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ON THE GLUING OF GERMS OF COMPLEX ANALYTIC SPACES, BETTI NUMBERS AND THEIR STRUCTURE

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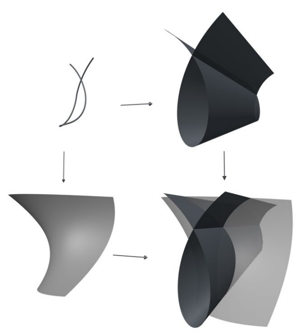
Abstract. In this paper, we introduce new classes of gluing of complex analytic space germs, called weakly large, large, and strongly large. We describe their Poincaré series and, as applications, we give numerical criteria to determine when these classes of gluing of germs of complex analytic spaces are smooth, singular, complete intersections and Gorenstein, in terms of their Betti numbers. In particular, we show that the gluing of the same germ of complex analytic space along any subspace is always a singular germ.

1. Introduction

In modern algebraic geometry, gluing constructions are a relevant topic of investigation by several authors over the years (for instance [6], [7], [10] and [20]). In the case that $(X,x) \subset (\mathbb{C}^n, x)$, $(Y,y) \subset (\mathbb{C}^m, y)$ and $(Z,z) \subset (\mathbb{C}^l, z)$ are germs of complex analytic spaces, in [6] the authors have shown that the gluing $(X,x) \sqcup_{(Z,z)} (Y,y)$ is also a germ of a complex analytic space, provided $\mathcal{O}_{X,x} \to \mathcal{O}_{Z,z}$ and $\mathcal{O}_{Y,y} \to \mathcal{O}_{Z,z}$ are both surjective homomorphisms. Also, it is given the description of some algebraic/geometric and topological invariants such as the degree of a finite map germ, multiplicity and the Milnor number.



Gluing of two surfaces along a curve

The study of the structure of a germ of a complex analytic space deserves special attention from the Singularity Theory viewpoint ([8], [9], [16], [17] and [19]). Some results concerning the Cohen-Macaulayness of the gluing $(X,x) \sqcup_{(Z,z)} (Y,y)$ are provided in [6], when (Z,z) is a reduced point, and show that the structure of the gluing of germs of analytic spaces may have severe changes, depending on how this gluing is being made. For instance, the gluing of two Cohen-Macaulay surfaces can not be Cohen-Macaulay, and the gluing of two germs of analytic spaces that are complete intersections with isolated singularities (ICIS) is not

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always a complete intersection (see [6, Proposition 4.1, Theorem 4.3]). When (Z, z) is not a reduced point, results concerning the structure of the gluing (when is singular, smooth, complete intersection or Gorenstein) are not known.

The main focus of the present paper is to define classes of gluing of germs of a complex analytic space, called weakly large, large and strongly large gluing, and give numerical criteria to determine when it is smooth, singular, hypersurface, complete intersection and Gorenstein. The class of strongly large gluing contains, for instance, the gluing $(X, x) \sqcup_{(Z,z)} (Y, y)$ when (Z, z) is a reduced point, and the gluing $(X, x) \sqcup_{(Z,z)} (X, x)$. For this purpose, we give a description of their Poincaré series and the Betti numbers in terms of the germs involved.

We briefly describe the contents of the paper. In Section 2, we recall the main definitions and results for the rest of the work. Section 3 is devoted to defining the classes of weakly large, large and strongly large gluing of complex analytic space germs, to give a shape of their Poincaré series (see Lemma 3.9 and Theorem 3.13) and, as the main consequence, the description of their Betti numbers (Corollaries 3.12 and 3.15).

The last section contains the applications of the paper. Actually, using the obtained Betti numbers, we give numerical criteria to determine when the defined classes of gluing in Section 2 are singular, hypersurfaces, complete intersections and Gorenstein. For instance, large gluing of germs of complex analytic spaces can be smooth, but any strongly large gluing is singular (Proposition 4.3 and Theorem 4.6 (i)). Despite the defined classes have a subtle difference, these results also illustrate that, for instance, the Betti numbers of strongly large gluing of complex analytic space germs provide a better understanding concerning their structure. As the main consequence of this section, we derive that the gluing $(X, x) \sqcup_{(Z,z)} (X, x)$ is always singular (see Corollary 4.9).

2. Setup and Background

In this section, we recall the main concepts and results for the rest of the paper. For the basic definitions see [12] and [13].

Definition 2.1. Let $\Omega \subset \mathbb{C}^n$ be an open subset. A closed subset $X \subset \Omega$ is called *an analytic subset (or analytic set)* of Ω if for all $x \in X$, there is an open neighborhood $V \subset \Omega$ of x and a finite set of analytic functions $f_1, \dots, f_s \in \mathcal{O}_n(\Omega)$ defined on V such that

$$X \cap V = \{x \in V \mid f_1(x) = \cdots = f_s(x) = 0\}.$$

Definition 2.2. A *ringed space* $(X, \mathcal{O}_X)$ is a Hausdorff topological space X together with a sheaf of rings $\mathcal{O}_X$. In this case, $\mathcal{O}_X$ is a sheaf of commutative rings on an analytic set X. To simplify, we write X for the pair $(X, \mathcal{O}_X)$. In particular, if the stalk $\mathcal{O}_{X,x}$ is a local ring for every $x \in X$, we call $(X, \mathcal{O}_X)$ a *locally ringed space*.

A pair $(\varphi, \varphi^*) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is called a *morphism of ringed spaces* if the map $\varphi : X \to Y$ is continuous and $\varphi^* : \mathcal{O}_Y \to \varphi_*\mathcal{O}_X$ is a morphism of sheaves of rings. Also, $\varphi_*\mathcal{O}_X$ is the sheaf of commutative rings given by $\varphi_*\mathcal{O}_X(U) = \Gamma(\varphi^{-1}(U), \mathcal{O}_X)$, for any open subset $U \subset Y$.

A morphism of locally ringed spaces $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ is a morphism of ringed spaces $(\varphi, \varphi^*) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ such that for all $x \in X$, the homomorphism $\varphi_x^* : \mathcal{O}_{Y,\varphi(x)} \to \mathcal{O}_{X,x}$ induced from φ^* is a local homomorphism, i.e., ${\varphi^*}^{-1}(\mathfrak{m}_{X,x}) = \mathfrak{m}_{Y,\varphi(x)}$.

A morphism $(\varphi, \varphi^*) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is an isomorphism if φ is a homeomorphism and φ^* is an isomorphism of sheaves of rings.

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From now on, the ringed space $(X, \mathcal{O}_X)$ and the morphism of ringed spaces (φ, φ^*) will be denoted by X and φ, respectively.

Lemma 2.3. [13, Theorem 6.1.10] *Let $\varphi : X \to Y$ be a morphism of $\mathbb{C}$-ringed spaces. Then φ is an isomorphism if and only if φ is a homeomorphism and φ_x^* is an isomorphism for every $x \in X$.*

Definition 2.4. A ringed space $(X, \mathcal{O}_X)$ is called an *analytic space* if every $x \in X$ has a neighborhood U such that $(U, \mathcal{O}_X(U))$ is isomorphic to a local model $(V, \mathcal{O}_V)$ as locally ringed spaces, i.e., V is an analytic subset of an open set $\Omega \subset \mathbb{C}^n$ for some n, and $\mathcal{O}_V = (\mathcal{O}_n(\Omega)/\mathcal{I}_V)|_V$.

Definition 2.5. On the set $\mathfrak{A}$ of pairs (X, x) consisting of an analytic space X and its point x, we define a relation $\sim$ as follows:
$(X, x) \sim (Y, y) \Leftrightarrow$ there is a neighborhood $U \subset X$ of x, a neighborhood $V \subset Y$ of y and an isomorphism $f : U \cong V$ such that $f(x) = y$.
The relation becomes an equivalence relation; let the quotient set $\mathfrak{G} := \mathfrak{A}/\sim$. An element of $\mathfrak{G}$ is called a *germ of an analytic space*, denoted by (X, x).

A morphism of germs $(X, x) \to (Y, y)$ is a germ of an analytic spaces map $X \to Y$. For an open $U \subset X$, a point $x \in U$, and an analytic map $\varphi : U \to Y$ with $\varphi(x) = y$, we denote the induced germ by $\varphi_x : (X, x) \to (Y, y)$.

Remark 2.6. It should be noted that the elements of the stalks $\mathcal{O}_{X,x}$ are seen as germs at x of holomorphic functions on X. Each germ is represented by a holomorphic function $f \in \mathcal{O}_X(U)$, defined on an open neighborhood U of x. Conversely, each $f \in \mathcal{O}_X(U)$ defines a unique germ at $x \in U$, which is denoted by f_x. Hence, since $(X, \mathcal{O}_X)$ is an analytic space and $x = (a_1, \ldots, a_n) \in X \subset \Omega \subset \mathbb{C}^n$, one has the isomorphism

$$\mathcal{O}_{X,x} \cong \mathcal{O}_{\mathbb{C}^n,x}/\mathcal{I}_{X,x} \cong \mathbb{C}\{x_1 - a_1, \ldots, x_n - a_n\}/\mathcal{I}_{X,x},$$

where $\mathcal{I}_{X,x} = \{f_x \in \mathcal{O}_{\mathbb{C}^n,x} \mid \exists\ f \in \mathcal{O}_{\mathbb{C}^n}(U)$ representing f_x and $f|_{U\cap X} = 0\}$. Now, the fact that $\mathcal{O}_{\mathbb{C}^n,x}$ is Noetherian gives that the ideal $\mathcal{I}_{X,x}$ is finitely generated, and so there exists $f_1, \ldots, f_k \in \mathcal{O}_{\mathbb{C}^n,x}$ such that $\mathcal{I}_{X,x} = \langle f_1, \ldots, f_k\rangle$. For this paper, $\mathcal{I}_{X,x}$ is an ideal that defines the germ (X, x) of an analytic space. Note that $\mathcal{O}_{X,x}$ is an analytic $\mathbb{C}$-algebra and is a local ring with maximal ideal $\mathfrak{m}_{X,x} = \{f \in \mathcal{O}_{X,x} \mid f(x) = 0\}$.

Set $X \coprod Y$ as the co-product or disjoint union of sets X and Y.

Definition 2.7. Let $\alpha : Z \to X$ and $\beta : Z \to Y$ be morphisms of ringed spaces. Set

$$X \sqcup_Z Y = X \coprod Y/\sim,$$

where the relation $\sim$ is generated by relations of the form $x \sim y$ ($x \in X$, $y \in Y$), provided there exists $z \in Z$ such that $\alpha(z) = x$ and $\beta(z) = y$.

Namely, it is the smallest equivalence relation on $X \coprod Y$ such that after passing to the quotient $X \coprod Y/\sim$ the following square becomes commutative

$$\begin{array}{ccc} Z & \xrightarrow{\alpha} & X \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{g} & X \sqcup_Z Y. \end{array} \tag{2.1}$$

where f and g are the continuous natural maps.

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Since $(X, \mathcal{O}_X)$, $(Y, \mathcal{O}_Y)$ and $(Z, \mathcal{O}_Z)$ are ringed spaces, [20, Proposition 2.2] provides that $(X \sqcup_Z Y, \mathcal{O}_{X \sqcup_Z Y})$ is also a ringed space, and so f and g becomes morphisms of ringed spaces. Also, note that this definition satisfies the universal property by [20, Theorem 2.3].

Analogous to the morphisms of germs $\alpha_z : (Z, z) \to (X, \alpha(z))$ and $\beta_z : (Z, z) \to (Y, \beta(z))$, the previous definition can be made for germs of analytic spaces (X, x), (Y, y) and (Z, z), and denoted by $(X, \alpha(z)) \sqcup_{(Z,z)} (Y, \beta(z))$. In the rest of the paper $(X, \alpha(z)) \sqcup_{(Z,z)} (Y, \beta(z))$ will be denoted by $(X, x) \sqcup_{(Z,z)} (Y, y)$, where $\alpha(z) = x$ and $\beta(z) = y$. When a germ (Z, z) is a reduced point, i.e, $(Z, z) = (z, z)$, we will denote (Z, z) by $\{z\}$.

Now, we recall an important definition for this paper.

Definition 2.8. The fiber product of homomorphisms $\alpha_z^* : \mathcal{O}_{X,x} \to \mathcal{O}_{Z,z}$, $\beta_z^* : \mathcal{O}_{Y,y} \to \mathcal{O}_{Z,z}$ of $\mathbb{C}$-algebras is defined by

$$\mathcal{O}_{X,x} \times_{\mathcal{O}_{Z,z}} \mathcal{O}_{Y,y} := \{(s, t) \in \mathcal{O}_{X,x} \times \mathcal{O}_{Y,y} \mid \alpha_z^*(s) = \beta_z^*(t)\}.$$

By [2, Lemma 1.2], the fiber product is also a commutative and local ring with maximal ideal given by $\mathfrak{m} = \mathfrak{m}_{X,x} \times_{\mathfrak{m}_{Z,z}} \mathfrak{m}_{Y,y}$, where $\mathfrak{m}_{X,x}$, $\mathfrak{m}_{Y,y}$ and $\mathfrak{m}_{Z,z}$ are the maximal ideals of $\mathcal{O}_{X,x}$, $\mathcal{O}_{Y,y}$ and $\mathcal{O}_{Z,z}$, respectively. Also it is a subring of $\mathcal{O}_{X,x} \times \mathcal{O}_{Y,y}$ and universal with respect to the commutative diagram

$$\begin{array}{ccc} \mathcal{O}_{X,x} \times_{\mathcal{O}_{Z,z}} \mathcal{O}_{Y,y} & \xrightarrow{\pi_1} & \mathcal{O}_{X,x} \\ \downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle \alpha_z^*} \\ \mathcal{O}_{Y,y} & \xrightarrow{\beta_z^*} & \mathcal{O}_{Z,z}, \end{array} \tag{2.2}$$

where $\pi_1(s, t) = s$ and $\pi_2(s, t) = t$ are natural surjections. Also in [2, Section 1 (1.0.3)] and [5, Lemma 2.1] the authors have shown that $\mathcal{O}_{X,x} \times_{\mathcal{O}_{Z,z}} \mathcal{O}_{Y,y}$ is a Noetherian local ring if both α_z^* and β_z^* are surjective maps. It is important to realize that the assumptions over the maps are crucial for the Noetherianess of the fiber product ring [6, Example 2.9]. Also, if (X, x), (Y, y) and (Z, z) are germs of analytic spaces, then $\mathcal{O}_{X,x} \times_{\mathcal{O}_{Z,z}} \mathcal{O}_{Y,y}$ is a reduced ring ([3, Proposition 4.2.18]). For the fiber product $\mathcal{O}_{X,x} \times_{\mathcal{O}_{Z,z}} \mathcal{O}_{Y,y}$ we assume that $\mathcal{O}_{X,x} \neq \mathcal{O}_{Z,z} \neq \mathcal{O}_{Y,y}$. Note that every $\mathcal{O}_{X,x}$-module (or $\mathcal{O}_{Y,y}$-module) is an $\mathcal{O}_{X,x} \times_{\mathcal{O}_{Z,z}} \mathcal{O}_{Y,y}$-module via Diagram 2.2.

Remark 2.9. [6, Remark 2.11] Let $(X, x) \subset (\mathbb{C}^n, x)$ and $(Y, y) \subset (\mathbb{C}^m, y)$ be two germs of analytic spaces, where $x = (a_1, \ldots, a_n)$ and $y = (b_1, \ldots, b_m)$. Let $\mathcal{I}_{X,x}$ and $\mathcal{I}_{Y,y}$ be defining ideals of (X, x) and (Y, y), respectively. Consider $R = \frac{\mathcal{O}_{n+m,(x,y)}}{(\mathcal{I}_{X,x} + \mathcal{I}_{Y,y} + ((x_i - a_i)(y_j - b_j)))}$, $i = 1, \ldots, n$ and $j = 1, \ldots, m$, and let $I = (x_1 - a_1, \ldots, x_n - a_n)$ and $J = (y_1 - b_1, \ldots, y_m - b_m)$ be two ideals of R. Note that $I \cap J = 0$ and therefore

$$\frac{\mathcal{O}_{n+m,(x,y)}}{(\mathcal{I}_{X,x} + \mathcal{I}_{Y,y} + ((x_i - a_i)(y_j - b_j)))} \cong \mathcal{O}_{X,x} \times_{\mathbb{C}} \mathcal{O}_{Y,y}.$$

In particular, $\mathcal{O}_{X,x} \times_{\mathbb{C}} \mathcal{O}_{Y,y}$ is an analytic $\mathbb{C}$-algebra and the ideal $\mathcal{I}_{X,x} + \mathcal{I}_{Y,y} + ((x_i - a_i)(y_j - b_j))$ defines $(X, x) \sqcup_{\{z\}} (Y, y)$.

Below we summarize the key results shown in [6] which establishes the good structure of the gluing of complex analytic space germs.

Lemma 2.10. [6, Proposition 2.10] *Let $\alpha : Z \to X$ and $\beta : Z \to Y$ be holomorphic mappings of analytic spaces. Then,*

$$\mathcal{O}_{(X,\alpha(z)) \sqcup_{(Z,z)} (Y,\beta(z))} \cong \mathcal{O}_{X,\alpha(z)} \times_{\mathcal{O}_{Z,z}} \mathcal{O}_{Y,\beta(z)}.$$

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Lemma 2.11. [6, Lemma 3.1 and Corollary 3.3(b)] *Let $(X,x) \subset (\mathbb{C}^n, x)$, $(Y,y) \subset (\mathbb{C}^m, y)$ and $(Z,z) \subset (\mathbb{C}^l, z)$ be germs of analytic spaces such that $\mathcal{O}_{X,x} \to \mathcal{O}_{Z,z}$ and $\mathcal{O}_{Y,y} \to \mathcal{O}_{Z,z}$ are both surjective homomorphisms.*

(i) *Then, $\mathcal{O}_{X,x} \times_{\mathcal{O}_{Z,z}} \mathcal{O}_{Y,y}$ is an analytic $\mathbb{C}$-algebra.*
(ii) *There is a germ (W,w) and an isomorphism $\mathcal{O}_{W,w} \to \mathcal{O}_{X,x} \times_{\mathcal{O}_{Z,z}} \mathcal{O}_{Y,y}$ of local analytic $\mathbb{C}$-algebras. In particular $(X,x) \sqcup_{(Z,z)} (Y,y) \cong (W,w)$.*

Theorem 2.12. [6, Theorem 3.4] *Let X, Y and Z be analytic spaces such that $\mathcal{O}_X \to \mathcal{O}_Z$ and $\mathcal{O}_Y \to \mathcal{O}_Z$ are both surjective homomorphisms. Then, $X \sqcup_Z Y$ is an analytic space.*

3. Poincaré series and Betti numbers of gluing of Germs of analytic spaces

The main focus of this section is to define new classes of gluing of germs of complex analytic spaces and give the shape of their Poincaré series and Betti numbers. For this purpose, two important definitions are necessary:

Definition 3.1. Let $(X,x) \subset (\mathbb{C}^n, x)$ be a germ of an analytic space. Let M be a finitely generated $\mathcal{O}_{X,x}$-module. The Poincaré series of M is given by

$$P_M^{\mathcal{O}_{X,x}}(t) := \sum_{i \geq 0} \dim_{\mathbb{C}} \operatorname{Tor}_i^{\mathcal{O}_{X,x}}(M, \mathbb{C})\, t^i,$$

where $\mathbb{C} := \frac{\mathcal{O}_{X,x}}{\mathfrak{m}_{X,x}}$ is the residue field. The number $\beta_i^{\mathcal{O}_{X,x}}(M) := \dim_{\mathbb{C}} \operatorname{Tor}_i^{\mathcal{O}_{X,x}}(M, \mathbb{C})$ is called i-th Betti number of M. Let I be an ideal of $\mathcal{O}_{X,x}$. The Poincaré series of $\mathcal{O}_{X,x}/I$ is denoted by

$$P_{(Z,z)}^{(X,x)}(t) := P_{\mathcal{O}_{X,x}/I}^{\mathcal{O}_{X,x}}(t),$$

where (Z,z) is subspace of (X,x) defined by the reduced ideal I of $\mathcal{O}_{X,x}$. The i-th Betti number of (Z,z) is defined by $\beta_i^{(X,x)}(Z,z) := \beta_i^{\mathcal{O}_{X,x}}\left(\frac{\mathcal{O}_{X,x}}{I}\right)$.

Remark 3.2. Let (Z,z) be a subspace of (X,x) defined by the reduced ideal I. Set $\mu(Z,z)$ as the minimal number of generators of $\mathcal{O}_{X,x}/I$. Then

$$P_{(Z,z)}^{(X,x)}(t) = \mu(Z,z) + tP_{(\Omega_1,\omega_1)}^{(X,x)}(t),$$

where (Ω_1, ω_1) is the subspace that represents the first syzygy of $\mathcal{O}_{X,x}/I$ over $\mathcal{O}_{X,x}$ (see [4]).

The next definition is motivated by the work of Levin [15].

Definition 3.3. Let $f : (Y,y) \to (X,x)$ be a morphism of germs of complex analytic spaces, such that the induced map $\mathcal{O}_{X,x} \to \mathcal{O}_{Y,y}$ is a surjective homomorphism. Then f is said to be *large* provided, for any (Z,z) subspace of (Y,y) considered as a subspace of (X,x), the following equality happens

$$P_{(Z,z)}^{(X,x)} = P_{(Z,z)}^{(Y,y)} P_{(Y,y)}^{(X,x)}.$$

Now, we are able to define new classes of gluing of germs of analytic spaces.

Definition 3.4. Let $(X,x) \subset (\mathbb{C}^n, x)$, $(Y,y) \subset (\mathbb{C}^m, y)$ and $(Z,z) \subset (\mathbb{C}^l, z)$ be germs of analytic spaces such that $\mathcal{O}_{X,x} \to \mathcal{O}_{Z,z}$ and $\mathcal{O}_{Y,y} \to \mathcal{O}_{Z,z}$ are both surjective homomorphisms.

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(i) We say that the gluing $(X,x)\sqcup_{(Z,z)}(Y,y)$ is *weakly large*, provided

$$P_{(K,k)}^{(X,x)\sqcup_{(Z,z)}(Y,y)}(t)=P_{(K,k)}^{(X,x)}(t)P_{(X,x)}^{(X,x)\sqcup_{(Z,z)}(Y,y)}(t)\ ,$$

where (K,k) is the subspace of (X,x) that represents the kernel of the map α_z^* as $\mathcal{O}_{X,x}$-module (see Diagram 2.2).

(ii) The gluing $(X,x)\sqcup_{(Z,z)}(Y,y)$ is called *large* provided the map f is large (see Diagram 2.1 and Definition 3.3). In addition, if the map g is also large, we call the gluing $(X,x)\sqcup_{(Z,z)}(Y,y)$ as *strongly large* gluing of germs of analytic spaces.

It easy to see that every strongly large gluing is large and therefore weakly large. The next example and remark show that these new classes of gluing of germs of analytic spaces are non-empty and contain interesting types of gluing.

Example 3.5. *Let $(X,x)\subset(\mathbb{C}^n,x)$, $(Y,y)\subset(\mathbb{C}^m,y)$ and $(Z,z)\subset(\mathbb{C}^l,z)$ be germs of analytic spaces such that $\mathcal{O}_{X,x}\to\mathcal{O}_{Z,z}$ and $\mathcal{O}_{Y,y}\to\mathcal{O}_{Z,z}$ are both surjective homomorphisms.*

(i) *If the germ (Z,z) is a reduced point, the gluing $(X,x)\sqcup_{\{z\}}(Y,y)$ is strongly large. In fact, by* [14, Proposition 3.1] *the maps f and g are large.*

(ii) *Suppose that there are surjective ring homomorphism $\mathcal{O}_{X,x}\to\mathcal{O}_{Y,y}$ and the kernel of $\mathcal{O}_{Y,y}\to\mathcal{O}_{Z,z}$ is a weak complete intersection ideal in $\mathcal{O}_{X,x}$. Then the gluing $(X,x)\sqcup_{(Z,z)}(Y,y)$ is large* [18, Theorem 3.12].

Remark 3.6. If we assume germs $(X,x)\subset(\mathbb{C}^n,x)$, $(Y,y)\subset(\mathbb{C}^m,y)$ and $(Z,z)\subset(\mathbb{C}^l,z)$ of analytic spaces such that $\mathcal{O}_{Y,y}\to\mathcal{O}_{X,x}\to\mathcal{O}_{Z,z}$ are both surjective homomorphisms, [18, 3.11] gives that the map g (Diagram 2.1) is large and therefore the gluing $(X,x)\sqcup_{(Z,z)}(Y,y)$ is large. In particular one has that the gluing $(X,x)\sqcup_{(Z,z)}(X,x)$ is strongly large.

Notation 3.7. Throughout this paper, in order to use the structural results given in Lemma 2.10, Lemma 2.11 and Theorem 2.12, we assume germs of analytic spaces $(X,x)\subset(\mathbb{C}^n,x)$, $(Y,y)\subset(\mathbb{C}^m,y)$ and $(Z,z)\subset(\mathbb{C}^l,z)$ such that $\mathcal{O}_{X,x}\to\mathcal{O}_{Z,z}$ and $\mathcal{O}_{Y,y}\to\mathcal{O}_{Z,z}$ are both surjective homomorphisms.

We pose the following conjecture:

Conjecture 3.8. Every gluing $(X,x)\sqcup_{(Z,z)}(Y,y)$ of complex analytic space germs is large.

The next result is a key ingredient for the rest of the paper and shows the explicit shape of the Poincaré series of certain gluing of germs of analytic spaces.

Lemma 3.9. *Let $(X,x)\subset(\mathbb{C}^n,x)$, $(Y,y)\subset(\mathbb{C}^m,y)$ and $(Z,z)\subset(\mathbb{C}^l,z)$ be germs of analytic spaces.*

(i) *If the gluing $(X,x)\sqcup_{(Z,z)}(Y,y)$ is weakly large, then*

$$P_{(X,x)}^{(X,x)\sqcup_{(Z,z)}(Y,y)}(t)=\frac{1-P_{(Y,y)}^{(X,x)\sqcup_{(Z,z)}(Y,y)}(t)}{1-P_{(Z,z)}^{(X,x)}(t)}.$$

(ii) *Suppose that the gluing $(X,x)\sqcup_{(Z,z)}(Y,y)$ is large. If (W,w) is a subspace of (X,x), then*

$$P_{(W,w)}^{(X,x)\sqcup_{(Z,z)}(Y,y)}(t)=\frac{P_{(W,w)}^{(X,x)}(t)\left(1-P_{(Y,y)}^{(X,x)\sqcup_{(Z,z)}Y,y}(t)\right)}{1-P_{(Z,z)}^{(X,x)}(t)}.$$

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Proof. (i) The exact sequence

$$0 \longrightarrow \ker(\alpha_z^*) \longrightarrow \mathcal{O}_{X,x} \times_{\mathcal{O}_{Z,z}} \mathcal{O}_{Y,y} \xrightarrow{\pi_2} \mathcal{O}_{Y,y} \longrightarrow 0 \tag{3.1}$$

and Remark 3.2 gives

$$P^{(X,x)\sqcup_{(Z,z)}(Y,y)}_{(Y,y)}(t) = 1 + tP^{(X,x)\sqcup_{(Z,z)}(Y,y)}_{(K,k)}(t), \tag{3.2}$$

where (K,k) is the subspace of (X,x) that represents the kernel of the map α_z^*. Since $(X,x)\sqcup_{(Z,z)}(Y,y)$ is weakly large, one obtains

$$tP^{(X,x)\sqcup_{(Z,z)}(Y,y)}_{(K,k)}(t) = tP^{(X,x)}_{(K,k)}(t)P^{(X,x)\sqcup_{(Z,z)}(Y,y)}_{(X,x)}(t) = \left(P^{(X,x)}_{(Z,z)}(t) - 1\right)P^{(X,x)\sqcup_{(Z,z)}(Y,y)}_{(X,x)}(t), \tag{3.3}$$

where the last equality follows by the exact sequence

$$0 \longrightarrow \ker(\alpha_z^*) \longrightarrow \mathcal{O}_{X,x} \xrightarrow{\ker(\alpha_z^*)} \mathcal{O}_{Z,z} \longrightarrow 0 \tag{3.4}$$

and Remark 3.2. Hence (3.2) and (3.3) provide

$$P^{(X,x)\sqcup_{(Z,z)}(Y,y)}_{(Y,y)}(t) = 1 + \left(P^{(X,x)}_{(Z,z)}(t) - 1\right)P^{(X,x)\sqcup_{(Z,z)}(Y,y)}_{(X,x)}(t), \tag{3.5}$$

and therefore

$$P^{(X,x)\sqcup_{(Z,z)}(Y,y)}_{(X,x)}(t) = \frac{1 - P^{(X,x)\sqcup_{(Z,z)}(Y,y)}_{(Y,y)}(t)}{1 - P^{(X,x)}_{(Z,z)}(t)}. \tag{3.6}$$

(ii) Since the gluing $(X,x)\sqcup_{(Z,z)}(Y,y)$ is large, by definition it is also weakly large. Hence, multiplying both sides of (3.6) by $P^{(X,x)}_{(W,w)}(t)$, one has

$$P^{(X,x)\sqcup_{(Z,z)}(Y,y)}_{(W,w)}(t) = P^{(X,x)}_{(W,w)}(t) + P^{(X,x)}_{(Z,z)}(t)P^{(X,x)\sqcup_{(Z,z)}(Y,y)}_{(W,w)}(t) - P^{(X,x)\sqcup_{(Z,z)}(Y,y)}_{(Y,y)}(t)P^{(X,x)}_{(W,w)}(t).$$

Therefore

$$P^{(X,x)\sqcup_{(Z,z)}(Y,y)}_{(W,w)}(t) = \frac{P^{(X,x)}_{(W,w)}(t)\left(1 - P^{(X,x)\sqcup_{(Z,z)}(Y,y)}_{(Y,y)}(t)\right)}{1 - P^{(X,x)}_{(Z,z)}(t)}.$$

□

As a consequence, we derive a formula to compute the Betti numbers of any subspace of the complex analytic germ (X,x) as a subspace of the large gluing $(X,x)\sqcup_{(Z,z)}(Y,y)$.

For the next two results, in order to simplify the notation, let $(\mathcal{V},v)$ denote the gluing of germs $(X,x)\sqcup_{(Z,z)}(Y,y)$ and $\beta_i^T(U) := \beta_i^{(T,t)}(U,u)$, for any germs (T,t) and (U,u).

Proposition 3.10. *Let $(X,x) \subset (\mathbb{C}^n,x)$, $(Y,y) \subset (\mathbb{C}^m,y)$ and $(Z,z) \subset (\mathbb{C}^l,z)$ be germs of analytic spaces. Suppose that the gluing $(\mathcal{V},v)$ is large. Then, for any (W,w) subspace of (X,x),*

$$\sum_{i=0}^{j-1} \beta_i^{\mathcal{V}}(W)\beta_{j-i}^X(Z) = \sum_{i=0}^{j-1} \beta_i^X(W)\beta_{j-1}^{\mathcal{V}}(Y),$$

for each $j \geq 1$ positive integer.

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Proof. Since the gluing $(\mathcal{V}, v)$ is large, Lemma 3.9 (ii) provides

$$P^{(\mathcal{V},v)}_{(W,w)}(t) = \frac{P^{(X,x)}_{(W,w)}(t)\left(1 - P^{(\mathcal{V},v)}_{(Y,y)}(t)\right)}{1 - P^{(X,x)}_{(Z,z)}(t)}.$$

Set $P^{(\mathcal{V},v)}_{(W,w)}(t) = \sum_i \beta^{\mathcal{V}}_i(W)t^i$, $P^{(X,x)}_{(W,w)}(t) = \sum_i \beta^X_i(W)t^i$, $P^{(X,x)}_{(Z,z)}(t) = \sum_i \beta^X_i(Z)t^i$ and $P^{(\mathcal{V},v)}_{(Y,y)}(t) = \sum_i \beta^{\mathcal{V}}_i(Y)t^i$. The previous equality yields

$$\sum_i \beta^{\mathcal{V}}_i(W)t^i \left(1 - \sum_i \beta^X_i(Z)t^i\right) = \sum_i \beta^X_i(W)t^i \left(1 - \sum_i \beta^{\mathcal{V}}_i(Y)t^i\right). \tag{3.7}$$

Note that

$$\sum_i \beta^{\mathcal{V}}_i(W)t^i \left(1 - \sum_i \beta^X_i(Z)t^i\right) = \sum_i \beta^{\mathcal{V}}_i(W)t^i - \sum_i \beta^{\mathcal{V}}_i(W)t^i \sum_i \beta^X_i(Z)t^i$$

$$= \sum_i \beta^{\mathcal{V}}_i(W)t^i - \sum_{j\geq 0}\left(\sum_{i=0}^{j} \beta^{\mathcal{V}}_i(W)\beta^X_{j-i}(Z)\right)t^j.$$

Similarly, the right side of equality (3.7) gives

$$\sum_i \beta^{\mathcal{V}}_i(W)t^i - \sum_{j\geq 0}\left(\sum_{i=0}^{j} \beta^{\mathcal{V}}_i(W)\beta^X_{j-i}(Z)\right)t^j = \sum_i \beta^X_i(W)t^i - \sum_{j\geq 0}\left(\sum_{i=0}^{j} \beta^X_i(W)\beta^{\mathcal{V}}_{j-i}(Y)\right)t^j. \tag{3.8}$$

Therefore, for each $j \geq 1$,

$$\beta^{\mathcal{V}}_j(W) - \sum_{i=0}^{j} \beta^{\mathcal{V}}_i(W)\beta^X_{j-i}(Z) = \beta^X_j(W) - \sum_{i=0}^{j} \beta^X_i(W)\beta^{\mathcal{V}}_{j-i}(Y).$$

The fact $\beta^X_0(Z) = 1 = \beta^{\mathcal{V}}_0(Y)$ furnishes

$$\sum_{i=0}^{j-1} \beta^{\mathcal{V}}_i(W)\beta^X_{j-i}(Z) = \sum_{i=0}^{j-1} \beta^X_i(W)\beta^{\mathcal{V}}_{j-1}(Y),$$

for each $j \geq 1$ positive integer and therefore, the desired conclusion follows. □

Remark 3.11. It is important to realize that $\beta^X_1(Z) \neq 0 \neq \beta^Y_1(Z)$, because otherwise, for instance, if $\beta^X_1(Z) := \beta^{\mathcal{O}_{X,x}}_1(\mathcal{O}_{Z,z}) = 0$, then $\mathcal{O}_{Z,z}$ is a free $\mathcal{O}_{X,x}$-module. The surjective map $\mathcal{O}_{X,x} \xrightarrow{\alpha^\star_z} \mathcal{O}_{Z,z}$ and the fact that $\mathcal{O}_{Z,z} = \mathcal{O}^{\oplus r}_{X,x}$, implies that $r = 1$, (i.e., $\mathcal{O}_{X,x} = \mathcal{O}_{Z,z}$). This is a contradiction because $\mathcal{O}_{X,x} \neq \mathcal{O}_{Z,z}$.

Corollary 3.12. *Let $(X, x) \subset (\mathbb{C}^n, x)$, $(Y, y) \subset (\mathbb{C}^m, y)$ and $(Z, z) \subset (\mathbb{C}^l, z)$ be germs of analytic spaces. Suppose that the gluing $(X, x) \sqcup_{(Z,z)} (Y, y)$ is large. Then, for any (W, w) subspace of (X, x),*

(i) $\beta^{\mathcal{V}}_0(W) = \dfrac{\beta^X_0(W)\beta^{\mathcal{V}}_1(Y)}{\beta^X_1(Z)}$.

(ii) $\beta^{\mathcal{V}}_1(W) = \dfrac{1}{\beta^X_1(Z)}\left[\dfrac{\beta^{\mathcal{V}}_1(Y)\left(\beta^X_1(W)\beta^X_1(Z) - \beta^X_0(W)\beta^X_2(Z)\right)}{\beta^X_1(Z)} + \beta^X_0(W)\beta^{\mathcal{V}}_2(Y)\right]$.

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(iii) $\beta_2^{\mathcal{V}}(W) = \frac{\beta_0^X(W)}{\beta_1^X(Z)}\left[\beta_3^{\mathcal{V}}(Y) + \frac{\beta_1^{\mathcal{V}}(Y)}{\beta_1^X(Z)}\left(\beta_2^X(Z)k - \beta_3^X(Z)\right) - k\beta_2^{\mathcal{V}}(Y)\right] + \frac{\beta_1^X(W)}{\beta_1^X(Z)}\left[\beta_2^{\mathcal{V}}(Y) - k\beta_1^{\mathcal{V}}(Y)\right] + \frac{\beta_2^X(W)\beta_1^{\mathcal{V}}(Y)}{\beta_1^X(Z)}$, *where* $k = \dfrac{\beta_2^X(Z)}{\beta_1^X(Z)}$.

Proof. (i) By Proposition 3.10 in the case $j = 1$, one has

$$\beta_0^{\mathcal{V}}(W)\beta_1^X(Z) = \beta_0^X(W)\beta_1^{\mathcal{V}}(Y),$$

which implies that

$$\beta_0^{\mathcal{V}}(W) = \frac{\beta_0^X(W)\beta_1^{\mathcal{V}}(Y)}{\beta_1^X(Z)}.$$

This gives (i). The proof of (ii) and (iii) follows analogous by taking $j = 2$ and $j = 3$ in Proposition 3.10, respectively, together with the fact obtained in (i). □

Theorem 3.13. *Let $(X,x)\sqcup_{(Z,z)}(Y,y)$ be the gluing of the germs of analytic spaces $(X,x) \subset (\mathbb{C}^n, x)$, $(Y,y) \subset (\mathbb{C}^m, y)$ and $(Z,z) \subset (\mathbb{C}^l, z)$, satisfying one of the following conditions:*

(i) *$(X,x)\sqcup_{(Z,z)}(Y,y)$ is weakly large and there is a surjective map $\mathcal{O}_{Y,y} \twoheadrightarrow \mathcal{O}_{X,x}$.*
(ii) *$(X,x)\sqcup_{(Z,z)}(Y,y)$ is strongly large.*

If (W,w) is a subspace of (Y,y), the Poincaré series of (W,w) as a subspace of the gluing $(X,x)\sqcup_{(Z,z)}(Y,y)$ is given by

$$P_{(W,w)}^{(X,x)\sqcup_{(Z,z)}(Y,y)}(t) = \frac{P_{(W,w)}^{(Y,y)}(t)P_{(Z,z)}^{(X,x)}(t)}{P_{(Z,z)}^{(X,x)}(t) + P_{(Z,z)}^{(Y,y)}(t) - P_{(Z,z)}^{(X,x)}(t)P_{(Z,z)}^{(Y,z)}(t)}.$$

Proof. (i) Note that Lemma 3.9 (i) (see (3.5)) furnishes

$$P_{(Y,y)}^{(X,x)\sqcup_{(Z,z)}(Y,y)}(t) = 1 + P_{(Z,z)}^{(X,x)}(t)P_{(X,x)}^{(X,x)\sqcup_{(Z,z)}(Y,y)}(t) - P_{(X,x)}^{(X,x)\sqcup_{(Z,z)}(Y,y)}(t). \tag{3.9}$$

From the exact sequence

$$0 \longrightarrow \ker(\beta_z^*) \longrightarrow \mathcal{O}_{X,x} \times_{\mathcal{O}_{Z,z}} \mathcal{O}_{Y,y} \longrightarrow \mathcal{O}_{X,x} \longrightarrow 0, \tag{3.10}$$

similarly to the proof of Lemma 3.9 (i), one obtains

$$P_{(X,x)}^{(X,x)\sqcup_{(Z,z)}(Y,y)}(t) = 1 + tP_{(K,k)}^{(X,x)\sqcup_{(Z,z)}(Y,y)}(t) = 1 + tP_{(K,k)}^{(Y,y)}(t)P_{(Y,y)}^{(X,x)\sqcup_{(Z,z)}(Y,y)}(t) = 1 + (P_{(Z,z)}^{(Y,y)}(t) - 1)P_{(Y,y)}^{(X,x)\sqcup_{(Z,z)}(Y,y)}(t), \tag{3.11}$$

where (K,k) is the subspace of (X,x) that represents the kernel of the map α_z^*, and the second equality follows by the hypothesis and Remark 3.6.

Replacing (3.11) in (3.9) one has

$$P_{(Y,y)}^{(X,x)\sqcup_{(Z,z)}(Y,y)}(t) = \frac{P_{(Z,z)}^{(X,x)}(t)}{P_{(Z,z)}^{(X,x)}(t) + P_{(Z,z)}^{(Y,y)}(t) - P_{(Z,z)}^{(X,x)}(t)P_{(Z,z)}^{(Y,y)}(t)}. \tag{3.12}$$

Again, by the hypothesis and Remark 3.6, multiplying both sides of equation (3.12) by $P_{(W,w)}^{(Y,y)}(t)$ the desired conclusion follows.

(ii) Since $(X,x)\sqcup_{(Z,z)}(Y,y)$ is strongly large, it is also weakly large. So, as in (3.9),

$$P_{(Y,y)}^{(X,x)\sqcup_{(Z,z)}(Y,y)}(t) = 1 + P_{(Z,z)}^{(X,x)}(t)P_{(X,x)}^{(X,x)\sqcup_{(Z,z)}(Y,y)}(t) - P_{(X,x)}^{(X,x)\sqcup_{(Z,z)}(Y,y)}(t). \tag{3.13}$$

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With an analogous argument used in (i) and a base change, it is possible to show that

$$P_{(X,x)}^{(X,x)\sqcup_{(Z,z)}(Y,y)}(t) = 1 + P_{(Z,z)}^{(Y,y)}(t)P_{(Y,y)}^{(X,x)\sqcup_{(Z,z)}(Y,y)}(t) - P_{(Y,y)}^{(X,x)\sqcup_{(Z,z)}(Y,y)}(t), \tag{3.14}$$

and therefore the statement is similarly obtained. □

Corollary 3.14. *Let $(X,x) \subset (\mathbb{C}^n, x)$ and $(Z,z) \subset (\mathbb{C}^l, z)$ be germs of analytic spaces. If (W,w) is a subspace of (X,x), then*

$$P_{(W,w)}^{(X,x)\sqcup_{(Z,z)}(X,x)}(t) = \frac{P_{(W,w)}^{(X,x)}(t)}{2 - P_{(Z,z)}^{(X,x)}(t)}.$$

Proof. The result is a consequence of Remark 3.6 and Theorem 3.13 (ii). □

The next result shows the explicit shape of certain Betti numbers of the subspace (W,w) of (Y,y) seen as a subspace of the gluing $(X,x) \sqcup_{(Z,z)} (Y,y)$. We omit the proof because it is similar to Corollary 3.12.

Corollary 3.15. *Let $(X,x) \subset (\mathbb{C}^n, x)$, $(Y,y) \subset (\mathbb{C}^m, y)$ and $(Z,z) \subset (\mathbb{C}^l, z)$ be germs of analytic spaces. Consider (W,w) a subspace of (Y,y). If the gluing $(X,x) \sqcup_{(Z,z)} (Y,y)$ satisfies one of the conditions of Theorem 3.13, then*

(i) $\beta_0^{(X,x)\sqcup_{(Z,z)}(Y,y)}(W,w) = \beta_0^{(Y,y)}(W,w)$.
(ii) $\beta_1^{(X,x)\sqcup_{(Z,z)}(Y,y)}(W,w) = \beta_0^{(Y,y)}(W,w)\beta_1^{(X,x)}(Z,z) + \beta_1^{(Y,y)}(W,w)$.
(iii) $\beta_2^{(X,x)\sqcup_{(Z,z)}(Y,y)}(W,w) = \beta_0^{(Y,y)}(W,w)\beta_1^{(Y,y)}(Z,z)\beta_1^{(X,x)}(Z,z) + \beta_0^{(Y,y)}(W,w)\beta_2^{(X,x)}(Z,z) + \beta_1^{(Y,y)}(W,w)\beta_1^{(X,x)}(Z,z) + \beta_2^{(Y,y)}(W,w)$.

4. Applications

This section is devoted to show some consequences of the previous Betti numbers obtained. First, we recall some basic definitions for the convenience of the reader.

Embedding dimension: For a Noetherian local ring $(R, \mathfrak{m})$, the minimal number of generators of $\mathfrak{m}$ will be denoted by $\operatorname{edim}(R) := \dim_{\mathbb{C}} \mathfrak{m}/\mathfrak{m}^2$ and is called the embedding dimension of R. Recall that, in general, $\operatorname{edim}(R) \geq \dim(R)$. If this inequality is an equality, then R is called a regular local ring. The *embedding dimension of a germ of an analytic space* (X,x), denoted by $\operatorname{edim}(X,x)$, means the embedding dimension of the local ring $\mathcal{O}_{X,x}$.

Again, let $(\mathcal{V}, v)$ denote the gluing $(X,x) \sqcup_{(Z,z)} (Y,y)$, $\beta_i^T(U) := \beta_i^{(T,t)}(U,u)$, $\dim(T,t) := \dim(T)$ and the embedding dimension $\operatorname{edim}(T,t) := \operatorname{edim}(T)$, for any germs (T,t) and (U,u). An important fact for the rest of this section is that ([2, Lemma 1.5 (1.5.2)])

$$\dim(\mathcal{V}) = \max\{\dim(X), \dim(Y)\}.$$

As a consequence of the characterization of the Betti numbers of the gluing of germs of complex analytic spaces (Corollary 3.12 (ii) and Corollary 3.15 (ii)), a formula for their embedding dimension is also provided.

Corollary 4.1. *Let $(X,x) \subset (\mathbb{C}^n, x)$, $(Y,y) \subset (\mathbb{C}^m, y)$ and $(Z,z) \subset (\mathbb{C}^l, z)$ be germs of analytic spaces.*

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(i) *If the gluing* $(\mathcal{V}, v)$ *is large, then*

$$\operatorname{edim}(\mathcal{V}) = \frac{1}{\beta_1^X(Z)}\left[\frac{\beta_1^{\mathcal{V}}(Y)\left(\operatorname{edim}(X)\beta_1^X(Z) - \beta_2^X(Z)\right)}{\beta_1^X(Z)} + \beta_2^{\mathcal{V}}(Y)\right].$$

(ii) *If the gluing* $(\mathcal{V}, v)$ *satisfies one of the conditions of Theorem* 3.13*, then*

$$\operatorname{edim}(\mathcal{V}) = \beta_1^X(Z) + \operatorname{edim}(Y).$$

Example 4.2. Let $X = \mathbb{C}^2$ and $Y = V(v^2 - u^3)$ be two analytic subspaces of $\mathbb{C}^2$ and consider $(X, 0)$ and $(Y, 0)$ its respective germs at the origin. By Remark 2.9 one has that the ideal that defines $(X, 0) \sqcup_{\{0\}} (Y, 0)$ is given by $\mathcal{I}_{(X,0)\sqcup_{\{0\}}(Y,0)} = (v^2 - u^3, xu, xv, yu, yv)$. In addition, Example 3.5 (i) gives that the gluing $(X, 0) \sqcup_{\{0\}} (Y, 0)$ satisfies the condition (ii) of Theorem 3.13. Note that $\operatorname{edim}((X, 0) \sqcup_{\{0\}} (Y, 0)) = 4$, $\operatorname{edim}((Y, 0)) = 2$, $\beta_1^X(0) = 2$, and this illustrates Corollary 4.1 (ii).

Proposition 4.3. *Let* $(X, x) \subset (\mathbb{C}^n, x)$*,* $(Y, y) \subset (\mathbb{C}^m, y)$ *and* $(Z, z) \subset (\mathbb{C}^l, z)$ *be germs of analytic spaces such that* $\dim(\mathcal{V}) = \dim(X)$*. Suppose that the gluing* $(\mathcal{V}, v)$ *is large. Then* $(\mathcal{V}, v)$ *is smooth if and only if the following equality holds*

$$\beta_1^X(Z)\left(\beta_1^{\mathcal{V}}(Y)\operatorname{edim}(X) - \dim(X)\beta_1^X(Z) + \beta_2^{\mathcal{V}}(Y)\right) = \beta_1^{\mathcal{V}}(Y)\beta_2^X(Z).$$

Proof. By definition, $(\mathcal{V}, v)$ is smooth if and only if $\operatorname{edim}(\mathcal{V}) = \dim(\mathcal{V})$. Hence, using Corollary 4.1 (i), one has that $(\mathcal{V}, v)$ is smooth if and only if

$$\dim(X) = \frac{\beta_1^{\mathcal{V}}(Y)\left(\operatorname{edim}(X)\beta_1^X(Z) - \beta_2^X(Z)\right)}{\beta_1^X(Z)^2} + \frac{\beta_2^{\mathcal{V}}(Y)\beta_1^X(Z)}{\beta_1^X(Z)^2}. \tag{4.1}$$

Solving (4.1) for $\beta_1^X(Z)$, one obtains that $(\mathcal{V}, v)$ is smooth if and only if

$$\beta_1^X(Z)\left(\beta_1^{\mathcal{V}}(Y)\operatorname{edim}(X) - \dim(X)\beta_1^X(Z) + \beta_2^{\mathcal{V}}(Y)\right) = \beta_1^{\mathcal{V}}(Y)\beta_2^X(Z),$$

and this shows the statement. □

As an immediate consequence of Proposition 4.3, we derive the following.

Corollary 4.4. *Let* $(X, x) \subset (\mathbb{C}^n, x)$*,* $(Y, y) \subset (\mathbb{C}^m, y)$ *and* $(Z, z) \subset (\mathbb{C}^l, z)$ *be germs of analytic spaces such that* $\dim(\mathcal{V}) = \dim(X)$*. Suppose that the gluing* $(\mathcal{V}, v)$ *is large and* $\beta_1^{\mathcal{V}}(Y) = \beta_1^X(Z) = 1$*. Then* $(\mathcal{V}, v)$ *is smooth if and only if*

$$\operatorname{edim}(X) - \dim(X) = \beta_2^X(Z) - \beta_2^{\mathcal{V}}(Y).$$

Proposition 4.5. *Let* $(X, x) \subset (\mathbb{C}^n, x)$*,* $(Y, y) \subset (\mathbb{C}^m, y)$ *and* $(Z, z) \subset (\mathbb{C}^l, z)$ *be germs of analytic spaces. Suppose that* $(\mathcal{V}, v)$ *is large and* $\beta_2^X(Z) = 0$*. Then* $(\mathcal{V}, v)$ *is a complete intersection if and only if*

$$m\dim(X) + \operatorname{edim}(X)\left[\beta_2^{\mathcal{V}}(Y)(1 - \frac{l}{m}) - \frac{l}{2}\right] = \frac{1}{2m}\left[l^2\operatorname{edim}(X)^2 + \beta_2^{\mathcal{V}}(Y)^2\right] + \frac{\beta_2^{\mathcal{V}}(Y)}{2} - \beta_3^{\mathcal{V}}(Y) - l\beta_2^X(0),$$

where $l = \beta_1^{\mathcal{V}}(Y)$ *and* $m = \beta_1^X(Z)$*.*

Proof. By [1, Theorem 7.3.3] (or [11, Proposition 2.8.4 (3)]), $(\mathcal{V}, v)$ is a complete intersection if and only if

$$\beta_2^{\mathcal{V}}(0) = \binom{\beta_1^{\mathcal{V}}(0)}{2} + \beta_1^{\mathcal{V}}(0) - \dim(\mathcal{V}). \tag{4.2}$$

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Since $\beta_2^X(Z)=0$, we obtain that the projective dimension of the germ (Z,z) over (X,x) is smaller than 1, and so $\beta_3^X(Z)=0$. Now, by Corollary 3.12 (ii)-(iii), the shape of the Betti numbers $\beta_1^{\mathcal{V}}(0)$ and $\beta_2^{\mathcal{V}}(0)$ are given by

$$\beta_1^{\mathcal{V}}(0)=\frac{\beta_1^{\mathcal{V}}(Y)\,\mathrm{edim}(X)+\beta_2^{\mathcal{V}}(Y)}{\beta_1^X(Z)}; \tag{4.3}$$

$$\beta_2^{\mathcal{V}}(0)=\frac{\beta_3^{\mathcal{V}}(Y)+\mathrm{edim}\,(X)\beta_2^{\mathcal{V}}(Y)+\beta_2^X(0)\beta_1^{\mathcal{V}}(Y)}{\beta_1^X(Z)}. \tag{4.4}$$

Replacing (4.3) in (4.2) and comparing with (4.4), the desired result follows immediately. ☐

It should be noted that the last results show that it is difficult to have large gluing of germs of analytic spaces that are smooth and complete intersection. For the cases of Theorem 3.13, the next result yields a better understanding of their structure.

Theorem 4.6. *Let $(X,x)\subset(\mathbb{C}^n,x)$, $(Y,y)\subset(\mathbb{C}^m,y)$ and $(Z,z)\subset(\mathbb{C}^l,z)$ be germs of analytic spaces such that $\dim(\mathcal{V})=\dim(Y)$. Suppose that the gluing $(\mathcal{V},v)$ satisfies one of the conditions of Theorem 3.13.*

- (i) *Then $(\mathcal{V},v)$ is singular.*
- (ii) *Suppose that $(\mathcal{V},v)$ is Cohen-Macaulay. Then $(\mathcal{V},v)$ is a hypersurface if and only if (Y,y) is smooth and $\beta_1^X(Z)=1$.*
- (iii) *Suppose that (X,x) is a complete intersection. Then $(\mathcal{V},v)$ is a complete intersection if and only if*
$$\frac{\beta_1^X(Z)^2+\beta_1^X(Z)}{\beta_1^X(Z)\beta_1^Y(Z)+\beta_2^X(Z)}=2.$$
- (iv) *Suppose that $(\mathcal{V},v)$ is Cohen-Macaulay. If $\beta_1^{\mathcal{V}}(W)\le\beta_0^{\mathcal{V}}(W)$ for some (W,w) subspace of (Y,y), then $(\mathcal{V},v)$ is Gorenstein if and only if (Y,y) is smooth.*

Proof. (i) Suppose that $(\mathcal{V},v)$ is smooth. Then, Corollary 4.1 (ii) gives

$$\dim(Y)=\dim(\mathcal{V})=\mathrm{edim}(\mathcal{V})=\beta_1^X(Z)+\mathrm{edim}(Y).$$

So $\beta_1^X(Z)=0$ because $\mathrm{edim}(Y)\ge\dim(Y)$, which is a contradiction (Remark 3.11).

(ii) By (i), since $(\mathcal{V},v)$ is singular one has $\mathrm{edim}(\mathcal{V})-\dim(\mathcal{V})>0$. Hence $(\mathcal{V},v)$ is a hypersurface if and only if $\mathrm{edim}(\mathcal{V})-\mathrm{depth}(\mathcal{V})=1$. Since $(\mathcal{V},v)$ is Cohen-Macaulay by hypothesis, Corollary 4.1 (ii) furnishes

$$\beta_1^X(Z)+\mathrm{edim}(Y)-\dim(Y)=1.$$

The facts $\beta_1^X(Z)\neq 0$ (Remark 3.11) and $\mathrm{edim}(X)\ge\dim(X)$ yield that $(\mathcal{V},v)$ is a hypersurface if and only if $\beta_1^X(Z)=1$ and $\mathrm{edim}(Y)=\dim(Y)$ (i.e., Y is smooth).

(iii) Set $d:=\dim(\mathcal{V})$. By [11, Proposition 2.8.4 (3)]), $(\mathcal{V},v)$ is a complete intersection if and only if

$$\beta_2^{\mathcal{V}}(0)=\binom{\overline{e}}{2}+\overline{e}-d, \tag{4.5}$$

where $\overline{e}:=\mathrm{edim}(\mathcal{V})=\beta_1^X(Z)+\mathrm{edim}(Y)$ (Corollary 4.1 (ii)). By Corollary 3.15 (iii) and the fact that (X,x) is a complete intersection ([11, Proposition 2.8.4 (3)]) yield

$$\beta_2^{\mathcal{V}}(0)=\beta_1^Y(Z)\beta_1^X(Z)+\beta_2^X(Z)+e_2\beta_1^X(Z)+\binom{e_2}{2}+e_2-d, \tag{4.6}$$

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where $e_2 := \operatorname{edim}(Y)$. Therefore, comparing (4.5) and (4.6) we obtains that $(\mathcal{V}, v)$ is a complete intersection if and only if

$$\beta_1^X(Z)^2 + \beta_1^X(Z) = 2\left(\beta_1^Y(Z)\beta_1^X(Z) + \beta_2^X(Z)\right).$$

The desired conclusion follows, because $\beta_1^Y(Z) \neq 0 \neq \beta_1^X(Z)$ (Remark 3.11).

(iv) Suppose that (Y, y) is smooth. Since $\beta_1^X(Z) \neq 0$ (Remark 3.11), the hypothesis and Corollary 3.15 (i)-(ii) provide $\beta_1^X(Z) = 1$ and $\beta_1^Y(W) = 0$. Therefore $(\mathcal{V}, v)$ is Gorenstein by (ii). The converse immediately follows from [5, Proposition 4.19]. ☐

Example 4.7. Let $X = \mathbb{C}$ and $Y = \mathbb{C}$ be two analytic spaces and consider $(X, 0)$ and $(Y, 0)$ its respective germs at the origin. Since $(X, 0)$ and $(Y, 0)$ are regular germs, Remark 2.9 provides that the ideal that defines $(X, 0) \sqcup_{\{0\}} (Y, 0)$ is given by $\mathcal{I}_{(X,0)\sqcup_{\{0\}}(Y,0)} = (xy)$. Therefore, the gluing $(X, 0) \sqcup_{\{0\}} (Y, 0)$ is a complete intersection, and strongly large (Example 3.5 (i)). In addition, it satisfies both conditions of Theorem 3.13. This example illustrates that even considering the most natural and simple smooth germs of analytic spaces, their gluing is singular, as stated in the previous result.

Example 4.8. Let $X = V(x^5)$, $Y = V(y^5)$ and $Z = V(z^2)$ be analytic subspaces of $\mathbb{C}$ and consider $(X, 0)$, $(Y, 0)$ and $(Z, 0)$ their respective germs at the origin. Note that $(X, 0)$ is a complete intersection and the gluing $(X, 0) \sqcup_{(Z,0)} (Y, 0)$ is defined by the ideal

$$\mathcal{I}_{(X,0)\sqcup_{(Z,0)}(Y,0)} = (u^5, uv^2, v^2 - u^2v)$$

in $\mathbb{C}\{u, v\}$, which is not a complete intersection ([2, Example 3.4]). By Example 3.5 (ii), the gluing $(X, 0) \sqcup_{(Z,0)} (Y, 0)$ is large, because the ideal (y^2) in $\mathbb{C}\{y\}/(y^5)$ is the kernel of the map $\mathbb{C}\{y\}/(y^5) \to \mathbb{C}\{z\}/(z^2)$ and it is a weak complete intersection ideal (by [18, Example 2.3 (ii)]). Since $\beta_1^Y(Z) = \beta_1^X(Z) = \beta_2^X(Z) = 1$, one has

$$\frac{\beta_1^X(Z)^2 + \beta_1^X(Z)}{\beta_1^X(Z)\beta_1^Y(Z) + \beta_2^X(Z)} \neq 2.$$

As mentioned in Remark 3.6, the gluing $(X, x) \sqcup_{(Z,z)} (X, x)$ is always strongly large. Since the dimension of $\dim (X, x) \sqcup_{(Z,z)} (X, x)$ and $\dim(X)$ are equal, as a consequence of Theorem 3.13 we derive the following result.

Corollary 4.9. *Let $(X, x) \subset (\mathbb{C}^n, x)$, and $(Z, z) \subset (\mathbb{C}^l, z)$ be germs of analytic spaces.*

(i) *Then $(X, x) \sqcup_{(Z,z)} (X, x)$ is singular.*
(ii) *If $(X, x) \sqcup_{(Z,z)} (X, x)$ is Cohen-Macaulay, then $(X, x) \sqcup_{(Z,z)} (X, x)$ is a hypersurface if and only if X is smooth and $\beta_1^X(Z) = 1$.*
(iii) *If (X, x) is a complete intersection, then $(X, x) \sqcup_{(Z,z)} (X, x)$ is a complete intersection if and only if $\beta_1^X(Z) = 1$ and $\beta_2^X(Z) = 0$.*
(iv) *Suppose that $(X, x) \sqcup_{(Z,z)} (X, x)$ is Cohen-Macaulay. If $\beta_1^{\mathcal{V}}(W) \leq \beta_0^{\mathcal{V}}(W)$ for some (W, w) subspace of (X, x), then $(\mathcal{V}, v)$ is Gorenstein if and only if (X, x) is smooth.*

Proof. The proof of (i), (ii) and (iv) are immediate consequences of Theorem 3.13 (i)-(ii)-(iv). For (iii), Theorem 3.13 (iii) furnishes

$$\beta_1^X(Z) - \beta_1^X(Z)^2 = 2\beta_2^X(Z). \tag{4.7}$$

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Note that, if $\beta_1^X(Z) > 1$, then left side of (4.7) is a negative number. Since $\beta_2^X(Z) \geq 0$, the equality (4.7) occurs if and only if $\beta_1^X(Z) = 1$ and $\beta_2^X(Z) = 0$ or $\beta_1^X(Z) = 0$ and $\beta_2^X(Z) = 0$. But $\beta_1^X(Z) \neq 0$ (Remark 3.11), and therefore the result follows. □

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