scalar) invariant measure. By [36, Theorem 3.5.1], and since $\sum_{\omega\in\Omega}\lambda_\omega(x)>0$ is bounded away from zero uniformly in x,

$$\pi(x)=\frac{\widetilde{\pi}(x)}{q_x}=\frac{\widetilde{\pi}(x)}{\sum_{\omega\in\Omega}\lambda_\omega(x)},\qquad x\in\mathbb{N}_0, \tag{10}$$

is a finite stationary measure for Y_t. Let

$$b=\lim_{x\to\infty}\frac{\sum_{\omega\in\Omega}\lambda_\omega(x)}{x^R}=\sum_{d_\omega=R}a_\omega.$$

Note that when $R=0$, we have $\gamma=0$, and hence $R=0$, $\gamma\neq 0$, and $\beta\le 0$ cannot occur (see Table 1). Thanks to Theorem 3, we only need to show the following:

- Y_t is positive recurrent if one of (C3), (C9), (C10), (C11) holds.
- Y_t is null recurrent if one of the following conditions holds:
  - $R=0$, $\alpha=0$;
  - $R=1$, $\alpha=\gamma=0$;
  - $R=2$, $\alpha=0$, $\beta=0$;
  - $R=1$, $\alpha=0$, $\beta\le 0\le\gamma$.

First, we show positive recurrence and exponential ergodicity. Notice that (C9)∪(C10) =(C18)∪(C21). Moreover, (C3) can be decomposed into (C17) and (C17)′ $R=0$, $\alpha<0$.

For (C11) or (C17)′, let $f(x)=x+1$; for (C21), let $f(x)=\log(x+1)$. Then one can verify (using the asymptotics of Qf in Appendix A.7) that there exists $\epsilon>0$ such that $Qf(x)\le-\epsilon$ for all large x. With an appropriate finite set F, by Proposition 11, X_t is positive recurrent and there exists a unique ergodic stationary distribution on $\mathbb{N}_0$.

For (C17), let $f(x)=x+1$; for (C18), let $f(x)=\log\log(x+2)$. Then one can similarly verify that there exists $\epsilon>0$ such that $Qf(x)\le-\epsilon f(x)$ for all large x. By Proposition 12, Y_t is positive recurrent and there exists a unique *exponentially* ergodic stationary distribution on $\mathbb{N}_0$.

Next, we show null recurrence case by case, applying Proposition 16 as well as nonexistence of moments of passage times in Theorem 5.

- Assume $R=0$, $\alpha=0$. Then

$$\lambda_\omega(x)\equiv a_\omega,\qquad \omega\in\Omega,\qquad \sum_{\omega\in\Omega}a_\omega\omega=0,$$

since λ_ω are polynomials. Let $c=-\min\Omega$, $g(x)=x+c$, $h(x)=(x+c)^{1/2}$, and $f(x)=1$. Hence it is easy to verify that Proposition 16(i) and Proposition 16(iii) hold. Let $\mathcal{F}_n$ be the filtration

https://doi.org/10.1017/apr.2022.20 Published online by Cambridge University Press

to which $\widetilde{Y}_n$ is adapted. Then

$$\begin{aligned}\mathbb{E}\big(h\big(\widetilde{Y}_{n+1}-h\big(\widetilde{Y}_n\big)|\mathcal{F}_n\big)\big) &= \sum_{\omega\in\Omega} a_\omega\left(\sqrt{\widetilde{Y}_n+\omega+c}-\sqrt{\widetilde{Y}_n+c}\right)\\ &= \sum_{\omega\in\Omega}\frac{a_\omega\omega}{\sqrt{\widetilde{Y}_n+\omega+c}+\sqrt{\widetilde{Y}_n+c}}\\ &= \sum_{\omega\in\Omega}\left(\frac{a_\omega\omega}{\sqrt{\widetilde{Y}_n+\omega+c}+\sqrt{\widetilde{Y}_n+c}}-\frac{a_\omega\omega}{\sqrt{\widetilde{Y}_n+c}+\sqrt{\widetilde{Y}_n+c}}\right)\\ &= \sum_{\omega\in\Omega}\frac{\sqrt{\widetilde{Y}_n+c}-\sqrt{\widetilde{Y}_n+\omega+c}}{2\sqrt{\widetilde{Y}_n+c}\left(\sqrt{\widetilde{Y}_n+\omega+c}+\sqrt{\widetilde{Y}_n+c}\right)}a_\omega\omega\\ &= \frac{1}{2\sqrt{\widetilde{Y}_n+c}}\sum_{\omega\in\Omega}\frac{a_\omega\omega^2}{\left(\sqrt{\widetilde{Y}_n+\omega+c}+\sqrt{\widetilde{Y}_n+c}\right)^2}\geq 0,\end{aligned}$$

which shows that Proposition 16(ii) also holds for the finite set $A=[0,c]_1$. Moreover,

$$\mathbb{E}\big(\widetilde{Y}_{n+1}-\widetilde{Y}_n|\mathcal{F}_n\big)=\sum_{\omega\in\Omega}a_\omega\omega\equiv 0<1=f(x)\quad P_z\text{-a.s.}\quad \text{on }\{\tau_A>n\},$$

which shows Proposition 16(iv) also holds. Since

$$\pi(j)=\frac{\widetilde{\pi}(j)}{\sum_{\omega\in\Omega}a_\omega},$$

by Proposition 16, we have $\sum_{j\in\mathbb{N}_0}\pi(j)=\infty$. By the uniqueness of stationary measures under the recurrence condition [35], we know Y_t is null recurrent.

- Assume $R=1$, $\alpha=\gamma=0$. Then

$$\mathbb{E}\big(\widetilde{Y}_{n+1}-\widetilde{Y}_n|\widetilde{Y}_n=x\big)=\frac{\sum_{\omega\in\Omega}\lambda_\omega(x)\omega}{\sum_{\omega\in\Omega}\lambda_\omega(x)}\equiv 0$$

for all large x. Applying [6, Theorem 3.5(iii)] with $\varepsilon_0=1/2$ and $a>1$, we have

$$\sum_{j\geq 2}\frac{\widetilde{\pi}(j)}{j\sqrt{\log j}}=\infty. \tag{11}$$

From (10), (11), and the fact that $\deg\big(\sum_{\omega\in\Omega}\lambda_\omega(x)\big)=1$ for large x, it follows that

$$\sum_{j\geq 2}\frac{\pi(j)}{\sqrt{\log j}}=\infty,$$

which implies that $\sum_{j\geq 2}\pi(j)=\infty$. The rest of the argument is the same as in the case $R=0$, $\alpha=0$.

https://doi.org/10.1017/apr.2022.20 Published online by Cambridge University Press

• Assume $R=2$, $\alpha=\beta=0$. Let $g_\delta(x)=(\log x)^\delta$ for some $1<\delta\le 2$. Choose $g=g_2$, $h=g_{3/2}$, and $A=[0, a]$ for some large $a>0$ to be determined. Then it is straightforward to verify (using the asymptotics of Qf in Appendix A.7) that

$$\mathbb{E}\big(g_\delta\big(\widetilde{Y}_{n+1}\big)-g_\delta\big(\widetilde{Y}_n\big)|\mathcal{F}_n\big)=\delta(\delta-1)\frac{\vartheta}{b}x^{-2}(\log x)^{\delta-2}+\mathrm{O}\big(x^{-3}(\log x)^{\delta-1}\big).$$

Hence it is easy to show that h and g satisfy Proposition 16(i)–(iii).

Moreover, it is easy to show that there exist $C=C(\vartheta, b)>0$ and $f(x)=Cx^{-2}$ such that

$$\mathbb{E}\big(g(\widetilde{Y}_{n+1}-g\big(\widetilde{Y}_n\big)|\mathcal{F}_n)\big)\le f\big(\widetilde{Y}_n\big)\quad \mathbb{P}_x\text{-a.s.}\quad \text{on}\quad \{\tau_A>n\}.$$

By Proposition 16,

$$\sum_{j\in\mathcal{Y}} f(j)\widetilde{\pi}(j)=\infty.$$

Since $\sum_{\omega\in\Omega}\lambda_\omega(x)=bx^2+\mathrm{O}(x)$, substituting (10) we obtain

$$\sum_{j\in\mathcal{Y}}\pi(j)=\infty.$$

Hence Y_t is null recurrent.

• Assume $R=1$, $\alpha=0$, $\gamma>0$. To prove null recurrence, it suffices to show that there exists $x\in\mathbb{N}_0$ such that $\mathbb{E}_x\big(\tau_x^+\big)=\infty$. Let $B\subsetneq\mathbb{N}_0$ be as in Theorem 5(ii) and $x=\max B\in\mathbb{N}_0$. Hence $Y_{J_1}\in\Omega+x\subsetneq\mathbb{N}_0$ a.s. and $(\Omega_++x)\cap B=\varnothing$. By the Markov property of Y_t,

$$\mathbb{E}_x\big(\tau_x-J_1|Y_{J_1}=j\big)=\mathbb{E}_j(\tau_x),\qquad \forall j\in\mathbb{N}_0\setminus\{x\}.$$

Hence by the law of total probability,

$$\begin{aligned}
\mathbb{E}_x\big(\tau_x^+\big)&=\mathbb{E}_x(J_1)+\sum_{j\in\mathbb{N}_0\setminus B}\mathbb{E}_j(\tau_x)\mathbb{P}_x\big(Y_{J_1}=j\big)+\sum_{j\in B}\mathbb{E}_j(\tau_x)\mathbb{P}_x\big(Y_{J_1}=j\big)\\
&\ge\sum_{j\in\mathbb{N}_0\setminus B}\mathbb{E}_j(\tau_x)\mathbb{P}_x\big(Y_{J_1}=j\big)\\
&\ge\mathbb{P}_x(Y_{J_1}\in\Omega_++x)\inf_{j\in\mathbb{N}_0\setminus B}\mathbb{E}_j(\tau_x)\\
&=\frac{\sum_{\omega\in\Omega_+}\lambda_\omega(x)}{\sum_{\omega\in\Omega}\lambda_\omega(x)}\cdot\infty=\infty,
\end{aligned}$$

since

$$\frac{\sum_{\omega\in\Omega_+}\lambda_\omega(x)}{\sum_{\omega\in\Omega}\lambda_\omega(x)}>0,$$

and $\mathbb{E}_j(\tau_B)=\infty$ for all $j\in\mathbb{N}_0\setminus B$, by Theorem 5(ii), under the respective conditions.

(ii) By assumption, $\partial=\{0\}$ and $\mathcal{Y}\setminus\partial=\mathbb{N}$. We first show nonexistence of QSDs. Construct an irreducible process Z_t on $\mathbb{N}_0$ with transition rate matrix $\widetilde{Q}$ as in the proof of Theorem 3. Applying the conclusion (i) to Z_t, Z_t is *not* positive recurrent when none of the conditions of (C3), (C9), (C10), (C11) holds. It thus suffices to show that the existence of a QSD for Y_t implies positive recurrence of Z_t. Assume that Y_t has a QSD on $\mathbb{N}$. By Proposition 14, there exists $\psi>0$ such that

$$\psi\mathbb{E}_i(\tau_0)=\mathbb{E}_i(\psi\tau_0)\le\mathbb{E}_i(\exp(\psi\tau_0))<\infty,\qquad \forall i\in\mathbb{N}_0.$$

https://doi.org/10.1017/apr.2022.20 Published online by Cambridge University Press

Let $\mathbb{E}_i^Z(\tau_0)$ be the expected hitting times for process Z_t. By Proposition 15, $\left(\mathbb{E}_i^Z(\tau_0)\right)_{i\in\mathbb{N}_0}$ is the minimal solution to the linear equations associated with $\widetilde{Q}$. By a similar argument as in the proof of Theorem 3, $\left(\mathbb{E}_i^Z(\tau_0)\right)_{i\in\mathbb{N}_0}$ is also the minimal solution to the linear equations associated with the transition matrix Q, and thus

$$\mathbb{E}_i^Z(\tau_0) = \mathbb{E}_i(\tau_0) < \infty, \qquad \forall i \in \mathbb{N}_0.$$

Since Z_t is irreducible, we have that Z_t is positive recurrent, owing to the classical fact that an irreducible CTMC that positively recurs to a finite set positively recurs everywhere (cf. [32]).

Next, we prove the ergodicity of the Q-process. For either (C18) or (C19), Y_t is non-explosive by Theorem 1; let $f(x) = (2 - x^{-1})\mathbb{1}_{\mathcal{Y}\setminus\partial}(x)$. Under the respective conditions, it is straightforward to verify that

$$\lim_{x\to\infty} \frac{Qf(x)}{f(x)} = -\infty,$$

which implies that the set

$$D = \{x \in \mathcal{Y} \setminus \partial : \frac{Qf(x)}{f(x)} \geq -\psi_0 - 1\}$$

is finite. Then, with such f, D, and δ, the conditions in Proposition 13 are satisfied and the conclusions follow. Note that $\mathrm{supp}\,\nu = \mathbb{N}$ comes from the fact that the support of the ergodic stationary distribution of the Q-process is $\mathbb{N}$ by the irreducibility.

5.8. Proof of Theorem 8

As discussed in the proof of Theorem 4, the desired upper estimates of $Qf(x)$ with the same f under the respective conditions still hold, by replacing the asymptotic expansions by one-sided inequalities in the proof of Theorem 7.

5.9. Proof of Theorem 9

First we prove implosivity. Assume (C18) or (C19) holds. Hence Y_t is recurrent by Theorem 3. Let $f(x) = 1 - (x+1)^{-1}$. One can show that the conditions in Proposition 18(i-1) are fulfilled, and implosivity is achieved.

Next we turn to non-implosivity. Assume neither (C18) nor (C19) holds. Since Y_t does not implode towards any transient state, it suffices to prove non-implosivity assuming a recurrence condition, i.e., (C3) or (C6), by Theorem 3. Let $f(x) = \log\log(x+2)$. It is easy to verify that the conditions (with $\delta = 2$) in Proposition 18(ii) are fulfilled, and Y_t is non-implosive.

5.10. Proof of Theorem 10

As discussed in the proof of Theorem 4, the same functon f under the condition (C19) also serves as a Lyapunov function as in the proof of Theorem 9.

Appendix A. Classical criteria for dynamics

Let Y_t be a CTMC on a state space $\mathcal{Y} \subseteq \mathbb{N}_0$ with transition matrix $Q = (q_{x,y})_{x,y\in\mathcal{Y}}$, and let $\left(\widetilde{Y}_n\right)_{n\in\mathbb{N}_0}$ be its embedded discrete-time Markov chain. Let $q_x = \sum_{y\neq x} q_{x,y}$, $\forall x \in \mathcal{Y}$. The transition probability matrix $P = (p_{x,y})_{x,y\in\mathcal{Y}}$ of $\widetilde{Y}_n$ is given by

$$p_{x,y} = \begin{cases} q_{x,y}/q_x & \text{if} \quad x \neq y,\ q_x \neq 0, \\ 0 & \text{if} \quad x \neq y,\ q_x = 0, \end{cases} \qquad p_{x,x} = \begin{cases} 0 & \text{if} \quad q_x \neq 0, \\ 1 & \text{if} \quad q_x = 0. \end{cases}$$

https://doi.org/10.1017/apr.2022.20 Published online by Cambridge University Press

Let $\mathfrak{F}$ be the set of all nonnegative (finite) functions on $\mathcal{Y}$ satisfying

$$\sum_{\omega\in\Omega}\lambda_\omega(x)|f(x+\omega)|<+\infty \qquad \forall x\in\mathcal{Y}.$$

Since $\mathcal{Y}$ is discrete, $\mathfrak{F}$ is indeed a subset of nonnegative continuous (and thus Borel measurable) functions on $\mathcal{Y}$. The associated infinitesimal generator is also denoted by Q:

$$Qf(x)=\sum_{\omega\in\Omega}\lambda_\omega(x)\left(f(x+\omega)-f(x)\right), \quad \forall x\in\mathcal{Y}, \quad f\in\mathfrak{F}.$$

By (**A3**), $\mathfrak{F}$ is a subset of the domain of Q. In particular, functions with sublinear growth rate are in $\mathfrak{F}$. When Ω is finite, $\mathfrak{F}$ is the whole set of all nonnegative (finite) functions on $\mathcal{Y}$.

Before presenting the proofs, we recall general Lyapunov–Foster-type criteria for the reader's convenience [12, 32, 33]. The proofs are mainly based on constructions of specific Lyapunov functions. To avoid tedious but straightforward verifications against the corresponding criteria, we simply provide the specific Lyapunov functions we apply and leave the straightforward verifications to the interested reader.

The next proposition is used to estimate Qf for a Lyapunov function f. Let $R_+=\max\{\deg(\lambda_\omega):\omega\in\Omega_+\}$ and recall $R=\max\{\deg(\lambda_\omega):\omega\in\Omega\}$. It holds that $R,\ R_+\le M$.

Proposition 1. *Assume* (**A1**)–(**A4**). *Let* $f_n(x)=\sum_{\omega\in\Omega_+,\ \omega\le n}\lambda_\omega(x)\omega$ *for* $n\in\mathbb{N}$. *Then* f_n *converges nondecreasingly to a polynomial* f *of degree* R_+ *on* $\mathcal{Y}\setminus[0,u[_1$,

$$f(x)=\sum_{\omega\in\Omega_+}\lambda_\omega(x)\omega, \quad x\in\mathcal{Y}\setminus[0,u[_1, \tag{12}$$

with u *as in* (**A4**). *Furthermore,* $\sum_{\omega\in\Omega}\lambda_\omega(x)\omega$ *is a polynomial of degree at most* R *on* $\mathcal{Y}\setminus[0,u[_1$, *and* $\sum_{\omega\in\Omega}\lambda_\omega(x)$ *is a polynomial of degree* R *on* $\mathcal{Y}\setminus[0,u[_1$. *Moreover, there exists* $u'\ge u$ *such that*

$$\lim_{n\to\infty}\sup_{x\ge u'}\frac{f(x)-f_n(x)}{f(x)}=0. \tag{13}$$

Proof. Assume without loss of generality that $u=0$. Otherwise consider $\lambda_\omega(\cdot+u)$. Furthermore, assume $\mathcal{Y}=\mathbb{N}_0$. Let $n_*=\min\{\omega\in\Omega_+:\deg(\lambda_\omega)=R_+\}$. Then $(f_n)_{n\ge n_*}$ is a nondecreasing sequence of polynomials on $\mathbb{N}_0$ of degree R_+, as the coefficient of x^{R_+} is nonnegative in $\lambda_\omega(x)$. By (**A3**)–(**A4**), f as defined in (12) is a nonnegative finite function on $\mathbb{N}_0$, and f_n converges to f pointwise on $\mathbb{N}_0$.

Write $f_n(x)=\sum_{j=0}^{R_+}\alpha_n^{(j)}x^{\underline{j}}$ as a sum of descending factorials. Since $f_n(j)\to f(j)$ for $j=0,\ldots,R_+$ by assumption, we find inductively in j that $\alpha_n^{(j)}\to\alpha^{(j)}$ for some $\alpha^{(j)}\in\mathbb{R}$, $j=0,\ldots,R_+$. Let $\widetilde{f}(x)=\sum_{j=0}^{R_+}\alpha^{(j)}x^{\underline{j}}$. Then $f_n\to\widetilde{f}$ pointwise on $\mathbb{N}_0$, which implies $f=\widetilde{f}$ and that f is a polynomial on $\mathbb{N}_0$. By the definition of n_* and the monotonicity of $(f_n)_{n\ge n_*}$, we have $\alpha_n^{(R_+)}\ge\alpha_{n_*}^{(R_+)}>0$ for $n\ge n_*$, and $\alpha^{(R_+)}=\lim_{n\to\infty}\alpha_n^{(R_+)}>0$. Hence $\deg(f)=R_+$. Similarly, by (**A2**), one can show that $\sum_{\omega\in\Omega}\lambda_\omega(x)\omega$ is a polynomial of degree at most R on $\mathbb{N}_0$, and $\sum_{\omega\in\Omega}\lambda_\omega(x)$ is a polynomial of degree R on $\mathbb{N}_0$. It remains to prove (13). Indeed, for all $x\in\mathbb{N}$,

$$0\le\frac{f(x)-f_n(x)}{f(x)}=\frac{\sum_{j=0}^{R_+}(\alpha^{(j)}-\alpha_n^{(j)})x^{\underline{j}}}{\sum_{j=0}^{R_+}\alpha^{(j)}x^{\underline{j}}}\le\frac{x^{R_+}\sum_{j=0}^{R_+}|\alpha^{(j)}-\alpha_n^{(j)}|}{\sum_{j=0}^{R_+}\alpha^{(j)}x^{\underline{j}}}.$$

https://doi.org/10.1017/apr.2022.20 Published online by Cambridge University Press

Since there exists $u' \geq u$ such that $f(x) \geq \frac{1}{2}\alpha^{(R_+)}x^{R_+}$ for all $x \geq u'$, we have

$$\sup_{x \geq u'} \frac{f(x) - f_n(x)}{f(x)} \leq \frac{2\sum_{j=0}^{R_+} |\alpha^{(j)} - \alpha_n^{(j)}|}{\alpha^{(R_+)}},$$

which implies (13). □

A.1. Criteria for explosivity and non-explosivity

Proposition 2. ([32, Theorem 1.12, Remark 1.13].) *Assume Y_t is irreducible on $\mathcal{Y}$. Suppose that there exists a triple (ϵ, A, f) consisting of a constant $\epsilon > 0$, a set A that is a proper finite subset of $\mathcal{Y}$ such that $\mathcal{Y} \setminus A$ is infinite, and a function $f \in \mathfrak{F}$, such that*

(i) *there exists $x_0 \in \mathcal{Y} \setminus A$ with $f(x_0) < \min_A f$, and*

(ii) *$Qf(x) \leq -\epsilon$ for all $x \in \mathcal{Y} \setminus A$.*

Then the expected lifetime $\mathbb{E}_x(\zeta) < +\infty$ for all $x \in \mathcal{Y}$.

Proposition 3. ([32, Theorem 1.14].) *Assume Y_t is irreducible on $\mathcal{Y}$. Let $f \in \mathfrak{F}$ be such that $\lim_{x\to\infty} f(x) = +\infty$. If*

(i) *there exists a nondecreasing function $g : [0, \infty[\to [0, \infty[$ such that $G(z) = \int_0^z \frac{\mathrm{d}y}{g(y)} < +\infty$ for all $z \geq 0$ but $\lim_{z\to\infty} G(z) = +\infty$, and*

(ii) *$Qf(x) \leq g(f(x))$ for all $x \in \mathcal{Y}$,*

then $\mathbb{P}_x(\zeta = +\infty) = 1$ for all $x \in \mathcal{Y}$.

We give Reuter's criterion on explosivity of a CTMC in terms of the transition rate matrix.

Proposition 4. ([38, Theorem 10], [9, Theorem 13.3.11].) *Assume Y_t is irreducible on $\mathcal{Y}$ with transition matrix Q. Then Y_t is explosive with positive probability if and only if there exists a nonzero nonnegative solution to*

$$Qx = \lambda x,$$

for some (and all) $\lambda > 0$.

A.2. Criteria for recurrence, transience, and certain absorption

To prove Theorem 3(i), we rely on the following equivalence in relation to recurrence and transience between a CTMC and its embedded discrete-time Markov chain.

Proposition 5. ([36, Theorem 3.4.1].) *Assume that Y_t is irreducible. Let $\widetilde{Y}_n$ be the embedded discrete-time Markov chain of Y_t. Then the following hold:*

(i) Y_t is recurrent if and only if $\widetilde{Y}_n$ is recurrent.

(ii) Y_t is transient if and only if $\widetilde{Y}_n$ is transient.

Apart from the above equivalence, we need the following two properties to prove recurrence and transience for an irreducible discrete-time Markov chain.

Proposition 6. ([29, Theorem 2.1].) *Let Z_n be an irreducible discrete-time Markov chain on a subset of $\mathbb{N}_0$. If*

$$\mathbb{E}\big(Z_{n+1} - Z_n | Z_n = x\big) \leq 0, \quad \forall n \in \mathbb{N}_0, \text{ for all large } x,$$

then Z_n is recurrent.

https://doi.org/10.1017/apr.2022.20 Published online by Cambridge University Press

Proposition 7. ([29, Theorem 2.2].) *Let Z_n be a discrete-time Markov chain on the real line. Assume that there exists a positive constant M such that*

$$0 \leq Z_n < M < \infty \qquad \forall n \in \mathbb{N}_0,$$
$$\mathbb{P}\Big(\limsup_{n\to\infty} Z_n = M\Big) = 1. \tag{14}$$

If there exists a constant $C < M$ such that

$$\mathbb{E}\big(Z_{n+1} - Z_n | Z_n = x\big) \leq 0 \qquad \forall n \in \mathbb{N}_0, \text{ for all } x \geq C,$$

then

$$\mathbb{P}\Big(\lim_{n\to\infty} Z_n = M\Big) = 1.$$

Recall the definition of λ_ω; the transition probabilities $P = (p_{x,y})_{x,y\in\mathcal{Y}}$ of $\widetilde{Y}_n$ are

$$p_{x,x+\omega} = \frac{\lambda_\omega(x)}{\sum_{\tilde\omega\in\Omega} \lambda_{\tilde\omega}(x)} \mathbb{1}_{\mathcal{Y}\cap\left(\underset{\tilde\omega\in\Omega}{\cup} \operatorname{supp} \lambda_{\tilde\omega}\right)}(x),$$
$$p_{x,x} = 1 - \mathbb{1}_{\mathcal{Y}\cap\left(\underset{\tilde\omega\in\Omega}{\cup} \operatorname{supp} \lambda_{\tilde\omega}\right)}(x), \qquad x \in \mathbb{N}_0, \ \omega \in \Omega.$$

Proposition 8. ([36, Theorem 3.3.1].) *Let $A \subseteq \mathcal{Y}$. The vector of hitting probabilities $(h_A(i))_{i\in\mathcal{Y}}$ is the minimal nonnegative solution to the following linear equations:*

$$\begin{cases} h_A(i) = 1, & i \in A, \\ \sum_{j\in\mathcal{Y}\setminus\{i\}} q_{ij}(h_A(i) - h_A(j)) = 0, & i \in \mathcal{Y} \setminus A. \end{cases}$$

(Minimality means that if x is another nonnegative solution, then $x_i \geq h_A(i)$ for all $i \in \mathcal{Y}_0$.)

Proposition 9. ([36, Theorem 1.5.7, Theorem 3.4.1].) *Assume Y_t is irreducible on $\mathcal{Y}$. Then the following hold:*

(i) Y_t is recurrent if and only if $h_j(i) = 1$ for all $i \in \mathcal{Y}$ and some (and all) $j \in \mathcal{Y}$.

(ii) Y_t is recurrent if and only if $h_A(i) = 1$ for all $i \in \mathcal{Y}$ and some (and every) nonempty subset $A \subseteq \mathcal{Y}$.

Proof. Recall that by irreducibility, Y_t is recurrent if and only if one (and every) state $i \in \mathcal{Y}$ is recurrent, which is equivalent to $h_i(i) = 1$. The conclusion (i) is a direct result of [36, Theorem 1.5.7, Theorem 3.4.1].

To show (ii), by irreducibility, $\mathbb{P}_i(\{Y_{\tau_A} = j\}) > 0$ for all $j \in A$ and $\tau_j = \tau_A$ conditional on $Y_{\tau_A} = j$. Hence, by the law of total probability,

$$\mathbb{P}_i(\tau_A < \infty) = \sum_{j\in A} \mathbb{P}_i(\{Y_{\tau_A} = j\})\mathbb{P}(\tau_j < \infty),$$

which implies that $h_A(i) = 1$ if and only if $h_j(i)$ for all $j \in A$. On the one hand, given any nonempty $A \subseteq \mathcal{Y}$, by (i), since Y_t is recurrent, we have $h_j(i) = 1$ for all $i \in \mathcal{Y}$ for all $j \in A$, and thus $h_A(i) = 1$. On the other hand, if $h_A(i) = 1$ for all $i \in \mathcal{Y}$ and some (and all) subsets $A \subseteq \mathcal{Y}$, then $h_j(i) = 1$ for all $j \in A$, and by (i) we know Y_t is recurrent. □

A.3. Criteria for existence and nonexistence of moments of hitting times

Proposition 10. ([32, Theorem 1.5].) *Assume Y_t is irreducible on $\mathcal{Y}$. Let $f \in \mathfrak{F}$ be such that* $\lim_{x\to\infty} f(x) = +\infty$.

(i) *If there exist positive constants c_1, c_2, and σ such that $f^\sigma \in \mathfrak{F}$ and*

$$Qf^\sigma(x) \le -c_2 f^{\sigma-2}(x) \qquad \forall x \in \{f > c_1\},$$

then $\mathbb{E}_x\left(\tau^\epsilon_{\{f\le c_1\}}\right) < +\infty$ for all $0 < \epsilon < \sigma/2$ and all $x \in \mathcal{Y}$.

(ii) *Let $g \in \mathfrak{F}$. If there exist*

(ii-1) *a constant $c_1 > 0$ such that $f \le c_1 g$,*

(ii-2) *constants $c_2,\ c_3 > 0$ such that $Qg(x) \ge -c_3$ for all $x \in \{g > c_2\}$,*

(ii-3) *constants $c_4 > 0$ and $\delta > 1$ such that $g^\delta \in \mathfrak{F}$ and $Qg^\delta(x) \le c_4 g^{\delta-1}(x)$ for all $x \in \{g > c_2\}$, and*

(ii-4) *a constant $\sigma > 0$ such that $f^\sigma \in \mathfrak{F}$ and $Qf^\sigma(x) \ge 0$ for all $x \in \{f > c_1c_2\}$, then $\mathbb{E}_x\left(\tau^\epsilon_{\{f\le c_2\}}\right) = +\infty$ for all $\epsilon > \sigma$ and all $x \in \{f > c_2\}$.*

A.4. Criteria for positive recurrence, ergodicity, and existence of QSDs

For the reader's convenience, we first recall the classical Lyapunov–Foster criteria.

Proposition 11. ([32, Theorem 1.7].) *Assume Y_t is irreducible on $\mathcal{Y}$ and recurrent. Then the following are equivalent:*

(i) *Y_t is positive recurrent.*

(ii) *There exists a triple (ϵ, A, f) with $\epsilon > 0$, A a finite nonempty subset of $\mathcal{X}$, and $f \in \mathfrak{F}$ satisfying $Qf(x) \le -\epsilon$ for all $x \in \mathcal{Y} \setminus A$.*

Proposition 12. ([33, Theorem 7.1].) *Assume Y_t is irreducible on $\mathcal{Y}$. Then Y_t is positive recurrent and there exists an exponentially ergodic stationary distribution, if there exists a triple (ϵ, A, f) with $\epsilon > 0$, A a finite subset of $\mathcal{Y}$, and $f \in \mathfrak{F}$ with $\lim_{x\to\infty} f(x) = \infty$, satisfying $Qf(x) \le -\epsilon f(x)$ for all $x \notin A$.*

Proposition 13. ([11, Theorem 1.1], [12, Theorem 5.1, Remark 11], [27, Theorem 2.1].) *Assume $\partial \ne \varnothing$ and the Q-process of Y_t is irreducible. Then there exists a finite subset $D \subseteq \mathcal{Y} \setminus \partial$, with $\mathbb{P}_x(Y_1 = y) > 0$ for all $x, y \in D$, such that the constant*

$$\psi_0 := \inf\left\{\psi \in \mathbb{R} :\ \liminf_{t\to\infty} e^{\psi t}\mathbb{P}_x(Y_t = x) > 0\right\}$$

is finite and independent of $x \in D$. If in addition there exist $\psi_1 > \max\{\psi_0, \sup_{x\in\mathcal{Y}\setminus\partial} \sum_{z\in\partial} q_{x,z}\}$ and a function $f \in \mathfrak{F}$ such that $f\big|_{\mathcal{Y}\setminus\partial} \ge 1$, $f\big|_\partial = 0$, $\sup_{\mathcal{Y}\setminus\partial} f < \infty$, and

$$\sum_{y\in(\mathcal{Y}\setminus\partial)\setminus\{x\}} q_{x,y} f(y) < \infty \qquad \forall x \in \mathcal{Y} \setminus \partial; \qquad Qf(x) \le -\psi_1 f(x) \qquad \forall x \in (\mathcal{Y} \setminus \partial) \setminus D,$$

https://doi.org/10.1017/apr.2022.20 Published online by Cambridge University Press

then there exists a unique QSD ν *on* $\mathcal{Y}\setminus\partial$ *with positive constants* C *and* $\delta<1$ *such that for all Borel probability measures* μ *on* $\mathcal{Y}\setminus\partial$,

$$\left\|\mathbb{P}_\mu(Y_t\in\cdot|t<\tau_\partial)-\nu\right\|_{\mathsf{TV}}\le C\delta^t,\quad \forall t\ge 0.$$

In addition, $\mathrm{d}\xi(x)=\zeta(x)\mathrm{d}\nu(x)$ *is the unique quasi-ergodic distribution for* Y_t*, as well as the unique stationary distribution of the Q-process, where* ζ *is the nonnegative function*

$$\zeta(x)=\lim_{t\to\infty}e^{\psi_0 t}\mathbb{P}_x(t<\tau_\partial)\,,\qquad x\in\mathcal{Y}\setminus\partial.$$

To show the nonexistence of QSDs, we rely on the following two classical results.

Proposition 14. ([18, Lemma 4.1].) *Assume* $\partial\neq\varnothing$ *and the Q-process of* Y_t *is irreducible. If there exists a QSD for* Y_t *supported on* ∂*, then the uniform exponential moment property holds:*

$$\text{there exists } \psi>0 \text{ such that } \mathbb{E}_x(\exp(\psi\tau_\partial))<\infty\quad\forall x\in\mathcal{Y}.$$

Proposition 15. ([36, Theorem 3.3.3].) *Let* $A\subseteq\mathcal{Y}$ *and* $k_A(i)=\mathbb{E}_i(\tau_A)$ *for all* $i\in\mathcal{Y}$*. Assume* $q_x\neq 0$ *for all* $x\in\mathcal{Y}\setminus A$*. Then the vector of expected hitting times* $(k_A(i))_{i\in\mathcal{Y}}$ *is the minimal nonnegative solution to the following linear equations:*

$$\begin{cases}k_A(i)=1 & \text{if } i\in A,\\ \sum_{j\in\mathcal{Y}\setminus\{i\}}q_{ij}(k_A(i)-k_A(j))=1 & \text{if } i\in\mathcal{Y}\setminus A.\end{cases}$$

A.5. Criterion for non-summability of functions with respect to stationary measures

Proposition 16. ([6, Theorem 1′, Remarks 3–4].) *Let* $\mathcal{Y}$ *be an unbounded countable subset of* $\mathbb{R}_{\ge 0}$ *and* $(\mathcal{Y},\mathcal{F},\mathbb{P})$ *a probability space with a filtration* $\{\mathcal{F}_n\}_{n\in\mathbb{N}_0}$*. Assume that* Z_n *is a discrete-time* $\mathcal{F}_n$*-adapted irreducible aperiodic Markov chain on* $\mathcal{Y}$*, which is recurrent with unique (up to a multiplicative constant) stationary measure* ν*. Let* f *be a nonnegative function defined on* $\mathcal{Y}$*. Then*

$$\sum_{x\in\mathcal{Y}}f(x)\nu(x)=\infty$$

if there exist some finite set A*, some* $z\in A$*, and some nonnegative functions* g *and* h *such that*

(i) $\lim_{x\to\infty}h(x)=\infty$ *and* $\lim_{x\to\infty}\frac{g(x)}{h(x)}=\infty$,

(ii) whenever $z'\in E\subset\mathcal{Y}\setminus A$*, the process* $\{h(Z_{n\wedge\tau_A})\}_{n\in\mathbb{N}_0}$ *is a* $\mathbb{P}_{z'}$*-submartingale,*

(iii) $\mathbb{E}_z(g(Z_n)\mathbb{1}_{\tau_A>n})$ *is finite for all* $n\in\mathbb{N}$*, and*

(iv) $\mathbb{E}(g(Z_{n+1})-g(Z_n)|\mathcal{F}_n)\le f(Z_n)$*,* $\mathbb{P}_z$*-a.s., on* $\tau_A>n$.

A.6. Criterion for implosivity and non-implosivity

Proposition 17. ([32, Proposition 2.14].) *Assume* Y_t *is irreducible on* $\mathcal{Y}$*. If there exists a nonempty proper subset* $B\subsetneq\mathcal{Y}$ *such that* Y_t *implodes towards* B*, then* Y_t *is implosive.*

Proposition 18. ([32, Theorem 1.15, Proposition 1.16].) *Assume Y_t is irreducible on $\mathcal{Y}$.*

(i) *The following are equivalent:*

(i-1) *There exists a triple (ϵ, F, f) consisting of a positive constant ϵ, a finite set F, and a function $f \in \mathfrak{F}$ such that $\sup_{x\in\mathcal{Y}} f(x) < +\infty$ and $Qf(x) \le -\epsilon$ whenever $x \in \mathcal{Y} \setminus F$.*

(i-2) *There exists $c > 0$, and for every finite $A \subseteq \mathcal{Y}$ there exists a positive constant C_A, such that $\mathbb{E}_x(\tau_A) \le C_A$ and $\mathbb{E}_x(\exp(c\tau_A)) < \infty$ whenever $x \in \mathcal{Y} \setminus A$. In particular, Y_t is implosive.*

(ii) *Let $f \in \mathfrak{F}$ be such that $\lim_{x\to\infty} f(x) = +\infty$, and assume there exist positive constants a, c, ϵ and $\delta > 1$ such that $f^\delta \in \mathfrak{F}$. If, in addition,*

$$Qf(x) \ge -\epsilon, \quad Qf^\delta(x) \le cf^{\delta-1}(x), \quad \text{whenever } x \in \{f > a\},$$

then the chain does not implode towards $\{f \le a\}$.

A.7. Asymptotic expansion of Qf for Lyapunov functions f used in the proofs

We provide an asymptotic expansion of $Qf(x)$ for all large x, for various Lyapunov functions f. Let $\delta \in \mathbb{R}$.

- Let $f(x) = x^\delta$. Then

$$Qf(x) = \delta x^\delta \left\{\alpha x^{R-1} + (\beta + \delta\vartheta)x^{R-2} + \mathrm{O}\Big(x^{R-3}\Big)\right\}.$$

- Let $f(x) = \big(x(\log x)^{-1}\big)^\delta$. Then

$$Qf(x) = \delta\big(x(\log x)^{-1}\big)^\delta \left\{\alpha\left(1 - (\log x)^{-1}\right)x^{R-1} + \left((\beta + \delta\vartheta) - (\beta + 2\delta\vartheta)(\log x)^{-1}\right)x^{R-2} + \mathrm{O}\Big(x^{R-2}(\log x)^{-2}\Big)\right\}.$$

- Let $f(x) = (x\log x)^\delta$. Then

$$Qf(x) = \delta(x\log x)^\delta \left\{\alpha\left(1 + (\log x)^{-1}\right)x^{R-1} + (\beta + \delta\vartheta)x^{R-2} + (\gamma + \delta\vartheta)x^{R-2}(\log x)^{-1} + \mathrm{O}\Big(x^{R-2}(\log x)^{-2}\Big)\right\}.$$

- Let $f(x) = (\log x)^\delta$. Then

$$Qf(x) = \delta(\log x)^{\delta-1}\left\{\alpha x^{R-1} + \beta x^{R-2} + (\delta - 1)\vartheta x^{R-2}(\log x)^{-1} + \mathrm{O}\Big(x^{R-3}\Big)\right\}.$$

- Let $f(x) = (\log\log x)^\delta$. Then

$$Qf(x) = \delta(\log\log x)^{\delta-1}(\log x)^{-1} \cdot \left\{\alpha x^{R-1} + \beta x^{R-2} - \vartheta x^{R-2}(\log x)^{-1} + \mathrm{O}\left(x^{R-2}(\log x)^{-1}(\log\log x)^{-1}\right)\right\}.$$

https://doi.org/10.1017/apr.2022.20 Published online by Cambridge University Press

Acknowledgements

This work was initiated, and most of it completed, when the first author was at the University of Copenhagen. The authors thank Philip Pollett for making us aware of the reference [39], Linard Hoessly for commenting on the manuscript, and the editors and referees for their comments, which helped improve the presentation of the paper. The authors acknowledge the support of the Erwin Schrödinger Institute (ESI) for the workshop Advances in Chemical Reaction Network Theory.

Funding information

C. X. acknowledges the TUM Foundation Fellowship and the Alexander von Humboldt Fellowship. C. W. acknowledges support from the Novo Nordisk Foundation (Denmark), grant NNF19OC0058354.

Competing interests

There were no competing interests to declare which arose during the preparation or publication process for this article.

References

[1] ANDERSON, D. F., CAPPELLETTI, D., KOYAMA, M. AND KURTZ, T. G. (2018). Non-explosivity of stochastically modeled reaction networks that are complex balanced. *Bull. Math. Biol.* **80**, 2561–2579.

[2] ANDERSON, D. F. AND KIM, J. (2018). Some network conditions for positive recurrence of stochastically modeled reaction networks. *SIAM J. Appl. Math.* **78**, 2692–2713.

[3] ANDERSON, D. F. AND KURTZ, T. G. (2011). Continuous time Markov chain models for chemical reaction networks. In *Design and Analysis of Biomolecular Circuits: Engineering Approaches to Systems and Synthetic Biology*, eds H. KOEPPL, G. SETTI, M. di Bernardo and D. Densmore, Springer, New York, pp. 3–42.

[4] ANDERSON, D. F. AND KURTZ, T. G. (2015). *Stochastic Analysis of Biochemical Systems*. Springer, Cham.

[5] ANDERSON, W. J. (1991). *Continuous-Time Markov Chains: an Applications-Oriented Approach*. Springer, New York.

[6] ASPANDIIAROV, S. AND IASNOGORODSKI, R. (1999). Asymptotic behaviour of stationary distributions for countable Markov chains, with some applications. *Bernoulli* **5**, 535–569.

[7] BARBOUR, A. D. (1976). Quasi-stationary distributions in Markov population processes. *Adv. Appl. Prob.* **8**, 296–314.

[8] BE'ER, S. AND ASSAF, M. (2016). Rare events in stochastic populations under bursty reproduction. *J. Statist. Mech. Theory Experiment*, article no. 113501.

[9] BRÉMAUD, P. (2020). *Markov Chains: Gibbs Fields, Monte Carlo Simulation and Queues*, 2nd edn. Springer, Cham.

[10] BREYER, L. A. AND ROBERTS, G. O. (1999). A quasi-ergodic theorem for evanescent processes. *Stoch. Process. Appl.* **84**, 177–186.

[11] CHAMPAGNAT, N. AND VILLEMONAIS, D. (2016). Exponential convergence to quasi-stationary distribution and *Q*-process. *Prob. Theory Relat. Fields* **164**, 243–283.

[12] CHAMPAGNAT, N. AND VILLEMONAIS, D. (2017). *General criteria for the study of quasi-stationarity*. Preprint. Available at https://arxiv.org/abs/1712.08092.

[13] CHEN, A., POLLET, P., ZHANG, H. AND CAIRNS, B. (2005). Uniqueness criteria for continuous-time Markov chains with general transition structures *Adv. Appl. Prob.* **37**, 1056–1074.

[14] CHEN, F.-M. (1999). Single birth processes. *Chinese Ann. Math. Ser. A* **20**, 77–82.

[15] CHEN, M.-F. (2004). *From Markov Chains to Nonequilibrium Particle Systems*, 2nd edn. World Scientific, Singapore.

[16] CHEN, R.-R. (1997). An extended class of time-continuous branching processes. *J. Appl. Prob.* **34**, 14–23.

[17] CHEN, X. AND JIA, C. (2020). Limit theorems for generalized density-dependent Markov chains and bursty stochastic gene regulatory networks. *J. Math. Biol.* **80**, 959–994.

[18] COLLET, P., MARTÍNEZ, S. AND SAN MARTÍN, J. (2013). *Quasi-stationary Distributions: Markov Chains, Diffusions and Dynamical Systems*. Springer, Heidelberg.

[19] ETHIER, S. N. AND KURTZ, T. G. (2009). *Markov Processes: Characterization and Convergence*. John Wiley, Hoboken, NJ.
[20] EWENS, W. J. (2004). *Mathematical Population Genetics 1: Theoretical Introduction*. Springer, New York.
[21] FALK, J., MENDLER, B. AND DROSSEL, B. (2017). A minimal model of burst-noise induced bistability. PLoS ONE **12**, article no. e0176410.
[22] FERNÁNDEZ, L. AND DE LA IGLESIA, M. D. (2021). Quasi-birth-and-death processes and multivariate orthogonal polynomials *J. Math. Anal. Appl.* **499**, article no. 125029.
[23] FERNÁNDEZ, L., KESTEN, H., MARTINEZ, S. AND PICCO, P. (1995). Existence of quasi-stationary distributions. A renewal dynamical approach. *Ann. Prob.* **23**, 501–521.
[24] GARDINER, C. W. (2009). *Stochastic Methods: a Handbook for Physics, Chemistry and the Natural Sciences*, 4th edn. Springer, Berlin.
[25] GIBBS, A. L. AND SU, F. E. (2002). On choosing and bounding probability metrics. *Internat. Statist. Rev.* **70**, 419–435.
[26] GROSS, D. AND HARRIS, C. M. (1998). *Fundamentals of Queuing Theory*, 3rd edn. Springer, New York.
[27] HE, G., ZHANG, H. AND ZHU, Y. (2019). On the quasi-ergodic distribution of absorbing Markov processes. *Statist. Prob. Lett.* **149**, 116–123.
[28] KARLIN, S. AND MCGREGOR, J. L. (1957). The classification of birth and death processes. *Trans. Amer. Math. Soc.* **86**, 366–400.
[29] LAMPERTI, J. (1960). Criteria for the recurrence or transience of stochastic process. I. *J. Math. Anal. Appl.* **1**, 314–330.
[30] LI, Y. AND LI, J. (2009). Criteria for Feller transition functions. *J. Math. Anal. Appl.* **359**, 653–665.
[31] MACKEY, M. C., TYRAN-KAMIŃSKA, M. AND YVINEC, R. (2013). Dynamic behavior of stochastic gene expression models in the presence of bursting. *SIAM J. Appl. Math.* **73**, 1830–1852.
[32] MENSHIKOV, M. AND PETRITIS, D. (2014). Explosion, implosion, and moments of passage times for continuous-time Markov chains: a semimartingale approach. *Stoch. Process. Appl.* **124**, 2388–2414.
[33] MEYN, S. P. AND TWEEDIE, R. L. (1993). Stability of Markovian processes III: Foster–Lyapunov criteria for continuous-time processes. *Adv. Appl. Prob.* **25**, 518–548.
[34] MEYN, S. P. AND TWEEDIE, R. L. (2009). *Markov Chains and Stochastic Stability*, 2nd edn. Cambridge University Press.
[35] MILLER, R. G. (1963). Stationary equations in continuous time Markov chains. *Trans. Amer. Math. Soc.* **109**, 35–44.
[36] NORRIS, J. R. (1998). *Markov Chains*. Cambridge University Press.
[37] PASTOR-SATORRAS, R., CASTELLANO, C., VAN MIEGHEM, P. AND VESPIGNANI, A. (2015). Epidemic processes in complex networks. *Rev. Modern Phys.* **87**, 925–979.
[38] REUTER, G. E. H. (1957). Denumerable Markov processes and the associated contraction semigroups on l. *Acta Math.* **97**, 1–46.
[39] REUTER, G. E. H. (1961). Competition processes. In *Proc. 4th Berkeley Symp. Math. Statist. Prob.*, Vol. II, University of California Press, Berkeley, pp. 421–430.
[40] SCHWABE A., RYBAKOVA, K. N. AND BRUGGEMAN, F. J. (2012). Transcription stochasticity of complex gene regulation models. *Biophys. J.* **103**, 1152–1161.
[41] SHAHREZAEI, V. AND SWAIN, P. S. (2008). Analytical distributions for stochastic gene expression. *Proc. Nat. Acad. Sci. USA* **105**, 17256–17261.
[42] VAN DOORN, E. A. (1985). Conditions for exponential ergodicity and bounds for the decay parameter of a birth–death process. *Adv. Appl. Prob.* **17**, 514–530.
[43] VAN DOORN, E. A. (1991). Quasi-stationary distributions and convergence to quasi-stationarity of birth–death processes. *Adv. Appl. Prob.* **23**, 683–700.
[44] WEIDLICH, W. AND HAAG, G. (1983). *Concepts and Models of a Quantitative Sociology: the Dynamics of Interacting Populations*. Springer, Berlin.
[45] WILKINSON, D. J. (2006). *Stochastic Modelling for Systems Biology*. Chapman & Hall/CRC, London.
[46] WIUF, C. AND XU, C. (2020). *Classification and threshold dynamics of stochastic reaction networks*. Preprint. Available at https://arxiv.org/abs/2012.07954.
[47] XU, C., HANSEN, M. C. AND WIUF, C. (2020). *The asymptotic tails of limit distributions of continuous time Markov chains*. Preprint. Available at https://arxiv.org/abs/2007.11390.
[48] XU, C., HANSEN, M. C. AND WIUF, C. (2021). Structural classification of continuous time Markov chains with applications. Stochastics. Available at https://doi.org/10.1080/17442508.2021.2017937.