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ON SURFACES OF ORDER THREE

BY
TIBOR BISZTRICZKY

A surface of order three F in the real projective three-space P^3 is met by every line, not in F, in at most three points.

In the present paper, we determine the existence and examine the distribution of elliptic, parabolic and hyperbolic points; that is, the differentiable points of F which do not lie on any line contained in F.

We define a topology of P^3 in the usual manner. We denote the planes, lines and points of P^3 by the letters $\alpha, \beta, \ldots; L, M, \ldots$; and $p, q, \ldots$ respectively. For a collection of flats $\alpha, L, p, \ldots; \langle \alpha, L, p, \ldots \rangle$ denotes the flat of P^3 spanned by them. For a set $\mathcal{M}$, $\langle \mathcal{M} \rangle$ denotes the flat of P^3 spanned by the points of $\mathcal{M}$.

1. **A (plane) curve** Γ is the union of a finite collection of sets $C_\lambda(M)$ where the C_λ's are continuous maps from a line $M = \{m, m', \ldots\}$ into a plane α. Let $p \in \alpha$. Then p is a *simple* [*double*] point of $C_\lambda(M)$ if the equation $p = C_\lambda(m)$ has exactly one solution [exactly two solutions] $m \in M$.

Let $C = C_\lambda$. The line $T_m = \lim \langle C(m), C(m') \rangle$, as $m' \neq m$ tends to m, is the *tangent* of C at m. Let C be *differentiable*; that is, T_m exists and $|T_m \cap C(M)| < \infty$ for every $m \in M$. We introduce (cf. [1] 1.3.3 and [4]) the characteristic of C at m and the multiplicity with which a line $L \subset \alpha$ meets C at m. Then C is of *order* n if n is the supremum of the number of points of M, counting multiplicities, mapped into collinear points by C.

If C is of order two [three], we denote $C(M)$ by $S^1[F^1_*]$. Every point of an S^1 is simple, an F^1_* contains at most one double point and a simple point of F^1_* is an ordinary, inflection or cusp point; cf. [1] 1.4 and [2]. If $C(M)$ is a line [point], then C is considered to have order one [two].

Γ is of *order* k if k is the supremum of the number of points of Γ, counting multiplicities on each C_λ, lying on any line not in Γ. If $k = 1$, then Γ is a straight line. If $k = 2$, then Γ is an S^1 or an isolated point or a pair of distinct lines. If $k = 3$, then Γ is (i) an F^1_* plus possibly an S^1 or an isolated point either disjoint from F^1_* or (ii) the union of a line and a Γ' of order two. We denote a Γ of order three satisfying (i) by F^1.

2. **A surface of order three** F in P^3 is a compact and connected set such that every intersection of F with a plane is a curve of order ≤ 3 and some plane intersection is an F^1.

Let F be a surface of order three; $p \in F$. Let α denote a plane through p.

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Then p is *regular* in $F[\alpha \cap F]$ if there is a line N in $P^3[\alpha]$ such that $p \in N$ and $|N \cap F| = 3$; otherwise, p is *irregular* in $F[\alpha \cap F]$. If $\alpha \cap F$ is an F^1, then there is at most one irregular point v in $\alpha \cap F$ and such a v is a cusp, double point or isolated point. Finally,

$$l(p, \alpha) = |\{L \subset \alpha \mid p \in L \subset F\}| \leq l(\alpha) = |\{L \subset \alpha \mid L \subset F\}| \leq 3.$$

A line T is a *tangent* of F at p if T is the tangent of some C_λ at m; $p = C_\lambda(m) \subset C_\lambda(M) \subset F$. Let $\tau(p)$ be the set of tangents of F at p. Then p is *differentiable* if p is regular in F and $\tau(p)$ is a plane $\pi(p)$; otherwise, p is *singular.*

Henceforth, we assume that every regular p in F is differentiable and $\pi(p)$ depends continuously on p.

Let p be differentiable. Then $p \in T \subset \pi(p)$ implies that either $T \subset F$ or $|T \cap F| \leq 2$. Thus $l(p) = |\{L \subset F \mid p \in L\}| = l(p, \pi(p))$ and p is irregular in $\pi(p) \cap F$. If $l(p) = 0$, then p is an isolated point, cusp or double point of $\pi(p) \cap F$ and we call p *elliptic, parabolic* or *hyperbolic* respectively.

Let $\mathscr{F}$ be a closed connected subset of S^1 or F^1_*. If the end points of $\mathscr{F}$ are distinct [equal], then $\mathscr{F}$ is a *subarc* [*subcurve*]. A subarc of F^1_*, containing only ordinary points in its interior, is of order two.

Let p be regular in F. Let $\mathscr{F}(p)$ be the set of all subarcs $\mathscr{F}$ of order two such that $p \in \mathscr{F} \not\subset \pi(p)$; $\{\mathscr{F}_1, \mathscr{F}_2\} \subset \mathscr{F}(p)$. Then $\mathscr{F}_1$ and $\mathscr{F}_2$ are *p-compatible* if there is a $\beta \subset P^3 \setminus \{p\}$ and an open neighbourhood $U(p)$ of p in P^3 such that $U(p) \cap (\mathscr{F}_1 \cup \mathscr{F}_2)$ is contained in a closed half-space of P^3 bounded by $\pi(p)$ and β; otherwise, $\mathscr{F}_1$ and $\mathscr{F}_2$ are *p-incompatible.*

A pair of subarcs $\mathscr{F}$ and $\mathscr{F}'$ are *compatible* [*incompatible*] if there is a $p \in \mathscr{F} \cap \mathscr{F}'$ such that $\{\mathscr{F}, \mathscr{F}'\} \subset \mathscr{F}(p)$ and $\mathscr{F}$ and $\mathscr{F}'$ are p-compatible [p-incompatible]. We consider a subcurve of order two as an element of $\mathscr{F}(p)$ if it contains a subarc $\mathscr{F}$ such that $p \in \mathscr{F} \in \mathscr{F}(p)$.

3. **We assume that *F* is non-ruled;** that is, F is not generated by lines. Then $l(F) = |\{l \subset P^3 \mid L \subset F\}| < \infty$ and F contains at most four irregular points; cf. [3]. We denote by E, I and H: the set of elliptic, parabolic and hyperbolic points of F respectively. We shall prove that for any (non-ruled) $F: H \neq \phi, I \neq \phi$ implies that $E \neq \phi$, E is open and I is nowhere dense in F.

By way of preparation, we have the following remarks:

3.1 Let $L \subset F$ and $p \in F \setminus L$ such that $\langle L, p \rangle \cap F$ consists of L and an S^1. We denote this S^1 by $S^1(L, p)$.

3.2 If a plane section of F is of order two, then it consists of a pair of lines. ([1], 2.2.3.)

3.3 If p is regular in F and isolated in $\alpha \cap F$, then $p \in E$ and $\alpha = \pi(p)$. ([1], 2.3.7.)

3.4 Let p be regular in F, $l(p) = 0$. Then $p \in H$ if and only if there exist incompatible $\mathscr{F}$ and $\mathscr{F}'$ in $\mathscr{F}(p)$ with $p \in \text{int}(\mathscr{F}) \cap \text{int}(\mathscr{F}')$. ([1], 2.5.7)

3.5 Let $\mathscr{F}' \in \mathscr{F}(p)$ for each $p \in \mathscr{F}'$. Let $L \subset F$ such that $L \not\subset \langle \mathscr{F}' \rangle$ and $S^1(L, p) \in \mathscr{F}(p)$ for each $p \in \mathscr{F}'$. Then $\mathscr{F}'$ and $S^1(L, p)$ are either compatible for all $p \in \mathscr{F}'$ or incompatible for all $p \in \mathscr{F}'$. ([1], 2.5.8.)

3.6 Let p_λ be a sequence converging to a differentiable p. If $p_\lambda \in I[E]$ for each λ, then $l(p) = 0$ implies that $p \in I[E \cup I]$ and $\pi(p) \cap F = L \cup S^1$ implies that $L \cap S^1 = \{p\}$. ([1], 2.4.6 and 2.4.9.)

3.7. Lemma. *Let G be an open region in F such that $\alpha_0 \cap \bar{G} = \phi$ for some α_0, $bd(F \backslash G) = bd(G)$, $\langle bd(G) \rangle$ is a plane and each $p \in G$ is regular in F. Then $G \cap E \neq \phi$.*

Proof. We note that any line in a plane $\langle F^1_* \rangle$ meets F^1_* and thus, any line in P^3 meets F by 3.2.

Let $p \in G$ and put $L = \alpha_0 \cap \langle bd(G) \rangle$. Then $L \cap \bar{G} = \phi$ implies that $L \cap (F \backslash \bar{G}) \neq \phi$ and $\langle L, p \rangle \cap G$ is an S^1 or an isolated point of $\langle L, p \rangle \cap F$. Obviously, $\alpha_0 \cap \bar{G} = \phi$ yields that there is a $p_0 \in G$ such that $\langle L, p_0 \rangle \cup G = \{p_0\}$. Then $p_0 \in E$ and $\pi(p_0) = \langle L, p_0 \rangle$ by 3.3.

We note that $\bar{E} \cap H = \phi$ and $\bar{I} \cap (E \cup H) = \phi$ by 3.6. It is clear that a limit of hyperbolic points may be parabolic but not elliptic. Thus, E is open in F and

$$\{p \in \bar{E} \cap \bar{H} \mid l(p) = 0 \text{ and } p \text{ is regular in } F\} \subseteq I.$$

3.8. Theorem. *If $l(F) > 0$, then I is nowhere dense in F and*

$$I = \{p \in \bar{E} \cap \bar{H} \mid l(p) = 0 \textit{ and } p \textit{ is regular in } F\}.$$

Proof. Let $L \subset F$ and $p_0 \in I$. Then there is a $T \subset \pi(p_0)$ such that $T \cap F = \{p_0\}$. Let $T \subset \beta \neq \pi(p_0)$. Then $l(p_0) = 0$, $T \cap F = \{p_0\}$ and 3.2 imply that $\beta \cap F$ is an F^1 with p_0 as an inflection point. Thus there are $\mathscr{F}$ and $\mathscr{F}'$ in $\mathscr{F}(p_0)$ such that $\mathscr{F} \cup \mathscr{F}' \subset \beta$ and $\mathscr{F} \cap \mathscr{F}' = \{p_0\}$. We note that $\mathscr{F}$ and $\mathscr{F}'$ are incompatible and for p close to p_0 in $\mathscr{F} \cup \mathscr{F}'$, p is regular in F. Since $l(p_0) = 0$ and $T \cap L = \phi$, we may assume that $l(p) = 0$ and $(\beta \cap \pi(p)) \cap L = \phi$ for each $p \in \mathscr{F} \cup \mathscr{F}'$. Then $\langle L, p \rangle \cap F = L \cup S^1(L, p)$ for each $p \in \mathscr{F} \cup \mathscr{F}'$ by 3.3.

(i) Since $\mathscr{F}$ and $\mathscr{F}'$ are incompatible, $S^1(L, p_0) \in \mathscr{F}(p_0)$ implies that $S^1(L, p_0)$ and say $\mathscr{F}$ are incompatible. By 3.5, $S^1(L, p)$ and $\mathscr{F}$ are incompatible for each $p \in \mathrm{int}(\mathscr{F})$. Thus $p_0 \in \bar{H}$ by 3.4.

(ii) Since $p_0 \in I$,

$$\pi(p_0) \cap F = \mathscr{F}_1 \cup \mathscr{F}_2$$

where $\mathscr{F}_1[\mathscr{F}_2]$ is a subarc of order two, $p_0 \in \mathscr{F}_1 \cap \mathscr{F}_2$ and $|\mathscr{F}_1 \cap \mathscr{F}_2| = 2$. Let p_λ in $\mathrm{int}(\mathscr{F})$ converge to p_0. Since $p_\lambda \in H$ for each λ,

$$\pi(p_\lambda) \cap F = \mathscr{L}_\lambda \cup \mathscr{F}_{1,\lambda} \cup \mathscr{F}_{2,\lambda}$$

where $\mathscr{L}_\lambda$ is a subcurve of order two, $\mathscr{F}_{1,\lambda}[\mathscr{F}_{2,\lambda}]$ is a subarc of order two, $\mathscr{L}_\lambda \cap (\mathscr{F}_{1,\lambda} \cup \mathscr{F}_{2,\lambda}) = \{p_\lambda\}$, $p_\lambda \in \mathscr{F}_{1,\lambda} \cap \mathscr{F}_{2,\lambda}$ and $|\mathscr{F}_{1,\lambda} \cap \mathscr{F}_{2,\lambda}| = 2$. We note that as

p_λ tends to p_0,

$$\lim \pi(p_\lambda)\cap F=\pi(p_0)\cap F$$

and $\lim \mathcal{L}_\lambda$ is a closed curve of order ≤ 2. It is easy to check that

$$\lim \mathcal{L}_\lambda=\{p_0\} \quad \text{and} \quad \lim \mathcal{F}_{1,\lambda}\cup\mathcal{F}_{2,\lambda}=\mathcal{F}_1\cup\mathcal{F}_2.$$

Since p_0 is parabolic, we can describe a sufficiently small neighbourhood of p_0 in F. In particular, it is easy to check that (for p_λ sufficiently close to p_0) $\mathcal{L}_\lambda$ is the boundary of an open region $F(\mathcal{L}_\lambda)\subset F$ such that $bd(F\setminus F(\mathcal{F}_\lambda))=bd(F(\mathcal{L}_\lambda))$ and $\lim \overline{F(\mathcal{L}_\lambda)}=\{p_0\}$. Finally, $l(p_0)=0$ and p_0 regular in F imply that $F(\mathcal{L}_\lambda)$ satisfies 3.7 for p_λ sufficiently close to p_0. Hence, $F(\mathcal{L}_\lambda)\cap E\neq\phi$ and $p_0\in\bar{E}$.

3.9. Theorem. *If $l(F)>0$, then F contains hyperbolic points.*

Proof. We note that $l(\beta)=0$ implies that $\beta\cap F$ contains an inflection point by 3.2. As $l(F)<\infty$, the set

$$\mathcal{Q}=\{q\in F \mid q \text{ is an inflection point}\}$$

is infinite. If $q\in\mathcal{Q}$ and $l(q)=0$, then $q\in I\cup H$. Since $I\neq\phi$ implies that $\bar{E}\cap\bar{H}\neq\phi$ (and thus $E\neq\phi\neq H$) by 3.8, we obtain that $q\in I\cup H$ yields that $H\neq\phi$.

Suppose that $H=\phi$. Then $l(q)>0$ for each $q\in\mathcal{Q}$ and there is an $L\subset F$ such that $L\cap\mathcal{Q}$ is an infinite set.

Case 1. There is an $M\subset F$ such that $L\cap M=\phi$.

Let $q\in L\cap\mathcal{Q}$. Then $\pi(q)=\langle L, T_q\rangle$ where $T_q\cap F=\{q\}$. As $\pi(q)\cap M\not\subset L$, this implies that either $\langle q, \pi(q)\cap M\rangle\subset F$ or $\pi(q)\cap F$ consists of L and an S^1 where $|L\cap S^1|=2$ and $q\in L\cap S^1$. In the latter case, 3.6 clearly implies that $q\in\bar{H}$ and $H\neq\phi$; a contradiction. Thus $\langle q, \pi(q)\cap M\rangle\subset F$ for each $q\in L\cap\mathcal{Q}$. Then $l(F)<\infty$ implies that $|L\cap\mathcal{Q}|<\infty$; a contradiction.

Case 2. Every $M\subset F$ meets L.

Clearly, there is a point $q_1\in\mathcal{Q}\setminus L$. Thus there is an $L_1\subset F$ such that $q_1\in L_1$ and $L_1\cap L$ is a point $v\neq q_1$.

If $M\subset F$ such that $M\cap L_1=\phi$, then $M_1=\langle q_1, \pi(q_1)\cap M\rangle\subset F$ by the preceding. Since $v\notin M_1$ and $M_1\cap L\neq\phi$, we obtain that $M_1\subset\langle L, L_1\rangle$. Then $\pi(q_1)\cap F=M_1\cup L_1\cup L$ where $M_1\cap L_1\cap L=\phi$ implies that $q_1\notin\mathcal{Q}$; a contradiction. Thus, every line in F passes through v.

Let $v\in\beta$. Then either $l(\beta)>0$ or ($l(\beta)=0$, $\beta\cap\mathcal{Q}=\{v\}$ and) β contains an irregular point of F. Since F contains a finite number of lines and irregular points, this is a contradiction.

4. **In each of the following examples,** F contains exactly one irregular point v and one line L. Recall that $\tau(v)$ is the set of tangents of F at v.

Let P^3 be suitably coordinatized.

The surface F defined by

$$0=x_0^3-(x_1^2+x_2^2)x_3 \qquad (L\equiv x_0=x_3=0,\ v\equiv(0,0,0,1))$$

contains neither elliptic nor parabolic points. $\tau(v)$ is a line $T\equiv x_1=x_2=0$ and v is the cusp [isolated point] of $\beta\cap F$ if $T\subset\beta[\beta\cap T=\{v\}]$.

The surface F defined by

$$0=x_0^3-(x_1^2+x_2^2-x_0^2)x_3 \qquad (L\equiv x_0=x_3=0,\ v\equiv(0,0,0,1))$$

contains elliptic but not parabolic points. $\tau(v)$ is a cone of order two with vertex v and $v\in T\subset\tau(v)$ if and only if $T\cap F=\{v\}$. If $\beta\cap\tau(v)=\{v\}$, then v is the isolated point of $\beta\cap F$; otherwise, v is the cusp or the double point of $\beta\cap F$. Finally, E and H are both connected with $\bar{E}=E\cup\{v\}$ and $\bar{H}=H\cup L\cup\{v\}$.

The surface F defined by

$$0=x_0^3+x_1x_2^2+x_1^2x_3 \qquad (L\equiv x_0=x_1=0,\ v\equiv(0,0,0,1))$$

contains parabolic points, $\tau(v)\equiv x_1=0$ and $\beta\cap L=\{v\}$ implies that v is the cusp of $\beta\cap F$. In this case; $I=S^1(L,\bar{p})\backslash\{v\}$ where $\bar{p}\equiv(0,1,0,0)$ or equivalently $I\equiv 0=x_0=x_2^2+x_1x_3$, E is connected with $\bar{E}=E\cup I\cup\{v\}$ and $\bar{H}=H\cup I\cup L$.

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Department of Mathematics and Statistics
The University of Calgary,
Calgary, Alberta. T2N 1N4