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A CONSTRUCTION OF MEROMORPHIC FUNCTIONS WITH PRESCRIBED ASYMPTOTIC BEHAVIOR

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Introduction

Although there are several constructions of meromorphic functions with prescribed asymptotic sets [e.g., 5, 6], it is usually difficult to determine or prescribe the nature of the asymptotic paths used in these constructions. On the other hand, there are several other constructions of meromorphic functions with prescribed asymptotic paths [e.g., 1, 10, 12], but the extent of the asymptotic values for these functions cannot always be restricted to the values approached along the given paths. Gross [3] has accomplished both results by prescribing paths for every value in the extended complex plane.

This paper presents another technique for constructing meromorphic functions where both the asymptotic paths and the asymptotic values can be prescribed in a fairly general way. In particular, if A is any F_σ set in the extended complex plane, a function $f(z)$ meromorphic in $|z|<R$ $(0<R\leq+\infty)$ is constructed where A is the set of asymptotic values for $f(z)$ and every value of A is obtained along a spiral asymptotic path. Other examples are constructed where A is the set of all asymptotic values and all the values of A are obtained along radial paths, all are obtained along arc paths, and all are obtained along boundary paths terminating at the same point on $|z|=R$.

Definitions and the Main Theorem

Let $0<R\leq+\infty$. For $0<r<R$, we shall denote by C_r the circle $|z|=r$, and by D_r the disk $|z|\leq r$. By a monotonic boundary path (*mb*-path) we shall mean a simple continuous curve, $z=z(s)$ $(0\leq s<1)$, in $|z|<R$, such that, as $s\to 1$, $|z(s)|\to R$ strictly monotonically.

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DEFINITION 1. In a complex τ-plane, let E be a non-empty set of points on the circle $|\tau| = 1$. To every $\tau \in E$, let there correspond an *mb*-path t_τ in $|z| < R$, such that the intersection of every pair of these *mb*-paths is the origin, and, for every r such that $0 < r < R$, the one-to-one correspondence $\tau \longleftrightarrow t_\tau \cap C_r$ $(\tau \in E)$ preserves cyclic order. The set $T = \{t_\tau | \tau \in E\}$ will be called *a modified tress in* $|z| < R$ *relative to* E, provided that the following conditions are satisfied:

(i) E is an F_σ of first category relative to $|\tau| = 1$.

(ii) There exists a decomposition of E,

$$E = \bigcup_{n=1}^{N^*} E_n, \quad N^* \leq \infty,$$

where each E_n is closed and nowhere dense relative to $|\tau| = 1$ such that:

a.)* for $\theta_0 = \min\{\theta | e^{i\theta} \in E_1\}$, $\theta_0 \leq \theta < \theta_0 + 2\pi$ for all $e^{i\theta} \in E$.

b.)* $\max\{\theta | e^{i\theta} \in E_n\} < \min\{\theta | e^{i\theta} \in E_{n+1}\}$, $n = 1, 2, \cdots, N^* - 1$.

c.) if $T_n = \bigcup_{\tau \in E_n} t_\tau$, then $T_n \cap D_r$ is closed and nowhere dense $(n = 1, 2, \cdots, N^*;\ 0 < r < R)$.

Remark 1. Every modified tress is a tress in the sense of Bagemihl and Seidel [1, p. 186, Definition 1]. Examples of modified tresses can be found on p. 187 of the abovementioned paper with the set E in examples 1–4 restricted as indicated in the definition above.

We shall use the notation $\Gamma(f)$ to denote the set of asymptotic values of a function $f(z)$ defined in $|z| < R$. $\alpha \in \Gamma(f)$ if and only if there exists a boundary path γ such that $\alpha = \lim_{\substack{|z| \to R \\ z \in \gamma}} f(z)$. Any boundary path along which $f(z)$ tends to a limit is called an asymptotic path for $f(z)$. If the end of γ (i.e., the set of limit points of γ on $|z| = R$) is a point (arc), α is said to be a point (arc) asymptotic value. The set of all point (arc) asymptotic values is denoted by $\Gamma_P(f)$ $(\Gamma_A(f))$. If $f(z)$ tends to a limit along every *mb*-path in a modified tress $T = \{t_\tau | \tau \in E\}$, f will be called a T-function, and we shall use $\Gamma_{T(E)}(f)$ to denote the set of all such limits. If $f(z)$ is a T-function, and if for every asymptotic path for $f(z)$ which is disjoint from T the limit attained by $f(z)$ on this path is also in $\Gamma_{T(E)}(f)$, f is said to be restricted on asymptotic paths disjoint from T.

* The direction here is arbitrary. Corresponding conditions in the clockwise direction can be substituted.

THEOREM 1. *Let $T=\{t_\tau|\tau\in E\}$ be a modified tress in $|z|<R$. Let $g_n(z)$ be a continuous complex-valued function defined on T_n such that for every $\tau\in E_n$, as $|z|\to R$ with $z\in t_\tau$, we have $\mathscr{R}g_n(z)\to\alpha(\tau)$ and $\mathscr{I}g_n(z)\to\beta(\tau)$ where $\alpha(\tau)$ and $\beta(\tau)$ are real-valued functions of $\tau\in E_n$ which may assume the value $\pm\infty$ $(n=1,2,\cdots,N^*)$. Then there exists a T-function $f(z)$ meromorphic in $|z|<R$ such that $\max_{z\in T_n}|f(z)-g_n(z)|\to 0$ uniformly as $|z|\to R$ for each $n=1,2,\cdots,N^*$. Furthermore, $\Gamma_{T(E)}(f)=\{G(\tau)=\alpha(\tau)+i\beta(\tau)|\tau\in E\}$ and f is restricted on asymptotic paths disjoint from T.*

Outline of Proof of Theorem

To accomplish this construction, we first define a point set S in $|z|<R$, called the skeleton, such that for each n $(1\leq n\leq N^*)$ S contains all the points of T_n whose modulus is greater than a fixed real number $(<R)$ which depends on n. In addition, S contains barriers which every boundary path disjoint from T must cut in every annulus $A_n=\{z|R-R/n<|z|<R\}$. A continuous function $h(z)$ is defined on S which agrees with $g_n(z)$ for every $z\in T_n\cap S$ $(n=1,2,\cdots,N^*)$. $f(z)$ is then constructed as an approximation to $h(z)$ on S and meromorphic in $|z|<R$ such that $\max_{z\in S}|f(z)-h(z)|\to 0$ uniformly as $|z|\to R$. The purpose of the enlargement of T to the skeleton S is to control the convergence on any boundary path in $\mathscr{C}T$ (i.e., the complement of T in $|z|<R$).

The Skeleton S and the Continuous Function $h(z)$ on S

Let $0<r_0<r_1<\cdots<r_n<\cdots<R$ where $\lim_{n\to\infty}r_n=R$. Let $\tau_0=e^{i\theta_0}$ where $\theta_0=\min\{\theta|e^{i\theta}\in E_1\}$. For each $n=1,2,\cdots,N^*-1$, let $\theta_n\in(\max\{\theta|e^{i\theta}\in E_n\}, \min\{\theta|e^{i\theta}\in E_{n+1}\})$. If $N^*<\infty$, define $\theta_{N^*}\in(\max\{\theta|e^{i\theta}\in E_{N^*}\}, \theta_0+2\pi)$. Now let $\tau_n=e^{i\theta_n}$ $(n=1,2,\cdots,N^*)$. Define t_{τ_n} to be an mb-path with 0 as initial point and disjoint from T. For each n satisfying $1\leq n\leq N^*$, define $E_n^*=E_n\cup\{\tau_{n-1},\tau_n\}$. Then T_n^* is defined as $\bigcup_{\tau\in E_n^*}t_\tau$ and $T^*=\bigcup_{n=1}^{N^*}T_n^*$.

Let R_n be the region bounded by $|z|=R$, $t_{\tau_{n-1}}$, t_{τ_n} and containing T_n in its interior. Let $W^n=R_n\cap\mathscr{C}T_n$. W^n is an open set in $|z|<R$ and so can be expressed as the union of a countable number of components $W^n=\bigcup_{m=1}^{M}W_m^n$ $(M=M(n)\leq\infty)$. Each component is bounded in $|z|<R$ by the union of two mb-paths in T_n^*, call them $t_m{}^n(1)$ and $t_m{}^n(2)$.

Let $p_m{}^n(k,i)=C_{r_k}\cap t_m{}^n(i)$ $(n=1,\cdots,N^*;\ k=1,2,\cdots,;\ m=1,\cdots,M(n);\ i=1,2)$. Let $J_m{}^n(k)$ be a Jordan curve which is contained in $W_m{}^n\cap(D_{r_k}-D_{r_{k-1}})$

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and which is the union of two Jordan arcs joining $p_m{}^n(k,1)$ and $p_m{}^n(k,2)$ $(n=1,2,\cdots,N^*;\ m=1,2,\cdots,M(n);\ k=1,2,\cdots)$. Let α_n be the subarc of C_{r_n} joining $C_{r_n}\cap t_{\tau_n}$ to $C_{r_n}\cap t_{\tau_0}$ which does not meet $\bigcup_{i=1}^{n} T_i\ (1\leq n\leq N^*)$. For N^* finite and $n\geq N^*$, α_n is the subarc of C_{r_n} joining $C_{r_n}\cap t_{\tau_{N^*}}$ to $C_{r_n}\cap t_{\tau_0}$ which does not meet T^*.

Let $S_1=\alpha_1\cup(T_1{}^*\cap C_{r_1})$. For $1<n\leq N^*$, let $S_n=\alpha_n\cup[(D_{r_n}-D_{r_{n-1}})\cap(\bigcup_{i=1}^{n-1} T_i{}^*)]\cup(T_n{}^*\cap C_{r_n})\cup[\bigcup_{j=1}^{n-1}\bigcup_{i=1}^{n-j} J_i{}^j(n)]$. If N^* is finite and $n\geq N^*$, define $S_n=\alpha_n\cup[D_{r_n}-D_{r_{n-1}})\cap T^*]\cup[\bigcup_{j=1}^{N^*}\bigcup_{i=1}^{n-j} J_i{}^j(n)]$. The skeleton $S=\bigcup_{n=1}^{\infty} S_n$.

The function $g_1(z)$ is defined and continuous for all $z\in t_{\tau_0}$. For $1\leq n\leq N^*$, define $h(z)$ on t_{τ_n} by $h(t_{\tau_n}\cap C_{r_n})=g(t_{\tau_0}\cap C_{r_n})$ $(0\leq r<R)$. With this definition of $h(z)$ on $\bigcup_{n=0}^{\infty} t_{\tau_n}$, we can easily extend $h(z)$ to the arcs α_n. $h(t_{\tau_n}\cap C_{r_n})=h(t_{\tau_0}\cap C_{r_n})$ and so we define h on α_n to be this constant value for all $n=1,2,\cdots$. For N^* finite and $n>N^*$, $h(z)=h(t_{\tau_{N^*}}\cap C_{r_n})=h(t_{\tau_0}\cap C_{r_n})$ for all $z\in\alpha_n$.

Set $h(z)=g_n(z)$ for $z\in T_n\cap S$. Define $h(z)$ on $J_m{}^n(k)$ to be the homeomorphic image of a circle joining $h(p_m{}^n(k,1))$ and $h(p_m{}^n(k,2))$ with diameter $|h(p_m{}^n(k,1))-h(p_m{}^n(k,2))|$. Thus, $h(z)$ is now defined and continuous for all $z\in S$.

Relation between $f(z)$ and $h(z)$

Suppose $f(z)$ has been constructed meromorphic in $|z|<R$ such that $\max_{z\in S}|f(z)-h(z)|\to 0$ uniformly as $|z|\to R$. Since $z\in T_n$ implies that $z\in S$ for all $z\in T_n$, with modulus $\geq r_n$, and since $h(z)=g_n(z)$ for $z\in T_n\cap S$, it follows that $\max_{z\in T_n}|f(z)-g_n(z)|\to 0$ uniformly as $|z|\to R\ (1\leq n\leq N^*)$. Thus $f(z)$ is a T-function and $\Gamma_{T(E)}(f)=\bigcup_{n=1}^{N^*}\{G(\tau)|\tau\in E_n\}=\{G(\tau)|\tau\in E\}$.

Consider now the asymptotic paths for f disjoint from T. It is clear that any asymptotic path that meets T^*-T in every annulus A_n must have asymptotic value equal to $\lim_{\substack{z\in t_{\tau_0}\\ |z|\to R}} g_1(z)$. Thus, it is sufficient to consider only asymptotic paths which are disjoint from T^*. Let $\lambda=\lambda(s)$ $(0\leq s<1)$ be such a path. Without loss of generality we may assume $|\lambda(s)|>|\lambda(0)|$ for all $0<s<1$. For some k, $r_k\leq|\lambda(0)|<r_{k+1}$. The only route to the boundary open to λ that does not meet T^* is the interior of $W_m{}^n$ if $\lambda(0)\in W_m{}^n$ for some pair (n,m), or else λ must cut every arc α_n for $n>k$. This latter alternative again restricts the asymptotic value attained by $f(z)$ along λ to

$\lim\limits_{\substack{z\in t_{\tau_0}\\|z|\to R}} g_1(z)$ since $h(z)$ is constant on α_n and $h(z)$ tends to this limit on α_n as $n\to\infty$.

Thus we are left with the possibility that $\lambda(s)\in W_m{}^n$ for all s where the pair (n,m) is fixed. λ must cut $J_m{}^n(q)$ for all integers $q>k$. There exists a q_0 such that for all $q>q_0$, $J_m{}^n(q)$ is a part (i.e., a subset) of the skeleton S. In particular λ must cut $J_m{}^n(q)$ for all $q>\max(k,q_0)$.

Let $a_i=\lim\limits_{\substack{|z|\to R\\ z\in t_m{}^n(i)}} g_n(z)$ $(i=1,2)$. $f(J_m{}^n(q))$ is a sequence of "circles" (i.e., approximations to circles) and can be written as $C^1{}_q\cup C^2{}_q$ where $C^1{}_q$ and $C^2{}_q$ are two "semicircles" joining $f(p_m{}^n(q,1))$ and $f(p_m{}^n(q,2))$. $f(\lambda(s))$ must cut every arc in the sequence $\{C^1{}_q\}$ and every arc in the sequence $\{C^2{}_q\}$ for $q>\max(k,q_0)$. Thus, if $f(\lambda(s))$ converges to a limit as $s\to1$, it must converge to a common limit point for both the sequences $\{C_q{}^1\}$ and $\{C_q{}^2\}$. Therefore $\lim\limits_{s\to1} f(\lambda(s))\subset\{a_1,a_2\}\subset\Gamma_{T(E_n)}(g_n)\subset\Gamma_{T(E)}(f)$.

It is appropriate to consider at this point the subset S^1 of S formed by $\{\bigcup\limits_{n=1}^{\infty}\alpha_n\}\cup\{S\cap\bigcup\limits_{n=0}^{\infty}t_{\tau_n}\}$. Since the values of h on $S^1\cap A_n$ are restricted to the values assumed by $h(z)=g_1(z)$ on $t_{\tau_0}\cap A_n$ and since $\lim\limits_{\substack{z\in t_{\tau_0}\\|z|\leftarrow R}} g_1(z)=G(\tau_0)$, it follows that any asymptotic path that meets S^1 in each annulus A_n must have $G(\tau_0)$ as asymptotic value. This leads to the following characterization of the maverick asymptotic paths $\lambda=\lambda(s)$ $(0\leq s<1)$ along which the behavior of $f(z)$ cannot be guaranteed.

1.) $\lambda\subset W^m$ for some m

2.) $\lambda\cap T_m\cap A_n\neq\emptyset$ for all n

3.) $\lambda(s)\cap t=\emptyset$ for all $s>s(t)$ and for each $t\in T_m$.

Later we will consider what might be done to control the behavior of $f(z)$ on these paths.

The Construction of $f(z)$

The following construction is a variation on an approximation technique used by Bagemihl and Seidel [1, pp. 188–189]. Another method that could be used is due to Barth [2, pp. 326–327].

Let $F_n=D_{r_n}\cup S_{n+1}$ $n=0,1,2,\cdots$. Notice that $\mathscr{C}F_n$ has $1+2\sum\limits_{i=1}^{n} i$ components. We proceed by induction on n.

Put $\varphi_0(z) = \psi_0(z) = 0$ for $z \in D_{r_0}$. $\varphi_0(z) = \mathscr{R}h(z)$; $\psi_0(z) = \mathscr{I}h(z)$ for $z \in S_1$. The function $\varphi_0(z) + i\Psi_0(z)$ is continuous on F_0 and analytic at all interior points. By Mergelyan's theorem [13, p. 109, Theorem 13.3] there exists a rational function (a polynomial in fact) $r_0(z)$ such that

$$|r_0(z) - [\varphi_0(z) + i\psi_0(z)]| \leq 1, \quad z \in F_0$$

Suppose $n > 0$ and we have defined $\varphi_{n-1}(z)$ and $\psi_{n-1}(z)$ on F_{n-1} and rational functions $r_0(z), \cdots, r_{n-1}(z)$ so that these rational functions have no poles on S,

$$\varphi_{n-1}(z) = \mathscr{R}h(z), \quad \psi_{n-1}(z) = \mathscr{I}h(z), \quad z \in F_{n-1} \cap C_{r_n}$$

and $|r_0(z) + r_1(z) + \cdots + r_{n-1}(z) - h(z)| \leq 1/2^{n-1}$, $z \in F_{n-1} \cap C_{r_n}$.

Using this last inequality and Tietze's theorem [7, pp. 127–128], there exist real-valued functions $\xi_n(z)$ and $\eta_n(z)$, continuous in $|z| < R$ such that

$$\xi_n(z) = \mathscr{R} \sum_{j=0}^{n-1} r_j(z), \quad \eta_n(z) = \mathscr{I} \sum_{j=0}^{n-1} r_j(z), \quad z \in F_{n-1} \cap C_{r_n};$$

(1) $\xi_n(z) = \mathscr{R}h(z)$, $\eta_n(z) = \mathscr{I}h(z)$, $z \in F_n \cap C_{r_{n+1}}$;

(2) $|\xi_n(z) - \mathscr{R}h(z)| \leq 1/2^{n-1}$, $|\eta_n(z) - \mathscr{I}h(z)| \leq 1/2^{n-1}$, $z \in S$, $r_n < |z| < r_{n+1}$.

Set $\varphi_n(z) = \mathscr{R} \sum_{j=0}^{n-1} r_j(z)$, $\Psi_n(z) = \mathscr{I} \sum_{j=0}^{n-1} r_j(z)$, $z \in D_{r_n}$,

(3) $\varphi_n(z) = \xi_n(z)$, $\psi_n(z) = \eta_n(z)$, $z \in F_n \cap (D_{r_{n+1}} - D_{r_n})$.

The function $A(z) = \varphi_n(z) + i\psi_n(z) - \sum_{j=0}^{n-1} r_j(z)$ is continuous on F_n and analytic at all interior points of F_n. Again by Mergelyan's theorem there exists a rational function $r_n(z)$ which has no poles on S and

(4) $$|A(z) - r_n(z)| \leq 1/2^n, \quad z \in F_n.$$

This completes the induction.

Let $f(z) = \sum_{j=0}^{\infty} r_j(z)$ for $|z| < R$. If $z \in D_{r_n}$,

(5) $$|r_{n+j}(z)| \leq 1/2^{n+j} \quad (j = 0, 1, 2, \cdots).$$

Since $r_{n+j}(z)$ $(j = 0, 1, 2, \cdots)$ is holomorphic in the interior of D_{r_n}, this implies $\sum_{j=0}^{\infty} r_{n+j}(z)$ is holomorphic in the interior of D_{r_n}, and therefore $f(z)$ is meromorphic in the interior of D_{r_n}. Every point of $|z| < R$ is in the interior of D_{r_n} for some n. Hence $f(z)$ is meromorphic in $|z| < R$.

To show that $\max_{z\in S}|f(z)-h(z)|\to 0$ uniformly as $|z|\to R$, consider any $z\in S=\bigcup_{n=1}^{\infty}S_n$. $z\in S_{n+1}$ for some $n=0,1,\cdots$ which implies that $z\in F_n$ for this value of n. Then, by the definition of $f(z)$, we have

$$|\mathscr{R}f(z)-\mathscr{R}h(z)|\leq \nu_n+\delta_n+\rho_n$$

where

$$\nu_n=|\sum_{j=0}^{n}\mathscr{R}r_j(z)-\varphi_n(z)|$$

$$\delta_n=|\varphi_n(z)-\mathscr{R}h(z)|$$

$$\rho_n=\sum_{j=n+1}^{\infty}|\mathscr{R}r_j(z)|.$$

It follows from (4) that $\nu_n\leq 1/2^n$. Applying (1), (2), and (3), we see that $\delta_n\leq 1/2^{n-1}$. Finally, (5) yields

$$\rho_n\leq\sum_{j=n+1}^{\infty}1/2^j.$$

Thus we have $|\mathscr{R}f(z)-\mathscr{R}h(z)|\leq\sum_{j=n-1}^{\infty}1/2^j$ for $z\in S_{n+1}$. As $|z|\to R$, $z\in S$, this implies $n\to\infty$ and so $|\mathscr{R}f(z)-\mathscr{R}h(z)|\to 0$ as $|z|\to R$, $z\in S$. An analagous argument for $|\mathscr{I}f(z)-\mathscr{I}h(z)|$ yields the desired conclusion that

$$\max_{z\in S}|f(z)-h(z)|\to 0 \text{ uniformly as } |z|\to R.$$

The set $\{G(\tau)|\tau\in E\}$

It is well known that the set of asymptotic values of a meromorphic function is characterized as an analytic set [5, 6, 9]. The set $\{G(\tau)|\tau\in E\}$ can be characterized in the same way.

LEMMA 1. *Let* $T=\{t_\tau|\tau\in E\}$ *be a modified tress. If* $z\in\bigcup_{j=1}^{m}T_j-\{0\}$, *then the unique* $\tau=\tau(z)\in\bigcup_{j=1}^{m}E_j$ *such that* $z\in t_\tau$, *is a continuous function of* z *on the set* $\bigcup_{j=1}^{m}T_j-\{0\}$ $(m=1,2,\cdots)$.

Proof. Assume, to the contrary, that $\tau(z)$ is discontinuous at $z_0\in\bigcup_{j=1}^{m}T_j-\{0\}$ for some m $(1\leq m<\infty)$. Then for some $\varepsilon>0$, there exists a sequence of points $\{z_n\}\subset\bigcup_{j=1}^{m}T_j-\{0\}$ such that $\lim z_n=z_0$ and $|\tau(z_n)-\tau(z_0)|>\varepsilon$ for all

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$n = 1, 2, \cdots$. Without loss of generality we may assume $\arg \tau(z_n)$ is monotonically increasing with respect to n and $\arg \tau(z_n) < \arg \tau(z_0)$. Since $\tau(z_n) \in \bigcup_{j=1}^{m} E_j$ and since each E_j is closed for each j $(1 \leq j \leq N^*)$, this implies $\lim_{n\to\infty} \tau(z_n) \in \bigcup_{j=1}^{m} E_j$. Let $\tau_0 = \lim_{n\to\infty} \tau(z_n)$. Our assumption implies $|\tau_0 - \tau(z_0)| \geq \varepsilon$. $t_{\tau_0} \cup t_\tau(z_1)$ divides $|z| < R$ into two components. Since the correspondence $\tau \longleftrightarrow t_\tau \cap C_r(\tau \in E)$ preserves cyclic order for every r $(0 < r < R)$, one of these components contains $t_{\tau(z_0)}$, while $\bigcup_{n=2}^{\infty} t_{\tau(z_n)} - \{0\}$ is contained in the other components. In particular, all of the points z_n are excluded from the component that contains z_0, contrary to the assumption that $\lim_{n\to\infty} z_n = z_0$. Hence $\tau(z)$ must be continuous on $\bigcup_{j=1}^{m} T_j - \{0\}$ for all m $(1 \leq m < \infty)$.

Remark 2. Bagemihl and Seidel [1, p. 195, Definition 2] call any tress with the continuity property mentioned in the above lemma a restricted tress. The above proof also proves that every tress [1, p. 186, Definition 1] is a restricted tress. Hence the term restricted can be removed from the statements of their theorems 7 [1, p. 196] and 9 [1, p. 198].

THEOREM 2. *Let $T = \{t_\tau | t \in E\}$ be a modified tress in $|z| < R$. Suppose that $\alpha(\tau)$ and $\beta(\tau)$ are real-valued functions (which may assume the value $\pm\infty$) of $\tau \in E$. Then a necessary and sufficient condition that there exist a T-function $f(z)$, meromorphic in $|z| < R$, such that $\Gamma_{T(E)}(f) = \{\alpha(\tau) + i\beta(\tau) | \tau \in E\}$, $\lim_{\substack{|z|\to R \\ z \in t_\tau}} f(z) = \alpha(\tau) + i\beta(\tau)$, and f is restricted on asymptotic paths disjoint from T is that $\alpha(\tau)$ and $\beta(\tau)$ be of Baire class ≤ 1 on E.*

Proof. First note that if such a function exists, there exists a sequence $0 < \rho_1 < \rho_2 < \cdots < \rho_n < \cdots < R$ with $\lim_{n\to\infty} \rho_n = R$ such that $f(z)$ has no poles on C_{ρ_n}. The proof is then the same as the one used by Bagemihl and Seidel [1, p. 196, proof of Theorem 7] using Theorem 1 above instead of their Corollary 1.

THEOREM 3. *Let $T = \{t_\tau | \tau \in E\}$ be a modified tress in $|z| < R$, where E is of power $2^{\aleph_0}$. A necessary and sufficient condition that a non-empty set A in the extended complex plane be equal to $\Gamma_{T(E)}(f)$ for some T-function $f(z)$ meromorphic in $|z| < R$ and restricted on asymptotic paths disjoint from T is that A be an analytic set.*

Proof. This proof is essentially the same as [1, p. 198, proof of Theorem 9]. Given f and $A = \Gamma_{T(E)}(f)$, Theorem 2 states that A is egual to $\{\alpha(\tau) + i\beta(\tau) | \tau \in E\}$ where α and β are of Baire class ≤ 1. Since E is a Borel set, A is analytic [4, p. 301].

On the other hand, any non-empty analytic set A is the image of E under a mapping of Baire class ≤ 1 [7, p. 480, Theorem 1]. The real and imaginary parts of this function are then real-valued functions (which may assume the values $\pm\infty$) of Baire class ≤ 1 on E [7, p. 376, Theorem 2. If we let $\alpha(\tau)$ and $\beta(\tau)$ represent the real and imaginary parts respectively of this function, Theorem 2 then produces the desired T-function.

Some Applications

MacLane has provided an example of a meromorphic function in $|z|<\infty$ without asymptotic values [8, pp. 180–181, Theorem 3] and an example of a function meromorphic in $|z|<1$ without asymptotic values [8, p. 183, Theorem 5]. Thus, in the corollaries that follow, the statements will be made in terms of an arbitrary analytic set A with the case for $A=\emptyset$ already resolved.

COROLLARY 1. *Given any analytic set A in the extended complex plane and any closed nowhere dense subset E of $|z| = 1$ such that E has power $2^{\aleph_0}$, there exists a function $f(z)$ meromorphic in $|z| < R$ such that A is the set of radial (ray if $R = \infty$) limits (abbreviated $\Gamma_R(f)$) for $f(z)$ and the set of radial limits generated by f on radii terminating at points of E is A.* (For a characterization, see [1], p. 198, Theorem 10).

Proof. For $z = e^{i\theta} \in E$, define $t_z = re^{i\theta}$ $(0 \leq r < R)$. Then $T = \{t_z | z \in E\}$ is a modified tress with $N^* = 1$ and $E = E_1$. Theorem 3 now gives the desired conclusion.

DEFINITION 2. For any non-negative, monotonically increasing, real-valued function $\sigma(r)$ $(0 \leq r < R \leq +\infty)$ such that $\lim_{r\to R} \sigma(r) = +\infty$, define $S_\tau = \{z = re^{i\theta} | \theta = \tau + \sigma(r),\ 0 \leq r < R\}$ for each τ $(0 \leq \tau < 2\pi)$. Let $\mathscr{S}(\sigma) = \{S_\tau | 0 \leq \tau < 2\pi\}$.

$\mathscr{S}(\sigma)$ is then a class of spirals which are mutually disjoint except for the origin which is common to all of them. Furthermore, every point of $|z| < R$ ifferent from 0 is on one and only one spiral in $\mathscr{S}(\sigma)$. Let

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$\Gamma_{\mathscr{S}(\sigma)}(f)$ be the set of asymptotic values attained by $f(z)$ along paths in $\mathscr{S}(\sigma)$.

COROLLARY 2. *Given any analytic set A in the extended complex plane, any closed nowhere dense subset E of $|\tau| = 1$ such that E has power $2^{\aleph_0}$ and any class of spirals $\mathscr{S}(\sigma)$, there exists a function $f(z)$ meromorphic in $|z| < R$ such that $A = \Gamma_{\mathscr{S}(\sigma)}(f)$ and $A = \{\lim_{\substack{|z|\to R \\ z\in S_\tau}} f(z) | \tau \in E\}$.*

Proof. For $\tau \in E$, define $t_\tau = S_\tau$. Then $T = \{t_\tau | \tau \in E\}$ is a modified tress with $N^* = 1$ and $E = E_1$. The conclusion follows as an application of Theorem 3.

The Further Restriction of the Asymptotic Behavior of f

As previously mentioned the only asymptotic paths that might escape the restriction of their asymptotic values are those paths $\lambda = \lambda(s)$ $(0 \leq s < 1)$ satisfying:

1.) $\lambda \subset W^m$ for some m

2.) $\lambda \cap T_m \cap A_n \neq \emptyset$ for all n

3.) $\lambda(s) \cap t = \emptyset$ for all $s > s(t)$ and for each $t \in T_m$.

It is natural to ask whether additional conditions can be imposed that allow the construction of a meromorphic function in D whose asymptotic behavior is prescribed along any modified tress $T(E)$ and restricted to $\Gamma_{T(E)}(f)$ along *all* other asymptotic paths. The answer to this question depends on how much control can be imposed on boundary paths of the type described above.

Given a modified tress $T = \{t_\tau | \tau \in E\}$ in $|z| < R$ and a boundary path λ in $|z| < R$ satisfying 1.), 2.), and 3.) above, there exist boundary paths $t_1, t_2, \cdots$ in T_m such that $t_i \neq t_j$ for $i \neq j$ and $t_n \cap \lambda \cap A_n \neq \emptyset$ for each n. Let $z_n \in t_n \cap \lambda \cap A_n$ and let $\tau^n = \tau(z_n)$ (i.e., $t_n = t_{\tau^n}$). Without loss of generality, we may assume the sequence $\{\tau^n\}$ has a single limit point and let $\tau^0 = \lim_{n\to\infty} \tau^n$. Since E_m is closed and $\tau^n \in E_m$ for all n, $\tau^0 \in E_m$.

Now suppose that λ is an asymptotic path for some $f(z)$. The asymptotic value assumed by f along λ must be $\lim_{n\to\infty} f(z_n)$. Thus, control of the values which can be assumed along sequences of this kind yields control of all possible asymptotic values which can be assumed along λ. Since the z_n are elements of t_n, and since we have some control of f on t_n with the construction used in the proof of Theorem 1, our problem can be resolved by increasing the degree of this control along T_m. More precisely, if $\lim_{n\to\infty} f(z_n) = G(\tau^0)$ we would have the control we need.

In general we need this additional control on all such possible sequences. To this end, for any sequence $0 < r_1' < r_2' < \cdots < r_n' < \cdots < R$, define $f_{r_n'}(\tau) = f(t_\tau \cap C_{r_n'})$ for each $\tau \in E_m$. If $f_{r'_n}$ tends to G uniformly on E_m, then G is continuous on E_m since all the $f_{r'_n}$ are continuous. This guarantees that $\lim_{n\to\infty} f(z_n) = G(\tau^0)$ as we desire. Conversely, if $G(\tau)$ is any continuous function on E_m assuming values in the extended complex plane Ω, $g_m(z)$ can be defined on T_m so that for $g_m{}^n(\tau) = g_m(t_\tau \cap C_{r'_n})$, $\mathscr{R} g_m^n(\tau)$ and $\mathscr{I} g_m^n(\tau)$ are continuous, monotonically nondecreasing, real-valued functions for every sequence $\{r_n'\}$ and $\lim_{\substack{|z|\to R \\ z\in t_\tau}} g_m(z) = G(\tau)$ for $\tau \in E_m$. It follows from Dini's Theorem that $g_m{}^n(\tau) \to G(\tau)$ uniformly on E_m. Since $f(z)$ is constructed in Theorem 1 to approximate $g_m(z)$ uniformly on T_m, $f_{r'_n}$ must also tend to G uniformly on E_m and again $\lim_{n\to\infty} f(z_n) = G(\tau^0)$.

We have thus proven

THEOREM 4. *Let $T = \{t_\tau | \tau \in E\}$ be a modified tress in $|z| < R$ where $E = \bigcup_{n=1}^{N^*} E_n$ as required by Definition* 1. *Let $G_n(\tau)$ be any continuous function defined on E_n $(1 \leq n \leq N^*)$ taking values in Ω. Then there exists a function $f(z)$ meromorphic in $|z| < R$ with $\Gamma(f) = \Gamma_{T(E)}(f) = \bigcup_{n=1}^{\infty} G_n(E_n)$.*

THEOREM 5. *Let A be any F_σ subset of Ω. Let $T = \{t_\tau | \tau \in E\}$ be any modified tress where $E = \bigcup_{n=1}^{\infty} E_n$ and E_n is homeomorphic to the Cantor ternary set. Then there exists a meromorphic function $f(z)$ in $|z| < R$ such that $\Gamma(f) = \Gamma_{T(E)}(f) = A$.*

Proof. Any closed subset F_n of Ω is the continuous image of E_n [11, p. 146, Theorem 78]. Hence any F_σ in Ω can be represented as $\bigcup_{n=1}^{\infty} G_n(E_n)$ for G_n continuous on E_n. The conclusion follows from Theorem 4 setting $A = \bigcup_{n=1}^{\infty} G_n(E_n)$.

Further Applications

Let E_n be a homeomorphic image of the Cantor ternary set in the open arc on $|\tau| = 1$ which is in the upper half plane and is bounded by $e^{\frac{i\pi}{n}}$ and $e^{\frac{i\pi}{n+1}}$ $(n = 1, 2, \cdots)$. Let $E = \bigcup_{n=1}^{\infty} E_n$. Using Theorem 5 and the modified tresses defined in the proofs of Corollaries 1 and 2 with respect to this $E = \bigcup_{n=1}^{\infty} E_n$, we can now add:

COROLLARY 3. *Given any F_σ subset A of Ω, there exists a function $f(z)$ meromorphic in $|z| < R$ such that $\Gamma(f) = \Gamma_R(f) = \Gamma_{T(E)}(f) = A$.*

COROLLARY 4. *Given any F_σ subset A of Ω and any class of spirals $\mathscr{S}(\sigma)$ (cf. Definition 2), there exists a function $f(z)$ meromorphic in $|z| < R$ such that $\Gamma(f) = \Gamma_{\mathscr{S}(\sigma)}(f) = \{\lim_{\substack{|z|\to R \\ z\in S_\tau}} f(z) | \tau \in E\} = A$.*

COROLLARY 5. *Given any F_σ subset A of Ω, there exists a function $f(z)$ meromorphic in $|z| < R$ $(R < +\infty)$ such that $\Gamma(f) = A$ and each value of A is an arc asymptotic value for an arc which is a proper subset of the whole boundary $|z| = R$.*

Proof. Let $E^* = E - \bigcup_{k=1}^{K^*(R)} E_k$ where E_k contains a value τ with arg $\tau > 1/R$. For every $\tau = e^{i\varphi} \in E^*$, define t_τ to be the set $z = re^{i\theta}$ where $\theta = \sin\frac{1}{R-r} + \varphi(R-r)\ (0 \leq r < R)$. The set $T = \{t_\tau | \tau \in E^*\}$ is a modified tress in $|z| < R$, and all mb-paths of T have the arc $\{z = Re^{i\alpha} | -1 \leq \alpha \leq 1\}$ as end. The desired result follows from Theorem 5.

COROLLARY 6. *Given any F_σ subset A of Ω, there exists a function $f(z)$ meromorphic in $|z| < R$ $(R < +\infty)$ such that $\Gamma(f) = A$ and each value of A is a point asymptotic value at the point $z = R$.*

Proof. Let $E^* = \bigcup_{n=4}^{\infty} E_n$. For $\tau = e^{i\varphi} \in E$, define t_τ to be the union of the line segments from 0 to $\frac{1}{2}e^{i\varphi}$ and from $\frac{1}{2}e^{i\varphi}$ to R. The set $T = \{t_\tau | \tau \in E^*\}$ is a modified tress and t_τ is an mb-path tending to R as $|z| \to R$. Theorem 5 again produces the desired result.

Finally, an F_σ subset of Ω is the most general kind of set we can expect from the construction presented in this paper if we desire f to be restricted on all asymptotic paths. This follows because the set $\Gamma(f)$ must be the union of continuous images of the sets E_n. Since each one of these E_n is compact, its continuous image will be closed in Ω and so the union of these images must be an F_σ.

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