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ON THE GEOMETRY OF SPACELIKE MEAN CURVATURE FLOW SOLITONS IMMERSED IN A GRW SPACETIME

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Abstract

We investigate geometric aspects of complete spacelike mean curvature flow solitons of codimension one in a generalized Robertson–Walker (GRW) spacetime $-I \times_f M^n$, with base $I \subset \mathbb{R}$, Riemannian fiber M^n and warping function $f \in C^\infty(I)$. For this, we apply suitable maximum principles to guarantee that such a mean curvature flow soliton is a slice of the ambient space and to obtain nonexistence results concerning these solitons. In particular, we deal with entire graphs constructed over the Riemannian fiber M^n, which are spacelike mean curvature flow solitons, and we also explore the geometry of a conformal vector field to establish topological and further rigidity results for compact (without boundary) mean curvature flow solitons in a GRW spacetime. Moreover, we study the stability of spacelike mean curvature flow solitons with respect to an appropriate stability operator. Standard examples of spacelike mean curvature flow solitons in GRW spacetimes are exhibited, and applications related to these examples are given.

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1. Introduction

Given a spacelike hypersurface $\psi : \Sigma^n \hookrightarrow \mathbb{R}^{n+1}_1$ (which means that the induced metric of Σ^n via the immersion ψ is Riemannian) in the $(n+1)$-dimensional Minkowski space $\mathbb{R}^{n+1}_1$, we recall that the *spacelike mean curvature flow* associated to ψ is a family of smooth spacelike immersions $\Psi_t = \Psi(t, \cdot) : \Sigma^n \to \mathbb{R}^{n+1}_1$ with corresponding images $\Sigma^n_t = \Psi_t(\Sigma^n)$ satisfying the evolution equation,

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$$\begin{cases} \dfrac{\partial \Psi}{\partial t} = \vec{H}, \\ \Psi(0,x) = \psi(x), \end{cases}$$

on some time interval, where $\vec{H}$ stands for the (nonnormalized) mean curvature vector of the spacelike submanifold Σ_t^n in $\mathbb{R}_1^{n+1}$.

Mean curvature flow in the Minkowski space and, more generally, in a Lorentzian manifold has been extensively studied by several authors (see, for example, [28, 31, 41]) and an important justification for this interest is the fact that spacelike self-shrinkers and, in a more general setting, spacelike mean curvature flow solitons (which constitute singularities of the spacelike mean curvature flow) can be regarded as a natural way of foliating spacetimes by almost null-like hypersurfaces. Particular examples may give insight into the structure of certain spacetimes at null infinity and have possible applications in general relativity. For example, they were used in the first proof of the positive mass theorem [39, 40] and in the analysis of the Cauchy problem for asymptotically flat spacetimes [23, 33].

In [22], Chen and Qiu proved that any complete m-dimensional spacelike self-shrinkers in pseudo-Euclidean spaces $\mathbb{R}_n^{m+n}$ of index n must be affine planes, and that there exists no complete m-dimensional spacelike translating soliton in $\mathbb{R}_n^{m+n}$. Subsequently, Xu and Liu [44] classified m-dimensional complete spacelike translating solitons in $\mathbb{R}_n^{m+n}$ by affine techniques and classical gradient estimates, and they obtained a Bernstein-type theorem when the translating vector is spacelike. In addition, Lambert and Lotay [32] proved long-time existence and convergence results for spacelike solitons to mean curvature flow in $\mathbb{R}_n^{n+m}$ that are entire or defined on bounded domains and satisfy Neumann or Dirichlet boundary conditions.

Related to the Riemannian setting, Alías, de Lira and Rigoli [9] introduced the general definition of self-similar mean curvature flow in a Riemannian manifold $\overline{M}^{n+1}$ endowed with a conformal vector field K, and they established the corresponding notion of a mean curvature flow soliton. In particular, when $\overline{M}^{n+1}$ is a Riemannian warped product of the type $I \times_f M^n$ and $\mathcal{K} = f(t)\partial_t$, they applied weak maximum principles to guarantee that a complete n-dimensional mean curvature flow soliton is a slice of $\overline{M}^{n+1}$. In [24], Colombo *et al.* also studied some properties of mean curvature flow solitons in general Riemannian manifolds and in warped products, focusing on splitting and rigidity results under various geometric conditions, ranging from the stability of the soliton to the fact that the image of its Gauss map be contained in suitable regions of the sphere.

More recently, Alías *et al.* [10] established a natural framework for the stability of mean curvature flow solitons in warped product spaces. By regarding these solitons as stationary immersions for a weighted volume functional, they were able to find geometric conditions for finiteness of the index and some characterizations of stable solitons. When the ambient space is a Lorentzian product space, the first author jointly with Batista [15] established nonexistence results for complete spacelike translating solitons under suitable curvature constraints on the curvatures of the Riemannian base of the ambient space.

Here, our purpose is to investigate geometric aspects of complete spacelike mean curvature flow solitons of codimension one in a *generalized Robertson–Walker* (GRW) spacetime $-I \times_f M^n$, with base $I \subset \mathbb{R}$, Riemannian fiber M^n and warping function $f \in C^\infty(I)$. In this context and inspired by the techniques developed in [9, 15, 24], in Section 3 we apply suitable maximum principles to guarantee that such a mean curvature flow soliton is a slice of the ambient space and to obtain nonexistence results concerning these solitons. For example, we apply an extension of Hopf's theorem to prove the following rigidity result (see Theorem 3.4).

THEOREM 1.1. *Let $\overline{M}^{n+1} = -I \times_f M^n$ be a GRW spacetime that obeys the null convergence condition (3-20), with equality holding only in isolated points of I. Let $\psi : \Sigma^n \looparrowright \overline{M}^{n+1}$ be a complete spacelike mean curvature flow soliton with soliton constant $c \neq 0$ lying in a timelike bounded region $\mathcal{B}_{t_1,t_2}$. If its second soliton function $\tilde{\zeta}_c = |A|^2 + cf'(h)$ is nonnegative and $|\nabla H| \in \mathcal{L}^1(\Sigma^n)$, then Σ^n is a slice of $\overline{M}^{n+1}$.*

In the previous statement, $\mathcal{L}^1(\Sigma^n)$ stands for the space of Lebesgue integrable functions on Σ^n. Moreover, among other results, we apply Omori–Yau's maximum principle to prove the following nonexistence result, which can be regarded as a sort of extension of [22, Theorem 3] (see Theorem 3.20).

THEOREM 1.2. *Let $\overline{M}^{n+1} = -I \times_f M^n$ be a GRW spacetime satisfying the strong null convergence condition (3-25). There is no complete spacelike mean curvature flow soliton immersed in $\overline{M}^{n+1}$ with soliton constant $c \neq 0$ such that $cf'(h) \geq 0$ and $((n-1)f''(h) + cf(h)f'(h))/f(h)$ is bounded from below.*

In particular, in Section 3.3, we deal with entire graphs constructed over the Riemannian fiber M^n that are spacelike mean curvature flow solitons. Then, in Section 4, we explore the geometry of a conformal vector field to establish topological and further rigidity results for compact (without boundary) mean curvature flow solitons in a GRW spacetime. For example, we obtain the following result (see Theorem 4.9).

THEOREM 1.3. *Let $\overline{M}^{n+1} = -I \times_f M^n$ be a GRW spacetime and let $\psi : \Sigma^n \looparrowright \overline{M}^{n+1}$ be a compact spacelike mean curvature flow soliton with soliton constant $c \neq 0$. Suppose that Σ^n is totally umbilical and is not contained in a slice of $\overline{M}^{n+1}$. If Σ^n has finite fundamental group, then Σ^n is diffeomorphic to an Euclidean sphere.*

Furthermore, motivated by [10], in Section 5, we study the stability of spacelike mean curvature flow solitons with respect to an appropriate stability operator L_{cu}, which is defined in (5-11). In this setting, we deduce the following result (see Theorem 5.6).

THEOREM 1.4. *Let $\psi : \Sigma^n \looparrowright -I \times_f M^n$ be a spacelike mean curvature flow soliton with soliton constant $c \neq 0$.*

(a) *If $\zeta_c'(t) \leq 0$ on Σ^n, then $\psi : \Sigma^n \looparrowright -I \times_f M^n$ is L_{cu}-stable.*
(b) *If Σ^n is compact and $\zeta_c'(t) \geq 0$ on it, then $\psi : \Sigma^n \looparrowright -I \times_f M^n$ is L_{cu}-stable if and only if $\zeta_c(t)$ is constant on Σ^n.*
(c) *If Σ^n is compact and $\zeta_c'(t) > 0$ on it, then $\psi : \Sigma^n \looparrowright -I \times_f M^n$ cannot be L_{cu}-stable.*

According to the terminology introduced in [9, 24], $\zeta_c(t)$ denotes the soliton function and it is defined in (2-7). Standard examples of spacelike mean curvature flow solitons in GRW spacetimes are given in this paper (see Section 2.3) as well as applications related to these examples.

2. Background

In this section, we quote some basic concepts, facts and standard examples which will be used and addressed in the following sections.

2.1. Some preliminaries. Let $(M^n, \langle\, ,\rangle_M)$ be a connected, n-dimensional, oriented Riemannian manifold, let $I \subset \mathbb{R}$ be an open interval and let $f : I \to \mathbb{R}$ be a positive smooth function. Also, in the product manifold $\overline{M}^{n+1} = I \times M^n$ let π_I and π_M denote the canonical projections onto the factors I and M^n, respectively.

The class of Lorentzian manifolds of concern here is the one obtained by furnishing $\overline{M}^{n+1}$ with the Lorentzian metric $\langle\, ,\rangle$ given by

$$\langle\, ,\rangle = -dt^2 + f(t)^2 \langle\, ,\rangle_M,$$

where $-dt^2$ stands for the standard metric of $I \subset \mathbb{R}$. In this article, we simply write

$$\overline{M}^{n+1} = -I \times_f M^n. \tag{2-1}$$

According to the nomenclature established in [11], we say that $\overline{M}^{n+1}$ is a GRW spacetime with warping function f and Riemannian fiber M^n. When M^n has constant sectional curvature, (2-1) has been known in the mathematical literature as a Robertson–Walker (RW) spacetime, an allusion to the fact that, for $n = 3$, it is an exact solution of Einstein's field equations (see, for example, [35, Ch. 12]).

In this setting, we consider the timelike conformal closed vector field

$$\mathcal{K}(t, y) = f(t)\partial_t|_{(t,y)}, \quad (t, y) \in -I \times_f M^n, \tag{2-2}$$

globally defined on $\overline{M}$, where $\partial_t = \partial/\partial_t$ stands for the coordinate timelike vector field tangential to I. From the relationship between the Levi-Civita connections of $\overline{M}^{n+1}$ and those of I and M^n (see [35, Proposition 7.35]), it follows that

$$\overline{\nabla}_V \mathcal{K} = f'(\pi_I)V,$$

for all $V \in \mathfrak{X}(\overline{M})$, where $\overline{\nabla}$ is the Levi-Civita connection of $\overline{M}^{n+1}$.

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Let Σ^n be an n-dimensional connected manifold. A smooth immersion $\psi : \Sigma^n \looparrowright \overline{M}^{n+1}$ is said to be a *spacelike hypersurface* if Σ^n, furnished with the metric induced from $\langle\,,\rangle$ via ψ, is a Riemannian manifold. We denote by ∇ the Levi–Civita connection of Σ^n endowed with its induced metric (which will be denoted by $\langle\,,\rangle$). Since $\overline{M}$ is time-orientable, it follows from the connectedness of Σ^n that one can uniquely choose a globally defined timelike unit vector field $N \in \mathfrak{X}^{\perp}(\Sigma)$ that has the same time-orientation as ∂_t, that is, such that $\langle N, \partial_t\rangle < 0$. In this case, one says that N is the *future-pointing Gauss map* of Σ^n and we always assume such a timelike orientation for Σ^n. From the inverse Cauchy–Schwarz inequality (see [35, Proposition 5.30]), we have that $\langle N, \partial_t\rangle \le -1$, with the equality holding at a point $p \in \Sigma^n$ if and only if $N = \partial_t$ at p.

We denote by A and $H = -\mathrm{trace}(A)$ the shape operator and the mean curvature function of the spacelike hypersurface $\psi : \Sigma^n \looparrowright \overline{M}^{n+1}$ with respect to its future-pointing Gauss map N. Throughout this paper, the mean curvature H, taken with respect to such a choice of orientation N, will be called the *future mean curvature* of Σ^n. In particular, for a fixed $t_* \in I$, from [6, Example 5.6] we have that the *slice* $\{t_*\} \times M^n$ has constant future mean curvature

$$H = n\frac{f'(t_*)}{f(t_*)}$$

with respect to $N = \partial_t$.

It follows from (2-2) that

$$\overline{\nabla}_V \partial_t = \overline{\nabla}_V\Big(\frac{1}{f(t)}\mathcal{K}\Big) = -\frac{1}{f(t)^2}\langle V, \overline{\nabla}\,\overline{f}\rangle\mathcal{K} + \frac{1}{f(t)}f'(t)V \tag{2-3}$$

and a simple computation shows that

$$\overline{\nabla}\pi_I = -\langle\overline{\nabla}\pi_I, \partial_t\rangle\partial_t = -\partial_t. \tag{2-4}$$

So, from (2-3),

$$\overline{\nabla}_V \partial_t = \frac{f'(\pi_I)}{f(\pi_I)}\{V + \langle V, \partial_t\rangle\partial_t\}. \tag{2-5}$$

2.2. Spacelike mean curvature flow solitons in GRW spacetimes. We recall that the spacelike mean curvature flow $\Psi : [0, T) \times \Sigma^n \looparrowright \overline{M}^{n+1}$ related to a spacelike hypersurface $\psi : \Sigma^n \looparrowright \overline{M}^{n+1}$ in an $(n+1)$-dimensional Lorentzian manifold $\overline{M}^{n+1}$, satisfying $\Psi(0,\cdot) = \psi(\cdot)$, looks for solutions of the equation

$$\frac{\partial\Psi}{\partial t} = \vec{H},$$

where $\vec{H}(t,\cdot)$ is the (nonnormalized) mean curvature vector of $\Sigma^n_t = \Psi(t, \Sigma^n)$ (see, for example, [32]). In our context, according to [9, Definition (1.1)] and [24, Definition (1.1)], a spacelike hypersurface $\psi : \Sigma^n \looparrowright \overline{M}^{n+1}$ immersed in a GRW spacetime

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$\overline{M}^{n+1} = -I \times_f M^n$ is called a *spacelike mean curvature flow soliton* with respect to $\mathcal{K} = f(t)\partial_t$ and has *soliton constant* $c \in \mathbb{R}$ when its (nonnormalized) future-pointing mean curvature vector $\vec{H} = HN$ satisfies

$$\vec{H} = c\,\mathcal{K}^{\perp}, \tag{2-6}$$

where $\mathcal{K}^{\perp}$ stands for the orthogonal projection of $\mathcal{K}$ in the direction of the future-pointing Gauss map N. Adopting the terminology introduced in [9, 24], we also consider the *soliton function*

$$\zeta_c(t) = nf'(t) + cf(t)^2. \tag{2-7}$$

So, each slice $M_{t_*} = \{t_*\} \times M^n$ is a spacelike mean curvature flow soliton with respect to $\mathcal{K} = f(t)\partial_t$ and with soliton constant c given by

$$c = -n\frac{f'(t_*)}{f(t_*)^2}. \tag{2-8}$$

Moreover, t_* is implicitly given by the condition $\zeta_c(t_*) = 0$.

2.3. Standard examples. In this subsection, we quote standard examples of spacelike mean curvature flow solitons in GRW spacetimes.

EXAMPLE 2.1. Using a method similar to that of [27], for the Lorentzian product space $-I \times M^n$, from (2-8) we get that the slices $\{t\} \times M^n$ are spacelike mean curvature flow solitons with soliton constant $c = 0$ with respect to vector field $\mathcal{K} = \partial_t$. Similarly to what happens in the Minkowski space $\mathbb{R}^{n+1}_1 = -\mathbb{R} \times \mathbb{R}^n$, such solitons are called spacelike translating solitons.

EXAMPLE 2.2. As in [8, Section 4], the *future temporal cone* Λ^+ of the Minkowski space $\mathbb{R}^{n+1}_1$ is defined as the set

$$\Lambda^+ = \{x \in \mathbb{R}^{n+1}_1 : \langle x, x\rangle < 0 \text{ and } \langle x, e_1\rangle < 0\},$$

where $e_1 = (1, 0, \dots, 0)$. We observe that Λ^+ can be regarded as the GRW spacetime

$$-\mathbb{R}^+ \times_t \mathbb{H}^n,$$

where $\mathbb{H}^n = \{x \in \mathbb{R}^{n+1}_1 : \langle x, x\rangle = -1,\ x_1 > 0\}$ denotes the n-dimensional hyperbolic space. Indeed, it is not difficult to verify that the map $\Phi : -\mathbb{R}^+ \times_t \mathbb{H}^n \to \Lambda^+$, given by $\Phi(t, x) = tx$, is an isometry. In this setting, we have that the slices $\{\sqrt{-n/c}\} \times \mathbb{H}^n$ are spacelike mean curvature flow solitons with soliton constant $c < 0$ with respect to vector field $\mathcal{K} = t\partial_t$.

EXAMPLE 2.3. The four-dimensional Einstein–de Sitter spacetime $-\mathbb{R}^+ \times_{t^{2/3}} \mathbb{R}^3$, where $\mathbb{R}^3$ stands for the three-dimensional Euclidean space endowed with its canonical metric, is a classical exact solution to the Einstein field equation without a cosmological constant. It is an open Friedmann–Robertson–Walker model, which incorporates

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homogeneity and isotropy (the cosmological principle) and permitted expansion (for more details, see [35, Ch. 12]). Here, we consider the $(n+1)$-dimensional Einstein–de Sitter spacetime $-\mathbb{R}^+ \times_{t^{2/3}} \mathbb{R}^n$. From (2-8), we conclude that the slices $\{(-2n/3c)^{3/5}\} \times \mathbb{R}^n$ are spacelike mean curvature flow solitons with respect to $\mathcal{K} = t^{2/3}\partial_t$ and with soliton constant $c < 0$.

EXAMPLE 2.4. According to the terminology introduced by Albujer and Alías [2], a GRW spacetime $-\mathbb{R} \times_{e^t} M^n$ is called a *steady-state-type* spacetime. This terminology is due to the fact that the steady-state model of the universe $\mathcal{H}^4$, proposed by Bondi–Gold [17] and Hoyle [30] when looking for a model of the universe that looks the same not only at all points and in all directions (that is, is spatially isotropic and homogeneous) but also at all times, is isometric to the GRW spacetime $-\mathbb{R} \times_{e^t} \mathbb{R}^3$ (for more details, see [29]). From (2-8), we conclude that the slices $\{\ln(-n/c)\} \times M^n$ are spacelike mean curvature flow solitons with respect to $\mathcal{K} = e^t\partial_t$ and with soliton constant $c < 0$.

EXAMPLE 2.5. From [34, Example 4.2], the $(n+1)$-dimensional de Sitter space $\mathbb{S}^{n+1}_1$ is isometric to the GRW spacetime $-\mathbb{R} \times_{\cosh t} \mathbb{S}^n$, where $\mathbb{S}^n$ denotes the n-dimensional unit Euclidean sphere endowed with its standard metric. Taking into account the terminology introduced in [5], the open half-space $\mathbb{R}^+ \times \mathbb{S}^n \subset \mathbb{S}^{n+1}_1$ (respectively, $\mathbb{R}^- \times \mathbb{S}^n \subset \mathbb{S}^{n+1}_1$) is called the *chronological future* (respectively, *past*) of $\mathbb{S}^{n+1}_1$ with respect to the totally geodesic *equator* $\{0\} \times \mathbb{S}^n$. From (2-8), we see that the equator is a spacelike mean curvature flow soliton with respect to $\mathcal{K} = \cosh t\,\partial_t$ and with soliton constant $c = 0$ and that the slices $\{\sinh^{-1}((-n \pm \sqrt{n^2 - 4c^2})/2c)\} \times \mathbb{S}^n$ are spacelike mean curvature flow solitons with respect to $\mathcal{K} = \cosh t\,\partial_t$ and with soliton constant $0 < |c| \leq (n/2)$.

EXAMPLE 2.6. Taking into account once more [34, Example 4.2], we consider the open region of $\mathbb{S}^{n+1}_1$ that is isometric to the GRW spacetime $-\mathbb{R}^+ \times_{\sinh t} \mathbb{H}^n$, where $\mathbb{H}^n$ denotes the n-dimensional hyperbolic space endowed with its standard metric. From (2-8), we have that the slices $\{\cosh^{-1}((-n - \sqrt{n^2 + 4c^2})/2c)\} \times \mathbb{H}^n$ are spacelike mean curvature flow solitons with respect to $\mathcal{K} = \sinh t\,\partial_t$ and with soliton constant $c < 0$.

EXAMPLE 2.7. Motivated by [34, Example 4.3], we consider the open subset of the $(n+1)$-dimensional anti-de Sitter space $\mathbb{H}^{n+1}_1$ that is isometric to the GRW spacetime $-(-\pi/2, \pi/2) \times_{\cos t} \mathbb{H}^n$. Analogous to the nomenclature of the de Sitter space, the open half-space $(0, \pi/2) \times \mathbb{H}^n \subset \mathbb{H}^{n+1}_1$ (respectively, $(-\pi/2, 0) \times \mathbb{H}^n \subset \mathbb{H}^{n+1}_1$) will be called the *chronological future* (respectively, *past*) of $\mathbb{H}^{n+1}_1$ with respect to the totally geodesic *equator* $\{0\} \times \mathbb{H}^n$. From (2-8), we see that the equator is a spacelike mean curvature flow soliton with respect to $\mathcal{K} = \cos t\,\partial_t$ and with soliton constant $c = 0$, and that the slices $\{\sin^{-1}((-n \pm \sqrt{n^2 + 4c^2})/2c)\} \times \mathbb{H}^n$ are spacelike mean curvature flow solitons with respect to $\mathcal{K} = \cos t\,\partial_t$ and with soliton constant $c \neq 0$.

3. Nonexistence and rigidity of spacelike mean curvature flow solitons

In this section, we study the nonexistence and rigidity of complete spacelike mean curvature flow solitons immersed in a GRW spacetime. For this, we need to develop some previous computations.

3.1. Some previous computations. Let $\psi : \Sigma^n \looparrowright -I \times_f M^n$ be a spacelike mean curvature flow soliton, as described in Section 2. The *height function* of $\psi : \Sigma^n \looparrowright -I \times_f M^n$, denoted by h, is the restriction of the projection $\pi_I(t, y) = t$ to Σ^n: that is, $h : \Sigma^n \to I$ is given by

$$h = \pi_I|_{\Sigma^n} = \pi_I \circ \psi. \tag{3-1}$$

Thus, the hyperbolic angle Θ of Σ^n verifies

$$\Theta = \langle N, \partial_t \rangle \leq -1, \tag{3-2}$$

where N denotes the future-pointing Gauss map of Σ^n. From (2-4), we have that the gradient of π_I on $-I \times_f M^n$ is given by $\overline{\nabla}\pi_I = -\partial_t$. Then, the gradient of h on Σ^n is given by

$$\nabla h = (\overline{\nabla}\pi_I)^\top = -\partial_t^\top = -\partial_t - \Theta N, \tag{3-3}$$

where $\partial_t = \partial_t^\top + \partial_t^\perp$. Here, $\partial_t^\top \in \mathfrak{X}(\Sigma^n)$ and $\partial_t^\perp \in \mathfrak{X}^\perp(\Sigma^n)$ denote, respectively, the tangential and normal components of ∂_t.

Thus, (3-3) gives the relationship

$$|\nabla h|^2 = \Theta^2 - 1, \tag{3-4}$$

where $|\cdot|$ stands for the norm of a tangential vector field on Σ^n considered with its induced metric.

Hence, from (3-3) and (2-5), we deduce that, for any $X \in \mathfrak{X}(\Sigma^n)$, the Hessian of h in the metric $\langle\, , \rangle$ is given by

$$\begin{aligned} \nabla^2 h(X, X) &= \langle \nabla_X \nabla h, X \rangle \\ &= -\frac{f'(h)}{f(h)}\{|X|^2 + \langle X, \nabla h \rangle^2\} + \langle AX, X \rangle \Theta. \end{aligned} \tag{3-5}$$

In what follows, we also consider the function

$$u = g(h) \in C^\infty(\Sigma^n), \tag{3-6}$$

where $g : I \to \mathbb{R}$ is an arbitrary primitive of f. Since $g' = f > 0$, $u = g(h)$ can be thought as a reparametrization of the height function. In particular, from (3-3), we have that the gradient of u on Σ^n is given by

$$\nabla u = f(h)\nabla h = -f(h)\partial_t^\top = -\mathcal{K}^\top, \tag{3-7}$$

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where $\mathcal{K}^\top$ denotes the tangential component of the closed conformal vector field $\mathcal{K}$, defined in (2-2). Taking into account this previous digression, we obtain the following auxiliary result.

LEMMA 3.1. *Let $\psi : \Sigma^n \looparrowright -I \times_f M^n$ be a spacelike mean curvature flow soliton with respect to $\mathcal{K} = f(t)\partial_t$ and with soliton constant $c \neq 0$. Then,*

$$H\langle AX, Y\rangle - c\nabla^2 u(X, Y) = cf'(h)\langle X, Y\rangle, \tag{3-8}$$

for all $X, Y \in \mathfrak{X}(\Sigma)$. Furthermore,

$$\nabla H = cA(\nabla u).$$

PROOF. First, we note that

$$\begin{aligned}\nabla^2 u(X, X) &= \langle\nabla_X\nabla u, X\rangle\\ &= \langle\nabla_X(f(h)\nabla h), X\rangle\\ &= f(h)\langle\nabla_X\nabla h, X\rangle + \langle\nabla_X f(h)\nabla h, X\rangle\\ &= f(h)\nabla^2 h(X, X) + f'(h)\langle X, \nabla h\rangle^2.\end{aligned}$$

Thus, from (3-5), we get that

$$\begin{aligned}\nabla^2 u(X, X) =&f(h)\Big(-\frac{f'(h)}{f(h)}\{|X|^2 + \langle X, \nabla h\rangle^2\} + \langle AX, X\rangle\Theta\Big) + f'(h)\langle X, \nabla h\rangle^2\\ =& - f'(h)|X|^2 - f'(h)\langle X, \nabla h\rangle^2 + f(h)\langle AX, X\rangle\Theta + f'(h)\langle X, \nabla h\rangle^2\\ =& - f'(h)|X|^2 + f(h)\langle AX, X\rangle\Theta.\end{aligned}$$

On the other hand,

$$\begin{aligned}\frac{1}{c}\langle\nabla H, X\rangle &= \langle\overline{\nabla}_X\mathcal{K}, N\rangle + \langle\mathcal{K}, \overline{\nabla}_X N\rangle\\ &= -\langle A(X), \mathcal{K}\rangle = \langle X, A(\nabla u)\rangle,\end{aligned} \tag{3-9}$$

for every vector field $X \in \mathfrak{X}(\Sigma^n)$, so that, from (3-7), we conclude the desired result. □

REMARK 3.2. We point out that (3-8) is close to the definition of Ricci solitons and, therefore, it is interesting to make a study of mean curvature flow solitons with this point of view.

Naturally attached to $\psi : \Sigma^n \looparrowright -I \times_f M^n$, we can consider the support function

$$\begin{aligned}\varphi_{\mathcal{K}} : \Sigma^n &\to \mathbb{R}\\ q &\mapsto \varphi_{\mathcal{K}}(q)\langle\mathcal{K}(q), N(q)\rangle.\end{aligned} \tag{3-10}$$

Hence, from (3-2),

$$\varphi_{\mathcal{K}} = f(h)\langle N, \partial_t\rangle = f(h)\Theta \leq -f(h) < 0. \tag{3-11}$$

Furthermore, from [19, Proposition 2.1] and (3-4),

$$\Delta(\varphi_{\mathcal{K}}) = \{\overline{\mathrm{Ric}}(N,N) + |A|^2\}\varphi_{\mathcal{K}} - \{nN(f') - Hf'\} + \langle \mathcal{K}, \nabla H\rangle, \tag{3-12}$$

where ∇H is the gradient of H in the metric of Σ^n, $\overline{\mathrm{Ric}}$ is the Ricci tensor of $\overline{M}^{n+1}$ and $|A|$ is the Hilbert–Schmidt norm of A.

In addition, we get that

$$N(f') = -f''\Theta = -\frac{f''}{f}\varphi_{\mathcal{K}}. \tag{3-13}$$

On the other hand, since $N = N^* - \Theta\partial_t$, where $N^* = \pi_M(N)$ is the orthogonal projection of N onto M^n, it follows from [35, Corollary 7.43] that

$$\begin{aligned}\overline{\mathrm{Ric}}(N,N) &= \overline{\mathrm{Ric}}(N^*,N^*) + \Theta^2\overline{\mathrm{Ric}}(\partial_t,\partial_t)\\ &= \mathrm{Ric}_M(N^*,N^*) + \langle N^*,N^*\rangle\Big\{\frac{f''}{f} + (n-1)\frac{(f')^2}{f^2}\Big\} - \frac{nf''}{f}\Theta^2\\ &= \mathrm{Ric}_M(N^*,N^*) - \Big\{\frac{f''}{f} + (n-1)\frac{(f')^2}{f^2}\Big\} - (n-1)\Big(\frac{f'}{f}\Big)'\Theta^2,\end{aligned} \tag{3-14}$$

where Ric_M denotes the Ricci tensor of M^n. We note that the relationship $\langle N^*,N^*\rangle = \Theta^2 - 1$ is used in the last equality above.

Thus, inserting (3-13) and (3-14) into (3-12), we obtain

$$\begin{aligned}\Delta(\varphi_{\mathcal{K}}) &= \Big\{\mathrm{Ric}_M(N^*,N^*) + |A|^2 - \Big\{\frac{f''}{f} + (n-1)\frac{(f')^2}{f^2}\Big\} - (n-1)\Big(\frac{f'}{f}\Big)'\Theta^2\Big\}\varphi_{\mathcal{K}}\\ &\quad + \Big\{n\frac{f''}{f}\varphi_{\mathcal{K}} + Hf'\Big\} + \langle \mathcal{K}, \nabla H\rangle\\ &= \Big\{\mathrm{Ric}_M(N^*,N^*) + |A|^2 + \frac{f''f - (f')^2}{f^2} - (n-1)\Big(\frac{f'}{f}\Big)'\Theta^2\Big\}\varphi_{\mathcal{K}} + Hf' + \langle \mathcal{K}, \nabla H\rangle\\ &= \{\mathrm{Ric}_M(N^*,N^*) + (n-1)(\ln f)''(1-\Theta^2) + |A|^2\}\varphi_{\mathcal{K}} + Hf' + \langle \mathcal{K}, \nabla H\rangle\\ &= \{\mathrm{Ric}_M(N^*,N^*) - (n-1)(\ln f)''|\nabla h|^2 + |A|^2\}\varphi_{\mathcal{K}} + Hf' + \langle \mathcal{K}, \nabla H\rangle.\end{aligned} \tag{3-15}$$

From Equations (2-6) and (3-10), we have that $H = c\,\varphi_{\mathcal{K}}$, and from (3-7) we get $\nabla u = -\mathcal{K}^\top$, where u is the reparametrization of the height function h given in (3-6). Consequently, we can rewrite (3-15) as

$$\Delta(\varphi_{\mathcal{K}}) = \{cf'(h) + \mathrm{Ric}_M(N^*,N^*) - (n-1)(\ln f)''(h)|\nabla h|^2 + |A|^2\}\varphi_{\mathcal{K}} + \langle \nabla(cu), \nabla(\varphi_{\mathcal{K}})\rangle. \tag{3-16}$$

We recall that the *drift Laplacian* on Σ^n is defined by

$$\Delta_{cu}(\varphi) = \Delta(\varphi) - \langle\, \nabla(cu), \nabla\varphi\,\rangle \tag{3-17}$$

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for all $\varphi \in C^\infty(\Sigma^n)$. So, from (3-16) and (3-17), we conclude that the drift Laplacian Δ_{cu} acting on $\varphi_{\mathcal{K}}$ is given by

$$\Delta_{cu}(\varphi_{\mathcal{K}}) = \{\tilde{\zeta}_c + \mathrm{Ric}_M(N^*, N^*) - (n-1)(\ln f)''(h)|\nabla h|^2\}\varphi_{\mathcal{K}}, \tag{3-18}$$

where $\tilde{\zeta}_c \in C^\infty(\Sigma^n)$ is the function defined by

$$\tilde{\zeta}_c(q) = cf'(h(q)) + |A(q)|^2 \tag{3-19}$$

for every $q \in \Sigma^n$, which will be called the *second soliton function* associated to the spacelike mean curvature flow soliton $\psi : \Sigma^n \looparrowright -I \times_f M^n$. Such nomenclature for $\tilde{\zeta}_c$ is motivated by [9, Equation (6.11)].

3.2. Rigidity and nonexistence results. In this subsection, beyond nonexistence results, we establish several theorems that guarantee that, under some necessary hypothesis (see Remark 3.25), a spacelike mean curvature flow soliton must coincide with a slice of the ambient GRW spacetime. We observe that these results are related to each other in the sense that they ensure the rigidity of a complete spacelike mean curvature flow soliton with nonnegative second soliton function that lies in a timelike bounded region of a GRW spacetime and obeys a suitable convergence condition.

In what follows, we assume that the GRW spacetime $-I \times_f M^n$ satisfies the *null convergence condition* (NCC)

$$\mathrm{Ric}_M \geq (n-1)(ff'' - f'^2)\langle\, , \rangle_M, \tag{3-20}$$

which was originally established by Montiel [34], where Ric_M denotes the Ricci tensor of the Riemannian fiber M^n. It is not difficult to verify that all the GRW spacetimes described in Section 2.2 satisfy the NCC. For this, in the case of a steady-state-type spacetime (see Example 2.4), it is necessary to assume that its Riemannian fiber has nonnegative Ricci curvature.

Before we prove the first rigidity result, we start by quoting an extension of Hopf's theorem on a complete Riemannian manifold Σ^n due to Yau in [45]. For this, we adopt the notation

$$\mathcal{L}^1(\Sigma^n) = \left\{ \varphi \in C^\infty(\Sigma^n) : \int_{\Sigma^n} |\varphi|\, d\Sigma \ll +\infty \right\}$$

for the space of Lebesgue integrable functions on Σ^n, where $d\Sigma$ stands for the volume element induced by the metric of Σ^n, and we denote by $\mathcal{L}^1_{cu}(\Sigma^n)$ the set of Lebesgue integrable functions on Σ^n with respect to the modified volume element

$$d\mu = e^{cu}\, d\Sigma. \tag{3-21}$$

We also recall that a smooth function φ on Σ^n is said to be (cu)-*subharmonic* (respectively, (cu)-*superharmonic*) if $\Delta_{cu}(\varphi) \geq 0$ (respectively, $\Delta_{cu}(\varphi) \leq 0$) on Σ^n. So, it is not difficult to verify that, from [18, Proposition 2.1], we obtain the following auxiliary lemma.

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LEMMA 3.3. *Let Σ^n be an n-dimensional complete oriented Riemannian manifold. If $\varphi \in C^\infty(\Sigma^n)$ is a (cu)-subharmonic function (or a (cu)-superharmonic function) on Σ^n and $|\nabla\varphi| \in \mathcal{L}^1_{cu}(\Sigma^n)$, then $\Delta_{cu}(\varphi) = 0$ on Σ^n.*

Given a GRW spacetime $\overline{M}^{n+1} = -I \times_f M^n$ obeying the NCC (3-20), we require a suitable behavior of the second soliton function associated to a spacelike mean curvature flow soliton $\psi : \Sigma^n \looparrowright \overline{M}^{n+1}$, and of the norm of the gradient of its mean curvature function, to establish our first uniqueness result. For this, we consider a *timelike bounded region* of $\overline{M}^{n+1}$ defined by

$$\mathcal{B}_{t_1,t_2} := \{(t, p) \in -I \times_f M^n : t_1 \le t \le t_2 \text{ and } p \in M^n\}.$$

Taking into account that all spacelike mean curvature solitons that appear in this paper are considered with respect to the closed conformal vector field $\mathcal{K} = f(t)\partial_t$, we are in position to present our first main result.

THEOREM 3.4. *Let $\overline{M}^{n+1} = -I \times_f M^n$ be a GRW spacetime that obeys the NCC (3-20), with equality holding only in isolated points of I. Let $\psi : \Sigma^n \looparrowright \overline{M}^{n+1}$ be a complete spacelike mean curvature flow soliton with soliton constant $c \neq 0$ that lies in a timelike bounded region $\mathcal{B}_{t_1,t_2}$. If its second soliton function $\tilde{\zeta}_c = |A|^2 + cf'(h)$ is nonnegative and $|\nabla H| \in \mathcal{L}^1(\Sigma^n)$, then Σ^n is a slice of $\overline{M}^{n+1}$.*

PROOF. From (3-20), we obtain that

$$\begin{aligned}
&\mathrm{Ric}_M(N^*, N^*) - (n-1)(\ln f)''(h)|\nabla h|^2 \\
&\quad \ge (n-1)(f(h)f''(h) - f'(h)^2)|N^*|^2_M - (n-1)(\ln f)''(h)|\nabla h|^2 \\
&\quad = (n-1)(f(h)f''(h) - f'(h)^2)|N + \Theta\partial_t|^2_M - (n-1)\Big(\frac{f'}{f}\Big)'(h)|\nabla h|^2 \\
&\quad = (n-1)\Big\{(f(h)f''(h) - f'(h)^2)\frac{|\nabla h|^2}{f(h)^2} - \Big(\frac{f(h)f''(h) - f'(h)^2}{f(h)^2}\Big)|\nabla h|^2\Big\} = 0. \qquad (3\text{-}22)
\end{aligned}$$

Thus, since $\tilde{\zeta}_c \ge 0$ on Σ^n, from (3-18) and (3-22) we get that the support function φ_K defined in (3-10) satisfies

$$\Delta_{cu}(\varphi_{\mathcal{K}}) \le f(h)\tilde{\zeta}_c\Theta \le 0. \qquad (3\text{-}23)$$

On the other hand, since $\psi : \Sigma^n \looparrowright -I \times_f M^n$ is contained in a timelike bounded region $\mathcal{B}_{t_1,t_2}$ of $-I \times_f M^n$, h is bounded on Σ^n and, consequently, the same happens with $u = g(h)$ and e^{cu}. So, since $c \neq 0$, from (3-21), (2-6), (3-10) and $|\nabla H| \in \mathcal{L}^1(\Sigma^n)$ we get $|\nabla(\varphi_{\mathcal{K}})| \in \mathcal{L}^1_{cu}(\Sigma^n)$. Next, from Lemma 3.3 we obtain that $\Delta_{cu}(\varphi_{\mathcal{K}}) = 0$ on Σ^n. Since $f(h) > 0$ and $\Theta < 0$ on Σ^n, from (3-22) and (3-23) we must have on Σ^n that

$$\tilde{\zeta}_c = 0 \quad \text{and} \quad \mathrm{Ric}_M(N^*, N^*) - (n-1)(\ln f)''(h)|\nabla h|^2 = 0.$$

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But, taking into account that the equality in (3-20) occurs only in isolated points of I, we can conclude that $|\nabla h| = 0$ on Σ^n and, consequently, h is constant on Σ^n. Therefore, $\psi(\Sigma^n)$ is a slice. □

REMARK 3.5. From (2-5), we have that the slice M^n_t is a spacelike hypersurface whose shape operator (with respect to the orientation ∂_t) A_t is given by

$$\begin{aligned} \mathcal{A}_{t_*} : \mathfrak{X}(M^n_{t_*}) &\to \mathfrak{X}(M^n_{t_*}) \\ V &\mapsto \mathcal{A}_{t_*}(V) = -\overline{\nabla}_V(\partial_{t_*}) = -\frac{f'(t_*)}{f(t_*)}\,V. \end{aligned} \tag{3-24}$$

Thus, from (3-24) we obtain that the principal curvatures $\kappa_i^{t_*}$ of the shape operator $\mathcal{A}_{t_*}$ of a slice $M^n_{t_*} = \{t_*\} \times M^n$, $t_* \in I$, are given by $\kappa_i^{t_*} = -f'(t_*)/f(t_*)$ for all $i \in \{1, \ldots, n\}$. So, from (2-8) and (3-19),

$$\tilde{\zeta}_c = c\,f'(t_*) + |\mathcal{A}_{t_*}|^2 = \sum_{i=1}^{n} (\kappa_i^{t_*})^2 + \Big(-\frac{n\,f'(t_*)}{f^2(t_*)}\Big) f'(t_*) = 0$$

on $M^n_{t_*}$. Hence, our restriction on the values of the second soliton function $\tilde{\zeta}_c$ in Theorem 3.4 constitutes a mild hypothesis in the sense that it is natural to detect slices of $-I \times_f M^n$.

From Theorem 3.4, we derive the following consequence.

COROLLARY 3.6. *Let $\psi : \Sigma^n \looparrowright -\mathbb{R}^+ \times_{t^{2/3}} \mathbb{R}^n$ be a complete spacelike mean curvature flow soliton with soliton constant $c < 0$ lying in a timelike bounded region of the Einstein–de Sitter spacetime $-\mathbb{R}^+ \times_{t^{2/3}} \mathbb{R}^n$. If $|A|$ does not vanish and $h \ge -8c^3/27|A|^6$ and $|\nabla H| \in \mathcal{L}^1(\Sigma^n)$, then Σ^n is the slice $\{(-2n/3c)^{3/5}\} \times \mathbb{R}^n$.*

When the ambient space is a steady-state-type spacetime, Theorem 3.4 gives the following rigidity result.

COROLLARY 3.7. *Let $\overline{M}^{n+1} = -I \times_{e^t} M^n$ be a steady-state-type spacetime whose Riemannian fiber M^n has positive Ricci curvature. Let $\psi : \Sigma^n \looparrowright \overline{M}^{n+1}$ be a complete spacelike mean curvature flow soliton with soliton constant $c < 0$ and lying in a timelike bounded region. If $|A|$ does not vanish and $h \ge \ln(-|A|^2/c)$ and $|\nabla H| \in \mathcal{L}^1(\Sigma^n)$, then Σ^n is the slice $\{\ln(-n/c)\} \times M^n$.*

From Theorem 3.4 we also get the following nonexistence results.

COROLLARY 3.8. *There is no complete spacelike translating soliton lying in a timelike bounded region of a Lorentzian product space $-I \times M^n$, whose Riemannian fiber M^n has positive Ricci curvature, that has soliton constant $c \neq 0$ and is such that $|\nabla H| \in \mathcal{L}^1(\Sigma^n)$.*

COROLLARY 3.9. *There is no complete spacelike mean curvature flow soliton lying in a timelike bounded region of a steady-state-type spacetime $-I \times_{e^t} M^n$, whose*

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Riemannian fiber M^n has positive Ricci curvature, that has soliton constant $c > 0$ and is such that $|\nabla H| \in \mathcal{L}^1(\Sigma^n)$.

According to the classical terminology in linear potential theory, a Riemannian manifold Σ^n is called (cu)-*parabolic* if the constant functions are the only functions $\varphi \in C^2(\Sigma)$ that are bounded from below and satisfy $\Delta_{cu}(\varphi) \le 0$. Inspired by the ideas of Romero *et al.* [37, 38], Albujer *et al.* established in [4, Theorem 1] the following parabolicity criterion, which provides conditions for a complete spacelike hypersurface immersed in GRW spacetime $-I \times_f M^n$ to be (cu)-parabolic. For this, we consider the function $\tilde{u} := g(\pi_I) \circ \tilde{\pi}$, where $\tilde{\pi} : \widetilde{M}^n \to M^n$ is the universal covering map of the Riemannian fiber M^n.

LEMMA 3.10. *Let $\psi : \Sigma^n \looparrowright -I \times_f M^n$ be a complete spacelike hypersurface immersed in a GRW spacetime $-I \times_f M^n$, whose Riemannian fiber M^n has $(c\tilde{u})$-parabolic universal Riemannian covering for some constant $c \neq 0$. If the hyperbolic angle Θ is bounded from below, and the warping function f and the height function h are such that $\sup_{\Sigma^n} f(h) < +\infty$ and $\inf_{\Sigma^n} f(h) > 0$, then Σ^n is (cu)-parabolic.*

We can state the following rigidity result for spacelike mean curvature flow solitons in GRW spacetimes.

THEOREM 3.11. *Let $\overline{M}^{n+1} = -I \times_f M^n$ be a GRW spacetime obeying the NCC (3-20), with equality holding only in isolated points of I and such that the Riemannian fiber M^n has $(c\tilde{u})$-parabolic universal Riemannian covering for some constant $c \neq 0$. Let $\psi : \Sigma^n \looparrowright \overline{M}^{n+1}$ be a complete spacelike mean curvature flow soliton with soliton constant c, lying in a timelike bounded region $\mathcal{B}_{t_1,t_2}$. If the hyperbolic angle Θ is bounded from below and the second soliton function $\tilde{\zeta}_c = |A|^2 + cf'(h)$ is nonnegative, then Σ^n is a slice of $\overline{M}^{n+1}$.*

PROOF. From (3-23), we get that $\Delta_{cu}(\varphi_K) \le 0$ on Σ^n. Thus, since we are assuming that $\psi : \Sigma^n \looparrowright -I \times_f M^n$ is contained in a timelike bounded region, Lemma 3.10 guarantees that Σ^n is (cu)-parabolic and, consequently, $\varphi_{\mathcal{K}}$ is constant on Σ^n. At this point, we can reason as in the last part of the proof of Theorem 3.4 to conclude that there is $t \in I$ such that Σ^n is a slice of $\overline{M}^{n+1}$. □

From Theorem 3.11, we obtain the following applications.

COROLLARY 3.12. *Let $\overline{M}^{n+1} = -I \times_{e^t} M^n$ be a steady-state-type spacetime whose Riemannian fiber M^n has positive Ricci curvature and $(c\tilde{u})$-parabolic universal Riemannian covering for some constant $c < 0$. Let $\psi : \Sigma^n \looparrowright \overline{M}^{n+1}$ be a complete spacelike mean curvature flow soliton with soliton constant c, lying in a timelike bounded region $\mathcal{B}_{t_1,t_2}$. If Θ is bounded from below and $h \ge \ln(-|A|^2/c)$, then Σ^n is the slice $\{\ln(-n/c)\} \times M^n$.*

COROLLARY 3.13. *Let $\overline{M}^{n+1} = -I \times_{e^t} M^n$ be a steady-state-type spacetime whose Riemannian fiber M^n has positive Ricci curvature and $(c\tilde{u})$-parabolic universal*

Riemannian covering for some constant $c > 0$. There is no complete spacelike mean curvature flow soliton lying in a timelike bounded region of $\overline{M}^{n+1}$ that has soliton constant c and is such that Θ is bounded from below.

COROLLARY 3.14. *Let $\overline{M}^{n+1} = -I \times M^n$ be a Lorentzian product space, whose Riemannian fiber M^n has positive Ricci curvature and $(c\tilde{u})$-parabolic universal Riemannian covering for some constant $c \neq 0$. There is no complete spacelike translating soliton in $\overline{M}^{n+1}$ that has soliton constant c and is such that Θ is bounded from below.*

Considering the *strong null convergence condition* (SNCC)

$$K_M \geq \sup_I (ff'' - f'^2), \tag{3-25}$$

which was introduced by Alías and Colares [7], where K_M denotes the sectional curvature of the Riemannian fiber M^n, and adding a suitable control to the growing of the height function through the second soliton function of a spacelike mean curvature flow soliton, we get the following version of Omori–Yau's maximum principle.

PROPOSITION 3.15. *Let $\overline{M}^{n+1} = -I \times_f M^n$ be a GRW spacetime obeying the SNCC (3-25), and let $\psi : \Sigma^n \looparrowright -I \times_f M^n$ be a complete spacelike mean curvature flow soliton with soliton constant $c \neq 0$. If the function $((n-1)f''(h) + cf(h)f'(h))/f(h)$ is bounded from below on Σ^n, then Omori–Yau's maximum principle holds for the drift Laplacian Δ_{cu}: that is, for $\varphi \in C^2(\Sigma^n)$ with $\sup_\Sigma \varphi < +\infty$, there exists a sequence of points $\{p_k\}_{k\geq 1}$ in Σ^n such that*

$$\lim_k \varphi(p_k) = \sup_\Sigma \varphi, \quad \lim_k |\nabla\varphi(p_k)| = 0 \quad \text{and} \quad \lim_k \Delta_{cu}\varphi(p_k) \leq 0.$$

PROOF. We recall that the curvature tensor R of Σ^n can be described in terms of its Weingarten operator A and the curvature tensor $\overline{R}$ of the ambient $-I \times_f M^n$ by the so-called Gauss equation, which is given by

$$\langle R(X,Y)Z, W\rangle = \langle \overline{R}(X,Y)Z, W\rangle - \langle AX, Z\rangle\langle AY, W\rangle + \langle AX, W\rangle\langle AY, Z\rangle, \tag{3-26}$$

for every tangential vector field $X, Y, Z \in \mathfrak{X}(\Sigma^n)$. Here, as in [35], the curvature tensor R is given by

$$R(X,Y)Z = \nabla_{[X,Y]}Z - [\nabla_X, \nabla_Y]Z,$$

where [,] denotes the Lie bracket and $X, Y, Z \in \mathfrak{X}(\Sigma^n)$.

We consider $X \in \mathfrak{X}(\Sigma^n)$ and take a (local) orthonormal frame $\{E_1, \ldots, E_n\}$. It follows from the Gauss equation (3-26) that the Ricci curvature Ric of Σ^n satisfies

$$\mathrm{Ric}(X,X) = \sum_i \langle \overline{R}(X,E_i)X, E_i\rangle + |AX|^2 + H\langle AX, X\rangle. \tag{3-27}$$

Thus, from (3-8) and (3-27), we get

$$\mathrm{Ric}(X,X) - c\nabla^2 u(X,X) \geq \sum_i \langle \overline{R}(X,E_i)X, E_i\rangle + cf'(h)|X|^2. \tag{3-28}$$

To estimate the first summand on the right-hand side of inequality (3-28), we consider $X^* = (\pi_M)_*(X)$ and $E_i^* = (\pi_M)_*(E_i)$. So, from [35, Proposition 7.42] and (3-3),

$$\sum_i \langle \overline{R}(X, E_i)X, E_i\rangle = \sum_i \langle R_M(X^*, E_i^*)X^*, E_i^*\rangle + (n-1)((\ln f)'(h))^2|X|^2 \\ - (n-2)(\ln f)''(h)\langle X, \nabla h\rangle^2 - (\ln f)''(h)|\nabla h|^2|X|^2, \qquad (3\text{-}29)$$

where R_M denotes the curvature tensor of the Riemannian fiber M^n. By writing $X^* = X + \langle X, \partial_t\rangle\partial_t$, we can estimate the first summand on the right-hand side of (3-29) to get

$$\sum_i \langle R_M(X^*, E_i^*)X^*, E_i^*\rangle = f^2(h)(|X^*|_M^2|E^*|_M^2 - \langle X^*, E^*\rangle_M^2)K_M(X^*, E^*) \\ \geq \frac{1}{f^2(h)}((n-1)|X|^2 + |\nabla h^2||X|^2 \\ + (n-2)\langle X, \nabla h\rangle^2) \min_i K_M(X^*, E_i^*). \qquad (3\text{-}30)$$

Consequently, since our ambient space obeys (3-25), from (3-30),

$$\sum_i \langle R_M(X^*, E_i^*)X^*, E_i^*\rangle \\ \geq ((n-1)|X|^2 + |\nabla h|^2|X|^2 + (n-2)\langle X, \nabla h\rangle^2)(\ln f)''(h). \qquad (3\text{-}31)$$

Substituting (3-31) into (3-29) gives

$$\sum_i \langle \overline{R}(X, E_i)X, E_i\rangle \geq ((n-1)|X|^2 + |\nabla h|^2|X|^2 + (n-2)\langle X, \nabla h\rangle^2)(\ln f)''(h) \\ + (n-1)((\ln f)'(h))^2|X|^2 - (n-2)(\ln f)''(h)\langle X, \nabla h\rangle^2 \\ - (\ln f)''(h)|\nabla h|^2|X|^2 \\ = (n-1)\frac{f''(h)}{f(h)}|X|^2. \qquad (3\text{-}32)$$

Hence, from (3-28) and (3-32), we obtain

$$\mathrm{Ric} - c\nabla^2 u \geq \Big((n-1)\frac{f''(h)}{f(h)} + cf'(h)\Big)\langle\ ,\ \rangle.$$

Therefore, since the right-hand side of the above inequality is bounded from below, we conclude our proof by applying [22, Theorem 1]. □

To proceed, we use Proposition 3.15 to establish the following result.

THEOREM 3.16. *Let* $\overline{M}^{n+1} = -I \times_f M^n$ *be a GRW spacetime obeying the SNCC (3-25), and let* $\psi : \Sigma^n \hookrightarrow -I \times_f M^n$ *be a complete spacelike mean curvature flow soliton with soliton constant* $c \neq 0$ *such that* $((n-1)f''(h) + cf(h)f'(h))/f(h)$ *is bounded from*

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below. If $\inf_\Sigma f(h) > 0$*, the second soliton function* $\tilde{\zeta}_c = |A|^2 + cf'(h)$ *is nonnegative and the height function h satisfies*

$$|\nabla h| \le \inf_{\Sigma^n} \tilde{\zeta}_c \quad \text{on} \quad \Sigma^n, \tag{3-33}$$

then Σ^n *is a slice of* $\overline{M}^{n+1}$.

PROOF. Since $\varphi_{\mathcal{K}} < 0$ on Σ^n, Proposition 3.15 ensures the existence of a sequence of points $\{p_k\}_{k\in\mathbb{N}} \subset \Sigma^n$ such that

$$\lim_{k\to+\infty} \varphi_{\mathcal{K}}(p_j) = \sup_{\Sigma^n} \varphi_{\mathcal{K}} \quad \text{and} \quad \lim_{k\to+\infty} \Delta_{cu}\, \varphi_{\mathcal{K}}(p_k) \le 0.$$

Hence, from (3-23), we get

$$0 \ge \lim_{k\to+\infty} \Delta_{cu}(\varphi_{\mathcal{K}})(p_j) = \sup_{\Sigma^n} \varphi_{\mathcal{K}} \lim_{k\to+\infty} \tilde{\zeta}_c(p_k) \ge 0. \tag{3-34}$$

But, since we are assuming that $\inf_\Sigma f(h) > 0$, we have that $\sup_{\Sigma^n} \varphi_{\mathcal{K}} < 0$. Consequently, from (3-34) we must have $\lim_{j\to+\infty} \tilde{\zeta}_c(p_k) = 0$, and hence $\inf_{\Sigma^n} \tilde{\zeta}_c = 0$. Therefore, the result follows from hypothesis (3-33). □

REMARK 3.17. We note that in Theorem 3.16 the hypotheses that the expression $((n-1)f''(h) + cf(h)f'(h))/f(h)$ is bounded from below and $\inf_\Sigma f(h) > 0$ are automatically satisfied if we assume that the spacelike mean curvature flow soliton lies in a timelike bounded region of the ambient spacetime.

From Theorem 3.16, we obtain the following applications.

COROLLARY 3.18. *Let* $\psi : \Sigma^n \looparrowright -\mathbb{R}^+ \times_t \mathbb{H}^n$ *be a complete spacelike mean curvature flow soliton with soliton constant* $c < 0$*. If* $\inf_\Sigma h > 0$*, the second soliton function* $\tilde{\zeta}_c = |A|^2 + c$ *is nonnegative and* $|\nabla h| \le \inf_{\Sigma^n} \tilde{\zeta}_c$*, then* Σ^n *is a slice* $\{\sqrt{-n/c}\} \times \mathbb{H}^n$.

COROLLARY 3.19. *There is no complete spacelike mean curvature flow soliton* $\psi : \Sigma^n \looparrowright -\mathbb{R}^+ \times_t \mathbb{H}^n$ *with soliton constant* $c > 0$ *such that* $\inf_\Sigma h > 0$ *and* $|\nabla h| \le \inf_{\Sigma^n} |A|^2 + c$.

Our next result can be regarded as a sort of extension of [22, Theorem 3].

THEOREM 3.20. *Let* $\overline{M}^{n+1} = -I \times_f M^n$ *be a GRW spacetime satisfying the SNCC (3-25). There is no complete spacelike mean curvature flow soliton immersed in* $\overline{M}^{n+1}$ *with soliton constant* $c \ne 0$ *such that* $cf'(h) \ge 0$ *and* $((n-1)f''(h) + cf(h)f'(h))/f(h)$ *is bounded from below.*

PROOF. Let us suppose, by contradiction, the existence of such a complete spacelike mean curvature flow soliton Σ^n immersed in $\overline{M}^{n+1}$. Since we are supposing that $cf'(h) \ge 0$ and that $\overline{M}^{n+1}$ satisfies the SNCC (3-25), we conclude from (3-18) and (3-22) that

$$\Delta_{cu}\varphi_{\mathcal{K}} \le |A|^2 \varphi_{\mathcal{K}}.$$

From the above equation, we get

$$\Delta_{cu}\varphi_{\mathcal{K}}^2 \geq 2\varphi_{\mathcal{K}}\Delta_{cu}\varphi_{\mathcal{K}} \geq 2|A|^2\varphi_{\mathcal{K}}^2.$$

Since $\varphi_{\mathcal{K}} = H/c$,

$$\Delta_{cu}H^2 \geq 2H^2|A|^2 \geq 2\frac{H^4}{n}. \tag{3-35}$$

With a straightforward computation, we can verify that

$$\Delta_{cu}\Big(\frac{-1}{\sqrt{1+H^2}}\Big) = \frac{\Delta_{cu}H^2}{2(1+H^2)^{3/2}} - \frac{3}{4}\frac{|\nabla H^2|^2}{(1+H^2)^{5/2}}. \tag{3-36}$$

Hence, from (3-35) and (3-36), we obtain

$$\Delta_{cu}\Big(\frac{-1}{\sqrt{1+H^2}}\Big) \geq \frac{H^4}{n(1+H^2)^{3/2}} - \frac{3}{4}\frac{|\nabla H^2|^2}{(1+H^2)^{5/2}}.$$

Therefore, since $((n-1)f''(h)+cf(h)f'(h))/f(h)$ is bounded from below, from Proposition 3.15 we can apply Omori–Yau's maximum principle and reason as in the proof of [22, Theorem 3] to conclude that $H \equiv 0$, which corresponds to an absurdity. □

From Theorem 3.20, we get the following nonexistence results.

COROLLARY 3.21. *There is no complete spacelike translating soliton with soliton constant $c \neq 0$ immersed in $-I \times M^n$ whose Riemannian fiber M^n has nonnegative sectional curvature.*

COROLLARY 3.22. *There is no complete mean curvature flow soliton with soliton constant $c > 0$, lying in a timelike bounded region of the steady-state-type spacetime $-\mathbb{R} \times_{e^t} M^n$, whose Riemannian fiber M^n has nonnegative sectional curvature.*

COROLLARY 3.23. *There is no complete spacelike mean curvature flow soliton with soliton constant $c > 0$ lying in a timelike bounded region of the Einstein–de Sitter spacetime $-\mathbb{R}^+ \times_{t^{2/3}} \mathbb{R}^n$.*

COROLLARY 3.24. *There is no complete mean curvature flow soliton with soliton constant $c \neq 0$ immersed in $-(-\pi/2, \pi/2) \times_{\cos t} \mathbb{H}^n \subset \mathbb{H}_1^{n+1}$ such that $c\sin(h) \leq 0$.*

REMARK 3.25. Fixing a constant $c \in \mathbb{R}$ with $0 < |c| < 1$, from [26, Example 4.4] we have that

$$\Sigma^n = \{(c\ln x_n, x_1, \dots, x_n) : x_n > 0\} \subset -\mathbb{R} \times \mathbb{H}^n$$

is a complete spacelike translating soliton of the mean curvature flow with respect to ∂_t that has soliton constant c and constant future mean curvature

$$H = \frac{c}{\sqrt{1-c^2}} = c\,\Theta.$$

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Moreover, we also get that

$$|\nabla h| = \frac{|c|}{\sqrt{1-c^2}} = |A|.$$

Hence, since the static GRW spacetime $-\mathbb{R} \times \mathbb{H}^n$ obeys neither the NCC (3-20) nor the SNCC (3-25), we can verify that it works as a counterexample related to our previous theorems. Consequently, we conclude that some hypothesis is needed.

Now, we deal with compact (without boundary) mean curvature flow solitons.

THEOREM 3.26. *Let $\overline{M}^{n+1} = -I \times_f M^n$ be a GRW spacetime and let $\psi : \Sigma^n \looparrowright \overline{M}^{n+1}$ be a compact mean curvature flow soliton with soliton constant $c \neq 0$. If $c > 0$, then*

$$\min_{\Sigma} H^2 \leq -cnf'(h_*) \quad \text{and} \quad \max_{\Sigma} H^2 \geq -cnf'(h^*),$$

where h_* and h^* are the minimum and maximum of the height function on Σ^n. Similarly, if $c < 0$, then

$$\min_{\Sigma} H^2 \leq -cnf'(h^*) \quad \text{and} \quad \max_{\Sigma} H^2 \geq -cnf'(h_*).$$

PROOF. From (3-8),

$$c\Delta u = -ncf'(h) - H^2. \tag{3-37}$$

We consider $c > 0$ and let p_0 be a minimum point of the height function h. Since a primitive g of f is an increasing function, we have that $h(p_0) = h_*$ is a minimum point of the function $u = g(h)$, and hence $\Delta u(p_0) \geq 0$. Thus, from (3-37), we get that

$$\min_{\Sigma} H^2 \leq H^2(p_0) \leq -ncf'(h_*).$$

Analogously, taking a maximum point of h, we are able to conclude that

$$\max_{\Sigma} H^2 \geq -cnf'(h^*).$$

The proof of the case $c < 0$ follows the same steps as the case $c > 0$. □

From the above result, we conclude directly the following nonexistence result.

COROLLARY 3.27. *There exists no compact spacelike translating soliton with soliton constant $c \neq 0$ immersed in $-I \times M^n$.*

We finish this subsection by establishing a rigidity result derived from Theorem 3.26.

COROLLARY 3.28. *Let $\overline{M}^{n+1} = -I \times_f M^n$ be a GRW spacetime and let $\psi : \Sigma^n \looparrowright \overline{M}^{n+1}$ be a compact mean curvature flow soliton with soliton constant $c \neq 0$. Assume that $f''(t) \leq 0$ for $h_* \leq t \leq h^*$, where h_* and h^* are the minimum and maximum on Σ^n of its height function h, respectively. If H is constant, then Σ^n is a slice of $\overline{M}^{n+1}$.*

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PROOF. Indeed, since $f''(t) \leq 0$, we have that f' is nondecreasing. In addition, since H is constant, from Theorem 3.26 we conclude that

$$-ncf'(t) = H^2$$

for $h_* \leq t \leq h^*$. Thus, from the above equation jointly with (3-37), we have that $\Delta u = 0$. Therefore, since Σ^n is compact, we conclude that u is constant, which means that Σ^n is a slice of $\overline{M}^{n+1}$. □

3.3. The spacelike mean curvature flow soliton equation. Let $\Omega \subseteq M^n$ be a connected domain and let $z \in C^\infty(\Omega)$ be a smooth function such that $z(\Omega) \subseteq I$. Then $\Sigma^n(z)$ will denote the (vertical) graph over Ω determined by z, that is,

$$\Sigma^n(z) = \{(z(p), p) : p \in \Omega\} \subset \overline{M}^{n+1} = -I \times_f M^n.$$

The graph is said to be entire if $\Omega = M^n$. Observe that $h(z(p), p) = z(p)$, $p \in \Omega$. Hence, h and z can be identified in a natural way. The metric induced on Ω from the Lorentzian metric of the ambient GRW spacetime via $\Sigma^n(z)$ is

$$g_z = -dz^2 + f^2(z)g_M. \tag{3-38}$$

It follows from (3-38) that a graph $\Sigma^n(z)$ is a spacelike hypersurface if and only if $|Dz|_M < f(z)$, where Dz stands for the gradient of z in M^n and $|Dz|_M$ denotes its norm, both with respect to the metric g_M. On the other hand, in the case where M^n is a simply connected manifold, from [11, Lemma 3.1] we have that every complete spacelike hypersurface $\psi : \Sigma^n \looparrowright -I \times_f M^n$ such that the warping function f is bounded on Σ^n is an entire spacelike graph over M^n. In particular, this happens for complete spacelike hypersurfaces lying in a timelike bounded region of $-I \times_f M^n$. It is also interesting to point out that, in contrast to the case of graphs in a Riemannian space, an entire spacelike graph $\Sigma^n(z)$ in a GRW spacetime is not necessarily complete, in the sense that the induced Riemannian metric (3-38) is not necessarily complete on M^n. For example, Albujer [1, Section 3] constructed explicit examples of noncomplete entire maximal spacelike graphs (that is, whose mean curvature is identically zero) in the Lorentzian product space $-\mathbb{R} \times \mathbb{H}^2$.

The future-pointing Gauss map of a spacelike graph $\Sigma^n(z)$ over Ω is given by the vector field

$$N(p) = \frac{f(z(p))}{\sqrt{f^2(z(p)) - |Dz(p)|_M^2}}\left(\partial_t|_{(z(p),p)} + \frac{Dz(p)}{f^2(z(p))}\right) \quad \forall p \in \Omega. \tag{3-39}$$

From (3-39), we have that the shape operator related to the future-pointing Gauss map (3-39) is given by

$$\begin{aligned} AX = &-\frac{1}{f(u)\sqrt{f^2(z)-|Dz|_M^2}}D_X Dz - \frac{f'(z)}{\sqrt{f^2(z)-|Dz|_M^2}}X \\ &+\Big(\frac{-g_M(D_X Dz, Dz)}{f(z)(f^2(z)-|Dz|_M^2)^{3/2}} + \frac{f'(z)g_M(Dz,X)}{(f^2(z)-|Dz|_M^2)^{3/2}}\Big)Dz, \end{aligned} \tag{3-40}$$

for any vector field X tangential to Ω, where D denotes the Levi–Civita connection of (M^n, g_M). Consequently, if $\Sigma^n(z)$ is a spacelike graph defined over a domain $\Omega \subseteq M^n$, it is not difficult to verify from (3-40) that the future mean curvature function $H(z)$ of $\Sigma^n(z)$ is given by

$$H(z) = \operatorname{div}_M\Big(\frac{Dz}{nf(z)\sqrt{f^2(z)-|Dz|_M^2}}\Big) + \frac{f'(z)}{n\sqrt{f^2(z)-|Dz|_M^2}}\Big(n + \frac{|Dz|_M^2}{f^2(z)}\Big), \tag{3-41}$$

where div_M stands for the divergence operator computed in the metric g_M.

Hence, from (2-6) and (3-41), we have that $\Sigma^n(z)$ is a spacelike mean curvature flow soliton with respect to $K = f(t)\partial_t$ and with soliton constant c if and only if $|Dz|_M < f(z)$ and z is a solution of the nonlinear differential equation

$$\operatorname{div}_M\Big(\frac{Dz}{f(z)\sqrt{f(z)^2-|Dz|_M^2}}\Big) = -\frac{1}{\sqrt{f(z)^2-|Dz|_M^2}}\Big\{cf(z)^2 + f'(z)\Big(n + \frac{|Dz|_M^2}{f(z)^2}\Big)\Big\}. \tag{3-42}$$

We say that $z \in C^\infty(M)$ has finite C^2 norm when

$$\|z\|_{C^2(M)} := \sup_{|k|\le 2}|D^k z|_{L^\infty(M)} < +\infty.$$

In this context, we obtain the following result.

Theorem 3.29. *Let $\overline{M}^{n+1} = -I \times_f M^n$ be a GRW spacetime obeying the SNCC (3-20), with equality occurring only in isolated points of I and whose Riemannian fiber M^n is complete. Let $z \in C^\infty(M)$ be an entire solution of Equation (3-42) for $c \neq 0$, with finite C^2 norm, such that $|Dz|_M \le \alpha f(z)$, for some constant $0 < \alpha < 1$, and the second soliton function $\zeta_c(z) = |A|^2 + cf'(z)$ is nonnegative. If $|Dz|_M \in \mathcal{L}^1(M)$, then $z \equiv t_*$ for some $t_* \in I$, which is implicitly given by the condition $\zeta(t_*) = 0$.*

Proof. Let $z \in C^\infty(M)$ be such a solution of Equation (3-42). It follows from (3-40) that the shape operator A of $\Sigma^n(z)$ is bounded provided that z has finite C^2. We note also that the finiteness of the C^2 norm of z implies, in particular, that z is bounded, which, in turn, guarantees that $\Sigma^n(z)$ is contained in a bounded timelike region of $\overline{M}^{n+1}$. Consequently, since we are also assuming that $|Dz|_M \le \alpha f(z)$, for some constant $0 < \alpha < 1$, we get that

$$|Dz|_M^2 \le f^2(z) - \beta$$

for $\beta = (1-\alpha^2)\inf_{\Sigma(z)} f^2(z)$. Thus, we can apply [3, Proposition 1] to conclude that $\Sigma^n(z)$ is complete.

We also have that $N = N^* - \Theta\partial_t$, where N^* denotes the projection of N onto the fiber M^n. Consequently, from (3-39), we get

$$|\nabla z|^2 = \langle N^*, N^*\rangle = f^2(z)\langle N^*, N^*\rangle_M. \tag{3-43}$$

Thus, from (3-39) and (3-43), we obtain

$$|\nabla z|^2 = \frac{|Dz|_M^2}{f^2(z) - |Dz|_M^2}. \tag{3-44}$$

On the other hand, it follows from (3-38) that $d\Sigma = \sqrt{|G|}dM$, where dM and $d\Sigma^n$ stand for the Riemannian volume elements of (M^n, g_M) and $(\Sigma^n(z), g_z)$, respectively, and $G = det(g_{ij})$ with

$$g_{ij} = g_z(E_i, E_j) = f^2(z)\delta_{ij} - E_i(z)E_j(z).$$

Here, $\{E_1, \dots, E^n\}$ denotes a local orthonormal frame with respect to the metric g_M. So, it is not difficult to verify that

$$|G| = f^{2(n-1)}(z)(f^2(z) - |Dz|_M^2).$$

Consequently,

$$d\Sigma = f^{n-1}(z)\sqrt{f^2(z) - |Dz|_M^2}dM. \tag{3-45}$$

Thus, from (3-44) and (3-45), we get

$$|\nabla z| d\Sigma = f(z)^{n-1}|Dz|_M dM. \tag{3-46}$$

Hence, since z is bounded and $|Dz|_M \in \mathcal{L}^1(M)$, from Equation (3-46) we conclude that $|\nabla z| \in \mathcal{L}^1(\Sigma^n(z))$. Consequently, from (3-9) we get that $|\nabla(\varphi_{\mathcal{K}})| \in \mathcal{L}^1_{cu}(\Sigma^n(z))$. Therefore, we can reason as in the last part of the proof of Theorem 3.4 to conclude the result. □

From Theorem 3.11, we obtain the following consequence.

THEOREM 3.30. *Let $\overline{M}^{n+1} = -I \times_f M^n$ be a GRW spacetime obeying the SNCC (3-20), with equality occurring only in isolated points of I and whose Riemannian fiber M^n is complete with a $(c\tilde{u})$-parabolic universal Riemannian covering for some constant $c \neq 0$. If $z \in C^\infty(M)$ is an entire solution of Equation (3-42) for c, with finite C^1 norm, such that $|Dz|_M \leq \alpha f(z)$, for some constant $0 < \alpha < 1$, and the second soliton function $\tilde{\zeta}_c(z) = |A|^2 + cf'(z)$ is nonnegative, then $z \equiv t_*$ for some $t_* \in I$, which is implicitly given by the condition $\zeta(t_*) = 0$.*

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PROOF. Observing that h satisfies (3-2) and (3-43), from (3-40) we obtain

$$|\nabla h|^2 = \frac{|Dz|_M^2}{f(z)^2 - |Dz|_M^2}. \tag{3-47}$$

Hence, since we are assuming that z has finite C^1 norm and taking into account once more that $\Theta^2 = |\nabla h|^2 + 1$, with the aid of (3-47) we conclude that Θ is bounded. Therefore, the result follows by applying Theorem 3.11. □

From Theorem 3.16, we also obtain the following result.

THEOREM 3.31. *Let $\overline{M}^{n+1} = -I \times_f M^n$ be a GRW spacetime obeying the SNCC (3-20) whose Riemannian fiber M^n is complete. Let $z \in C^\infty(M)$ be a bounded entire solution of Equation (3-42) for some constant $c \neq 0$ such that $|Dz|_M \leq \alpha f(z)$, for some constant $0 < \alpha < 1$, and the second soliton function $\tilde{\zeta}_c(z) = |A|^2 + cf'(z)$ is nonnegative. If*

$$|Dz|_M \leq \inf_M \tilde{\zeta}_c, \tag{3-48}$$

then $z \equiv t_*$ for some $t_* \in I$, which is implicitly given by the condition $\zeta(t_*) = 0$.

PROOF. From (3-47) and (3-48), we see that hypothesis (3-33) is satisfied. Therefore, the result follows by applying Theorem 3.16. □

We close this subsection with the following application of Theorem 3.20

THEOREM 3.32. *Let $\overline{M}^{n+1} = -I \times_f M^n$ be a GRW spacetime obeying the SNCC (3-25) whose Riemannian fiber M^n is complete. For any constant $c \neq 0$, there is no bounded entire solution $z \in C^\infty(M)$ of Equation (3-42) such that $|Dz|_M \leq \alpha f(z)$, for some constant $0 < \alpha < 1$, and $cf'(z) \geq 0$.*

4. Further rigidity and topological results

Given a smooth function $\varphi : \Sigma^n \to \mathbb{R}$ and a $(0, 2)$-tensor T defined on a Riemannian manifold Σ^n, we recall that the φ-divergence of T is given by

$$\operatorname{div}_\varphi T = e^\varphi \operatorname{div}(e^{-\varphi} T) = \operatorname{div} T - T(\nabla\varphi, .). \tag{4-1}$$

From Equation (4-1), we get the following auxiliary result.

LEMMA 4.1. *Given a $(0, 2)$-tensor T and smooth functions $v, \varphi : \Sigma^n \to \mathbb{R}$,*

$$\operatorname{div}_\varphi(T(\nabla v)) = \langle T, \nabla^2 v\rangle + (\operatorname{div}_\varphi T)(\nabla v).$$

PROOF. Indeed, let $\{E_i\}$ be a geodesic referential at a point $p \in \Sigma^n$. Then

$$\begin{aligned}(\operatorname{div} T)(\nabla v) =& E_i(T(E_i, \nabla v)) - T(E_i, \nabla_{E_i} \nabla v)\\ &= \operatorname{div}[T(\nabla v)] - \langle T(E_i), \nabla^2 v(E_i)\rangle\\ &= \operatorname{div}[T(\nabla v)] - \langle T, \nabla^2 v\rangle.\end{aligned}$$

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Hence, since $\mathrm{div}_\varphi(T(\nabla v)) = \mathrm{div}(T(\nabla v)) - T(\nabla v, \nabla\varphi)$, we get the desired result from (4-1). □

We also need the following lemma, the proof of which can be found in [20, Equation (3.8)].

LEMMA 4.2. *Given a smooth function $v : \Sigma^n \to \mathbb{R}$, we have that*

$$\mathrm{div}(\nabla^2 v) = \nabla\Delta v + \mathrm{Ric}(\nabla v),$$

where Ric(.) *stands for the linear operator metrically equivalent to the Ricci tensor of* Σ^n.

We recall that the traceless tensor associated with a tensor T is defined by

$$\overset{\circ}{T} = T - \frac{\mathrm{tr}(T)}{n}\langle\, ,\rangle.$$

With this notation, from Equation (3-8), we can verify the equation

$$H\overset{\circ}{A} = c\,\overset{\circ}{\nabla^2} u,$$

where $u = g(h)$.

Now, we are able to state and prove our next result.

THEOREM 4.3. *Let $\overline{M}^{n+1} = -I \times_f M^n$ be a GRW spacetime and let $\psi : \Sigma^n \looparrowright \overline{M}^{n+1}$ be a compact spacelike mean curvature flow soliton with soliton constant $c \neq 0$. If $H^2 \geq n^2 f''(h)/2f(h)$ and $\mathrm{Ric}(\nabla h, \nabla h) \geq 0$, then Σ^n is a slice of $\overline{M}^{n+1}$.*

PROOF. We recall that, from (3-8),

$$HA - c\nabla^2 u = cf'(h)\langle\, ,\rangle. \tag{4-2}$$

Taking the trace of Equation (4-2), we get that

$$c\Delta u = -ncf'(h) - H^2. \tag{4-3}$$

On the other hand, taking the divergence of (4-2) and using the relationship given in Lemma 4.2, we obtain the divergence of HA as

$$\mathrm{div}HA - c\nabla\Delta u - c\mathrm{Ric}(\nabla u) = cf''(h)\nabla h. \tag{4-4}$$

Inserting (4-3) into (4-4) gives

$$\mathrm{div}HA = -cf''(h)\nabla h(n-1) - \nabla H^2 + c\mathrm{Ric}(\nabla u). \tag{4-5}$$

Since $H\overset{\circ}{A} = H(A - trA/n\langle\, ,\rangle) = H(A + H/n\langle\, ,\rangle)$, we reach

$$\mathrm{div}(H\overset{\circ}{A}) = \mathrm{div}(HA) + \frac{\nabla H^2}{n}. \tag{4-6}$$

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Hence, from (4-6) and (4-5), we obtain

$$\operatorname{div} H\mathring{A} = -cf''(h)\nabla h(n-1) - \frac{(n-1)}{n}\nabla H^2 + c\operatorname{Ric}(\nabla u). \tag{4-7}$$

Therefore, from (3-9) and (4-7), we deduce that

$$\operatorname{div} H\mathring{A} + 2\frac{(n-1)c}{n}H\,\mathring{A}\,(\nabla u) = \frac{2(n-1)c}{n^2}H^2\nabla u - cf''(h)\nabla h(n-1) + c\operatorname{Ric}(\nabla u).$$

From now on, we take $\varphi = -2(n-1)c/nu$. So, we conclude that

$$(\operatorname{div}_\varphi H\mathring{A})(c\nabla u) = \frac{2(n-1)c^2}{n^2}H^2|\nabla u|^2 - c^2 f''(h)f(h)|\nabla h|^2(n-1) + c^2\operatorname{Ric}(\nabla u, \nabla u). \tag{4-8}$$

But, from Lemma 4.1,

$$\begin{aligned}\operatorname{div}_\varphi[H\mathring{A}(c\nabla u)] &= \langle H\mathring{A}, c\nabla^2 u\rangle + (\operatorname{div}_\varphi H\mathring{A})(c\nabla u)\\ &= |H\,\mathring{A}\,|^2 + (\operatorname{div}_\varphi H\mathring{A})(c\nabla u).\end{aligned} \tag{4-9}$$

Thus, from Equations (4-9) and (4-8), we obtain

$$\int_{\Sigma^n}\Big((n-1)c^2|\nabla u|^2\Big(\frac{-f''(h)}{f(h)} + \frac{2H^2}{n^2}\Big) + c^2\operatorname{Ric}(\nabla u, \nabla u)\Big)e^{-\varphi}\,d\Sigma \le 0, \tag{4-10}$$

with equality holding if and only if Σ^n is totally umbilical. Hence, from our hypothesis we conclude that Σ^n is totally umbilical and that $\operatorname{Ric}_\Sigma(\nabla u, \nabla u) = 0$.

Consequently, since Σ^n is totally umbilical, we conclude from (4-9) that ∇u is a conformal vector field on Σ^n with

$$\nabla^2 u = \frac{\Delta u}{n}\langle\,,\rangle. \tag{4-11}$$

From Lemma 4.2 and Equation (4-11),

$$0 = \operatorname{Ric}(\nabla u, \nabla u) = -\frac{(n-1)}{n}\langle\nabla\Delta u, \nabla u\rangle. \tag{4-12}$$

Therefore, from (4-12) jointly with Stokes' theorem, we deduce that $\Delta u = 0$, and hence we conclude that h is constant on Σ^n. □

Proceeding, we present our next rigidity result for compact spacelike mean curvature flow solitons in a GRW spacetime.

THEOREM 4.4. *Let $\overline{M}^{n+1} = -I \times_f M^n$ be a GRW spacetime satisfying the SNCC (3-25) and let $\psi : \Sigma^n \looparrowright \overline{M}^{n+1}$ be a compact spacelike mean curvature flow soliton with soliton constant $c \neq 0$. If $\mathfrak{L}_{\nabla u}(H) \le 0$, where $\mathfrak{L}_{\nabla u}$ stands for the Lie derivative on Σ^n with respect to ∇u, then Σ^n is a slice of $\overline{M}^{n+1}$.*

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PROOF. From (3-27), (3-32) and (3-8), we conclude that

$$\begin{aligned}\mathrm{Ric}(\nabla u, \nabla u) - (n-1)\frac{f''(h)}{f(h)}|\nabla u|^2 &\geq H\langle A\nabla u, \nabla u\rangle \\ &= \frac{H}{c}\langle \nabla H, \nabla u\rangle.\end{aligned}$$

Since H/c is negative and $\mathfrak{L}_{\nabla u}(H) = \langle \nabla H, \nabla u\rangle \leq 0$, we conclude, from (4-10) and the above inequality, that Σ^n is totally umbilical.

Since Σ^n is totally umbilical, from (3-9),

$$cH\nabla u + n\nabla H = 0.$$

Thus,

$$cH\langle \nabla u, \nabla u\rangle = -n\langle \nabla H, \nabla u\rangle \geq 0.$$

Now, taking into account that $cH < 0$, we conclude from the above inequality that $\nabla u = 0$. Therefore, Σ^n must be a slice of $\overline{M}^{n+1}$. □

Motivated by [43, Theorem 10], we establish the following diameter estimate.

THEOREM 4.5. *Let $\overline{M}^{n+1} = -I \times_f M^n$ be a GRW spacetime and let $\psi : \Sigma^n \looparrowright \overline{M}^{n+1}$ be a compact spacelike mean curvature flow soliton with soliton constant $c \neq 0$ such that the mean curvature is not constant. If $f' \neq 0$, then we have the lower estimate for the diameter of Σ^n given by*

$$\mathrm{diam}(\Sigma) \geq |c|\frac{u_{\max} - u_{\min}}{\sqrt{H^2_{\max} - H^2_{\min}}},$$

where $H^2_{\max}$ and $H^2_{\min}$ are the maximum and minimum values of H^2 on Σ^n, respectively.

PROOF. First, from (3-4) we get that

$$-H^2 + c^2|\nabla u|^2 = -c^2 f^2(h). \tag{4-13}$$

Let $p \in \Sigma^n$ be a minimum point of $f^2(h)$. Then

$$c^2 f^2(h(p)) = H^2(p). \tag{4-14}$$

On the other hand,

$$c^2 f^2(h(p)) \leq c^2 f^2(h(x)) = H^2(x) - c^2|\nabla u|^2(x),$$

for any $x \in \Sigma^n$. From the above inequality and (4-14), we conclude that

$$H^2(x) - H^2(p) \geq c^2|\nabla u|^2(x). \tag{4-15}$$

In particular, p is a minimum point of H^2, that is, $H^2 = H^2_{\min}$. From (4-15) we conclude that

$$H^2_{\max} - H^2_{\min} \geq c^2|\nabla u|^2.$$

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From the above equation and the mean value theorem,

$$u_{\max} - u_{\min} \leq \max_{\Sigma} |\nabla u| \operatorname{diam}(\Sigma) \leq \frac{\sqrt{H_{\max}^2 - H_{\min}^2}}{|c|} \operatorname{diam}(\Sigma),$$

and hence we have the desired estimate. □

From the previous theorem, we obtain the following consequence.

COROLLARY 4.6. *Let $\overline{M}^{n+1} = -I \times_f M^n$ be a GRW spacetime and let $\psi : \Sigma^n \looparrowright \overline{M}^{n+1}$ be a compact spacelike mean curvature flow soliton with soliton constant $c \neq 0$ such that the mean curvature is constant. If $f'(h) \neq 0$, then Σ^n is a slice of $\overline{M}^{n+1}$.*

Our next result provides a sufficient condition to guarantee that a compact spacelike mean curvature flow soliton is totally umbilical.

THEOREM 4.7. *Let $\overline{M}^{n+1} = -I \times_f M^n$ be a GRW spacetime and let $\psi : \Sigma^n \looparrowright \overline{M}^{n+1}$ be a compact spacelike mean curvature flow soliton with soliton constant $c \neq 0$. If*

$$\int_{\Sigma^n} \operatorname{Ric}(\nabla u, \nabla u)\, d\Sigma \geq \frac{n-1}{nc^2} \int_{\Sigma} (H^2 + ncf'(h))^2\, d\Sigma,$$

then ∇u is a conformal vector field and, therefore, Σ^n is totally umbilical. Moreover, if H is constant, then Σ^n is a slice of $\overline{M}^{n+1}$.

PROOF. From Bochner's formula [16],

$$\frac{\Delta |\nabla u|^2}{2} = \operatorname{Ric}(\nabla u, \nabla u) + \langle \nabla \Delta u, \nabla u \rangle + |\nabla^2 u|^2. \tag{4-16}$$

But, we note that

$$|\nabla^2 u|^2 = |\mathring{\nabla}^2 u|^2 + \frac{(\Delta u)^2}{n}.$$

Substituting the above equality into (4-16) and applying Stokes' theorem gives

$$\int_{\Sigma^n} |\mathring{\nabla}^2 u|^2\, d\Sigma = \frac{n-1}{n} \int_{\Sigma^n} (\Delta u)^2\, d\Sigma - \int_{\Sigma^n} \operatorname{Ric}(\nabla u, \nabla u)\, d\Sigma.$$

From our hypothesis on the Ricci tensor and (4-3), we conclude that ∇u is a conformal vector field. Moreover, since $c \neq 0$, from equality (4-13) we have that $H \neq 0$. Finally, from (4-9), we conclude that Σ^n is totally umbilical.

On the other hand, since Σ^n is totally umbilical, from (3-9),

$$cH\nabla u + n\nabla H = 0.$$

From the above equation, a straightforward calculation shows that $He^{c/nu}$ is constant. Thus, supposing that H is a constant, we conclude that Σ^n is a slice of $\overline{M}$. □

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Inspired by [36], we also obtain the following result.

PROPOSITION 4.8. *Let $\overline{M}^{n+1} = -I \times_f M^n$ be a GRW spacetime and let $\psi : \Sigma^n \looparrowright \overline{M}^{n+1}$ be a compact spacelike mean curvature flow soliton with soliton constant $c \neq 0$. Suppose that Σ^n is totally umbilical and* Ric *is nonpositive. Then Σ^n is a slice of $\overline{M}^{n+1}$.*

PROOF. Using the same ideas as in the previous theorem, we conclude that ∇u is a conformal vector field. Thus, from (4-12),

$$\langle \nabla u, \nabla \Delta u \rangle \geq 0.$$

Applying Stokes' theorem, we conclude that

$$\int_{\Sigma^n} (\Delta u)^2 \, d\Sigma = 0$$

and, therefore, h is constant. □

The last result of this section provides an interesting topological characterization for a compact spacelike mean curvature flow soliton.

THEOREM 4.9. *Let $\overline{M}^{n+1} = -I \times_f M^n$ be a GRW spacetime and let $\psi : \Sigma^n \looparrowright \overline{M}^{n+1}$ be a compact spacelike mean curvature flow soliton with soliton constant $c \neq 0$. Suppose that Σ^n is totally umbilical and it is not contained in a slice of $\overline{M}^{n+1}$. If Σ^n has finite fundamental group, then Σ^n is diffeomorphic to an Euclidean sphere.*

PROOF. Since we are assuming that Σ^n is totally umbilical and that Σ^n it is not contained in a slice, we conclude that ∇u is a non-Killing conformal vector field. Therefore, since Σ^n has finite fundamental group, we conclude the desired result from [42, Theorem 2]. □

5. Stability of spacelike mean curvature flow solitons in GRW spacetimes

Let $\psi : \Sigma^n \looparrowright -I \times_f M^n$ be a complete spacelike mean curvature flow soliton with soliton constant c. We recall that a *variation with compact support and fixed boundary* of $\psi : \Sigma^n \looparrowright -I \times_f M^n$ is a smooth mapping

$$F : (-\epsilon, \epsilon) \times \Sigma^n \to -I \times_f M^n \tag{5-1}$$

such that:

(i) for $s \in (-\epsilon, \epsilon)$, the map $F_s : \Sigma^n \looparrowright -I \times_f M^n$ given by $F_s(q) = F(s, q)$ is an spacelike immersion with $F_0 = x$; and
(ii) $F_s|_{\partial\Sigma} = \psi|_{\partial\Sigma}$ for all $s \in (-\epsilon, \epsilon)$.

In all that follows, we let dM_s denote the volume element of the metric induced on Σ^n by F_s and let N_s denote the unit normal vector field along F_s. Moreover, we also consider in Σ^n the weighted volume form given by $d\mu_s = e^{-f} dM_s$. When $s = 0$, all of these objects coincide with the ones defined in Σ^n.

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The *variational field* associated to the variation F is the vector field $\partial F/\partial s|_{s=0}$. Letting

$$u_s = -\left\langle \frac{\partial F}{\partial s}, N_s \right\rangle,$$

we get

$$\frac{\partial F}{\partial s}\bigg|_{s=0} = u_0 N + \left(\frac{\partial F}{\partial s}\bigg|_{s=0} \right)^{\top},$$

where $(\cdot)^{\top}$ stands for tangential components.

Denoting the set of all smooth functions on Σ^n with compact support by $C_0^\infty(\Sigma^n)$, according to [14, Lemma 2.1] and [13, Lemma 2.1], every function $\varphi \in C_0^\infty(\Sigma^n)$ with

$$\int_{\Sigma^n} \varphi \, d\Sigma = 0 \tag{5-2}$$

induces a variation of $\psi : \Sigma^n \looparrowright -I \times_f M^n$ of the type (5-1), with variational normal field $\partial F/\partial s|_{s=0} = \varphi N$ and with *first variation* $\delta_\varphi \mathbb{A}$ of the *area functional*

$$\mathbb{A} : (-\epsilon, \epsilon) \to \mathbb{R}$$
$$s \mapsto \mathbb{A}(s) = \mathrm{Area}(F_s(\Sigma^n)) = \int_{\Sigma^n} d\Sigma_s,$$

given by

$$\delta_\varphi \mathbb{A} = \frac{d\mathbb{A}}{ds}(0) = \int_{\Sigma^n} \varphi H \, d\Sigma. \tag{5-3}$$

Here, N stands for a normal unit vector field globally defined on Σ^n, $d\Sigma_s$ denotes the volume element of Σ^n with respect to the metric induced by $F_s : \Sigma^n \looparrowright -I \times_f M^n$ and H is the mean curvature function of $\psi : \Sigma^n \looparrowright -I \times_f M^n$ with respect to N.

As a consequence of (5-3), *maximal* compact spacelike mean curvature flow solitons of $-I \times_f M^n$ (that is, with mean curvature identically zero) are characterized as critical points of the area functional $\mathbb{A}$, whereas any compact spacelike mean curvature flow soliton $\psi : \Sigma^n \looparrowright -I \times_f M^n$ with constant mean curvature H is a critical point of $\mathbb{A}$ restricted to functions $\varphi \in C^\infty(\Sigma^n)$ that satisfy condition (5-2). Geometrically, this additional condition means that the variations under consideration preserve a certain volume functional (for more details, see [13]).

For these critical points, [14, Proposition 2.3] asserts that the stability of the corresponding variational problem is given by the second variation of the area functional $\mathbb{A}$, which is given by

$$\delta_\varphi^2 \mathbb{A} = \frac{d^2\mathbb{A}}{ds^2}(0)(\varphi) = \int_{\Sigma^n} \{\Delta(\varphi) - \{\overline{\mathrm{Ric}}(N, N) + |A|^2\}\varphi\}\varphi \, d\Sigma,$$

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where Δ stands for the Laplacian operator on Σ^n, $\overline{\mathrm{Ric}}$ is the Ricci tensor of the GRW spacetime $-I\times_f M^n$ and $|A|$ denotes the length of the shape operator A of $\psi:\Sigma^n \looparrowright -I\times_f M^n$ with respect to N. In this setting, we establish the following definition.

DEFINITION 5.1. A compact spacelike mean curvature flow soliton $\psi:\Sigma^n \looparrowright -I\times_f M^n$ with constant mean curvature H is said to be strongly stable if $\delta^2_\varphi \mathbb{A}\le 0$ for every $\varphi\in C^\infty(\Sigma^n)$.

In our next result, we impose a suitable behavior on the warping function f to obtain a nonexistence result for strongly stable spacelike mean curvature flow solitons immersed in $-I\times_f M^n$.

THEOREM 5.2. *There is no strongly stable compact spacelike mean curvature flow soliton $\psi:\Sigma^n \looparrowright -I\times_f M^n$ with soliton constant $c\neq 0$ whose mean curvature H is constant and whose height function h satisfies $cf'(h)f(h)+nf''(h)>0$ on Σ^n.*

PROOF. By contradiction, let us suppose the existence of such a soliton $\psi:\Sigma^n \looparrowright -I\times_f M^n$. From the first equation in the proof of [19, Proposition 2.1], we have

$$\begin{aligned}\Delta(\varphi_{\mathcal{K}}) &= \{\overline{\mathrm{Ric}}(N,N)+|A|^2\}\varphi_{\mathcal{K}} - \{nN(f'(h)) - Hf'(h)\} + \langle \mathcal{K},\nabla H\rangle\\ &= \{\overline{\mathrm{Ric}}(N,N)+|A|^2\}\varphi_{\mathcal{K}} + c\,\varphi_{\mathcal{K}} f'(h) - nN(f'(h)),\end{aligned} \tag{5-4}$$

where $\varphi_{\mathcal{K}}\in C^\infty(\Sigma^n)$ is the support function defined in (3-10).

On the other hand, we also have

$$N(f'(h)) = -\langle f''(h)\partial_t, N\rangle = -\frac{f''(h)}{f(h)}\varphi_{\mathcal{K}}. \tag{5-5}$$

Thus, from (5-4) and (5-5), we get

$$\begin{aligned}\Delta(\varphi_{\mathcal{K}}) &= \{\overline{\mathrm{Ric}}(N,N)+|A|^2\}\varphi_{\mathcal{K}} + c\,\varphi_{\mathcal{K}} f'(h) - nN(f'(h))\\ &= \{\overline{\mathrm{Ric}}(N,N)+|A|^2\}\varphi_{\mathcal{K}} + \left\{c\,f'(h) + n\,\frac{f''(h)}{f(h)}\right\}\varphi_{\mathcal{K}}.\end{aligned} \tag{5-6}$$

Moreover, since $H=c\,\varphi_{\mathcal{K}}$ is constant and $c\neq 0$, we have that $\varphi_{\mathcal{K}}$ is also constant on Σ^n. Hence, from (5-6), we obtain

$$-\{\overline{\mathrm{Ric}}(N,N)+|A|^2\}\varphi_{\mathcal{K}} = \left\{c\,f'(h) + n\,\frac{f''(h)}{f(h)}\right\}\varphi_{\mathcal{K}}. \tag{5-7}$$

Now, from our hypothesis of strong stability and taking into account Definition 5.1, we currently have

$$\delta^2_\varphi \mathbb{A} = \int_{\Sigma^n}\{\Delta(\varphi) - \{\overline{\mathrm{Ric}}(N,N)+|A|^2\}\varphi\}\varphi\, d\Sigma \le 0$$

for every $\varphi\in C^\infty(\Sigma^n)$.

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Thus, making $\varphi = \varphi_{\mathcal{K}} < 0$ (see (3-11)), from the hypothesis that $cf'(h)f(h) + nf''(h) > 0$ jointly with (5-7) we get

$$0 < \int_{\Sigma^n} \left\{ c\, f'(h) + n\,\frac{f''(h)}{f(h)} \right\} \varphi_{\mathcal{K}}^2 = \int_{\Sigma^n} \{ \underbrace{\Delta(\varphi_{\mathcal{K}})}_{0} - \{ \overline{\mathrm{Ric}}(N,N) + |A|^2 \} \varphi_{\mathcal{K}} \} \varphi_{\mathcal{K}} \, d\Sigma \leq 0,$$

and we reach an absurdity. □

From Theorem 5.2, we get the following application.

COROLLARY 5.3. *There is no strongly stable compact spacelike mean curvature flow soliton in a steady-state-type spacetime* $-I \times_{e^t} M^n$ *with soliton constant* $c > 0$ *and constant mean curvature.*

In what follows, we consider the function

$$\bar{u} = -g(\pi_I) \in C^\infty(-I \times_f M^n),$$

where $g : I \to \mathbb{R}$ is the primitive of the warping function f that was used to define the reparametrization $u = -g(h)$ of the height function h of the spacelike mean curvature flow soliton $\psi : \Sigma^n \hookrightarrow -I \times_f M^n$ (see (3-6)). From (3-1), we observe that $\bar{u} = u$ on Σ^n, and hence $\bar{u}$ is a smooth extension of u. Following [12], we consider the *Bakry–Émery–Ricci tensor* $\overline{\mathrm{Ric}}_{c\bar{u}}$ of $-I \times_f M^n$, which is given by

$$\overline{\mathrm{Ric}}_{c\bar{u}} = \overline{\mathrm{Ric}} + c\,\overline{\nabla}^2\,\bar{u} = \overline{\mathrm{Ric}} - cf'(h)\langle\, , \rangle, \tag{5-8}$$

where $\overline{\mathrm{Ric}}$ and $\overline{\nabla}^2$ are the standard Ricci tensor and the Hessian in $-I \times_f M^n$, respectively. We also consider the modified volume element

$$d\bar{\mu} = e^{c\bar{u}} dV,$$

where dV denotes the standard volume element of $-I \times_f M^n$. We note that, on Σ^n, $d\bar{\mu}$ coincides with the modified volume element $d\mu$ previously defined in (3-21).

With all of these considerations, we have that any function $\varphi \in C_0^\infty(\Sigma^n)$ with

$$\int_{\Sigma^n} \varphi \, d\mu = 0$$

induces a variation of $\psi : \Sigma^n \hookrightarrow -I \times_f M^n$, having compact support and fixed boundary, with variational normal field $\partial F/\partial s|_{s=0} = \varphi N$ and with *first variation* $\delta_\varphi(\mathbb{A}_{cu})$ of the *modified area functional*

$$\begin{aligned} \mathbb{A}_{cu} : (-\epsilon, \epsilon) &\to \mathbb{R} \\ s &\mapsto \mathbb{A}_{cu}(s) = \int_{\Sigma^n} d\mu \end{aligned}$$

given by

$$\delta_\varphi(\mathbb{A}_{cu}) = \frac{d\mathbb{A}_{cu}}{ds}(0) = \int_{\Sigma^n} \varphi H_{c\bar{u}} \, d\mu \tag{5-9}$$

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(see, for example, [21, Lemma 3.2]), where $H_{c\bar{u}}$ is *the modified mean curvature* of $\psi : \Sigma^n \looparrowright -I \times_f M^n$ defined by

$$H_{c\bar{u}} = H - c\langle \overline{\nabla}(\bar{u}), N\rangle.$$

But, since $\psi : \Sigma^n \looparrowright -I \times_f M^n$ is a spacelike mean curvature flow soliton with respect to the closed conformal vector field $\mathcal{K} = f(t)\partial_t$ and with soliton constant $c \neq 0$, from (2-6) and (3-7) we get that

$$\begin{aligned} H_{c\bar{u}} &= cf(h)\Theta - c\langle \overline{\nabla}(\bar{u}), N\rangle = cf(h)\Theta - c\langle (\overline{\nabla}(\bar{u})^{\top} + \overline{\nabla}(\bar{u})^{\perp}), N\rangle \\ &= cf(h)\Theta - c\langle \overline{\nabla}(\bar{u})^{\perp}, N\rangle = cf(h)\Theta - c\langle (-g'(h)\overline{\nabla}\pi_I))^{\perp}, N\rangle. \\ &= cf(h)\Theta - cf(h)\langle \partial_t, N\rangle = 0. \end{aligned} \tag{5-10}$$

Therefore, from (5-9) and (5-10), we obtain that any spacelike mean curvature flow soliton $\psi : \Sigma^n \looparrowright -I \times_f M^n$ with respect to the closed conformal vector field $\mathcal{K} = f(t)\partial_t$ and with soliton constant $c \neq 0$ is a critical point of the modified area functional $\mathbb{A}_{cu}$. Furthermore, the *stability operator* $L_{cu} : C_0^\infty(\Sigma^n) \to C_0^\infty(\Sigma^n)$ for this variational problem is given by the second variation formula $\delta^2_\varphi(\mathbb{A}_{cu})$ of $\mathbb{A}_{cu}$, which, in our case, is written as (see, for example, [21, Proposition 3.5] for the case $H_{c\bar{u}} = 0$)

$$\delta^2_\varphi(\mathbb{A}_{cu}) = \frac{d^2\mathbb{A}_{cu}}{ds^2}(0)(\varphi) = \int_{\Sigma^n} \varphi L_{cu}(\varphi)\, d\mu,$$

with

$$L_{cu} = \Delta_{cu} - \{\overline{\mathrm{Ric}}_{c\bar{u}}(N, N) + |A|^2\}, \tag{5-11}$$

where Δ_{cu} is the drift Laplacian operator on Σ^n given in (3-17). So, using (5-8), we can rewrite the stability operator L_{cu} as

$$L_{cu} = \Delta_{cu} - \{\overline{\mathrm{Ric}}(N, N) - cf'(h) + |A|^2\}.$$

The following notion of stability concerning spacelike mean curvature flow solitons in GRW spacetime now makes sense.

DEFINITION 5.4. Let $\psi : \Sigma^n \looparrowright -I \times_f M^n$ be a spacelike mean curvature flow soliton with soliton constant $c \neq 0$. We say that $\psi : \Sigma^n \looparrowright -I \times_f M^n$ is L_{cu}-stable if $\delta^2_\varphi(\mathbb{A}_{cu}) \leq 0$ for all $\varphi \in C_0^\infty(\Sigma^n)$.

The next auxiliary result gives a sufficient condition to guarantee that a spacelike mean curvature flow soliton must be L_{cu}-stable (for its proof, see [25, Lemma 3.2]).

LEMMA 5.5. *Let $\psi : \Sigma^n \looparrowright -I \times_f M^n$ be a spacelike mean curvature flow soliton with soliton constant $c \neq 0$. If there exists a positive smooth function $\varphi \in C^\infty(\Sigma^n)$ such that $L_{cu}(\varphi) \leq 0$, then Σ^n is L_{cu}-stable.*

Now, we analyze the behavior of the warping function f along a spacelike mean curvature flow soliton to infer its L_{cu}-stability.

THEOREM 5.6. *Let $\psi : \Sigma^n \looparrowright -I \times_f M^n$ be a spacelike mean curvature flow soliton with soliton constant $c \neq 0$.*

(a) *If $\zeta_c'(t) \leq 0$ on Σ^n, then $\psi : \Sigma^n \looparrowright -I \times_f M^n$ is L_{cu}-stable.*
(b) *If Σ^n is compact and $\zeta_c'(t) \geq 0$ on it, then $\psi : \Sigma^n \looparrowright -I \times_f M^n$ is L_{cu}-stable if and only if $\zeta_c(t)$ is constant on Σ^n.*
(c) *If Σ^n is compact and $\zeta_c'(t) > 0$ on it, then $\psi : \Sigma^n \looparrowright -I \times_f M^n$ cannot be L_{cu}-stable.*

PROOF. From (5-6),

$$\Delta(\varphi_{\mathcal{K}}) = \{\overline{\mathrm{Ric}}(N,N) + |A|^2\}\varphi_{\mathcal{K}} + \left\{c\, f'(h) + n\,\frac{f''(h)}{f(h)}\right\}\varphi_{\mathcal{K}},$$

where $\varphi_{\mathcal{K}} \in C^\infty(\Sigma^n)$ is the support function defined in (3-10). So, by applying $\varphi_{\mathcal{K}}$ to the stability operator L_{cu} and using the last equation, we get

$$L_{cu}(\varphi_{\mathcal{K}}) = \Delta_{cu}(\varphi_{\mathcal{K}}) - \{\overline{\mathrm{Ric}}(N,N) - cf'(h) + |A|^2\}\varphi_{\mathcal{K}} = \left\{2cf'(h) + n\,\frac{f''(h)}{f(h)}\right\}\varphi_{\mathcal{K}}.$$

Hence,

$$L_{cu}(-\varphi_{\mathcal{K}}) = \{2cf'(h)f(h) + nf''(h)\}\,(-\varphi_{\mathcal{K}}), \tag{5-12}$$

where $-\varphi_{\mathcal{K}}$ is a positive smooth function on Σ^n and, with a direct application of Lemma 5.5, the result of item (a) is obtained directly.

Now, let us consider item (b). Note that, in this case, $C_0^\infty(\Sigma^n) = C^\infty(\Sigma^n)$. So, if $\psi : \Sigma^n \looparrowright -I \times_f M^n$ is L_{cu}-stable, from Definition 5.4 and Equation (5-12) we get

$$\begin{aligned} 0 \geq \delta^2_{(-\varphi_{\mathcal{K}})}(\mathbb{A}_{cu}) &= \int_{\Sigma^n} (-\varphi_{\mathcal{K}})L_{cu}(-\varphi_{\mathcal{K}})\,d\mu \\ &= \int_{\Sigma^n} \{2cf'(h)f(h) + nf''(h)\}(-\varphi_{\mathcal{K}})^2\,d\mu \geq 0, \end{aligned} \tag{5-13}$$

which guarantees that $\zeta_c(t)$ is constant on Σ^n. The converse follows from item (a).

Finally, we prove item (c). Assuming the opposite, if $\psi : \Sigma^n \looparrowright -I \times_f M^n$ is L_{cu}-stable, then, from the analysis of signs studied in (5-13),

$$0 \geq \int_{\Sigma^n} \{2cf'(h)f(h) + nf''(h)\}(-\varphi_{\mathcal{K}})^2\,d\mu \;>\; 0,$$

which is an absurdity. □

From Theorem 5.6, we obtain the following applications.

COROLLARY 5.7. *Every spacelike translating soliton immersed in the Lorentzian product space $-I \times M^n$ with soliton constant $c \neq 0$ is L_{cu}-stable.*

COROLLARY 5.8. *Every spacelike mean curvature flow soliton immersed in the future temporal cone $-\mathbb{R}^+ \times_t \mathbb{H}^n$ with soliton constant $c < 0$ and such that $h \geq \sqrt{-n/c}$ is L_{cu}-stable.*

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COROLLARY 5.9. *There is no L_{cu}-stable compact spacelike mean curvature flow soliton immersed in a steady-state-type spacetime $-I \times_{e^t} M^n$ with soliton constant $c > 0$.*

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