EXISTENCE OF SOLUTIONS AND OPTIMAL CONTROL PROBLEMS FOR HYPERBOLIC HEMIVARIATIONAL INEQUALITIES

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Abstract

In this paper we prove the existence of solutions for hyperbolic hemivariational inequalities and then investigate optimal control problems for some convex cost functionals.

1. Introduction

In this paper we shall study the following optimal control problem:

$$\text{Minimise} \quad J(y, u, v)$$

subject to a hyperbolic hemivariational inequality of the form

$$
\begin{cases}
y''(t) - \alpha \Delta y'(t) - \beta \Delta y(t) + \Xi(t) = Bu(t) & \text{a.e. } t \in (0, T), \\
y(t) = 0 & \text{on } \Gamma, \\
y(0) = y_0, \quad y'(0) = y_1, \\
\Xi(x, t) \in \varphi(x, t, v(x, t), y'(x, t)) & \text{a.e. } (x, t) \in Q,
\end{cases}
\tag{1.1}
$$

where Ω is a bounded domain in $\mathbb{R}^n (n \geq 1)$ with sufficiently smooth boundary $\Gamma = \partial\Omega$, $Q = \Omega \times (0, T)$, $y' = \partial y/\partial t$, $y'' = \partial^2 y/\partial t^2$, $\Delta y = \sum_{i=1}^{n} \partial^2 y/\partial x_i^2$, and α and β are positive real numbers. Here φ is a discontinuous and nonlinear multi-valued mapping by filling in jumps of a locally bounded function b, u and v denote the control variables, B is a bounded linear operator and the cost functional $J(y, u, v)$ is given by

$$
J(y, u, v) = \int_0^T \{g(y(t)) + h(u(t), v(t))\} \, dt,
\tag{1.2}
$$

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where g and h are convex functionals. From a physical point of view, y in (1.1) represents displacement, y' velocity and y'' acceleration. Recently, the theory of variational inequalities, which is closely related to the convexity of the energy functionals involved, has been considerably developed and optimal control problems for such variational inequalities have been investigated by many authors [1, 2, 8, 19]. It is well known that the existence of solutions for variational inequalities is based on monotonicity arguments and the derivation of the necessary optimality conditions is based on the subdifferentials of convex analysis [1]. On the other hand, following the work of Duvaut and Lions [4], several new types of variational problems in an inequality form have been investigated. The background of these variational problems is in physics, especially in solid mechanics, where nonconvex, nonmonotone and multivalued constitutive laws lead to hemivariational inequalities. We refer to [12] and [14] to see some applications of hemivariational inequalities. The existence of solutions for hemivariational inequalities has been proved by some authors [9, 10, 13, 16, 17, 18]. But there is not much literature dealing with optimal control problems for hemivariational inequalities and as far as we know there is no literature deriving the necessary optimality conditions for the corresponding optimal control problems because these are more complicated than those for variational inequalities due to the lack of convexity of the energy functionals. Haslinger and Panagiotopoulos [6] proved the existence of optimal controls for coercive hemivariational inequalities and Migórski and Ochal [11] showed the existence of optimal control pairs for parabolic hemivariational inequalities. Panagiotopoulos [15] considered an application problem for hyperbolic hemivariational inequalities with a multi-valued reaction-velocity law such as the last inequality of (1.1). Motivated by his work, we attempt to prove the existence of solutions for hyperbolic hemivariational inequalities of the form (1.1) and the existence of optimal control pairs for the optimal control problem (P) (see Section 4 below). The plan of this paper is as follows. In Section 2, assumptions and notation are given. In Section 3, the existence of a solution to the problem (1.1) is proved using the Faedo-Galerkin method and finally in Section 4 the existence of solutions to the optimal control problem (P) is investigated.

2. Assumptions and notation

Throughout this paper we denote

$$(y, z) = \int_\Omega y(x)z(x)\, dx \quad \text{and} \quad \|y\|^2 = \int_\Omega |y(x)|^2\, dx.$$

We denote by $\langle \cdot, \cdot \rangle$ the dual pairing between $H_0^1(\Omega)$ and $H^{-1}(\Omega)$. Let U be a real Hilbert space of variable u, $L^2(Q)$ a space of variable v, and $\mathscr{U}_{ad} \times \mathscr{W}_{ad}$ a nonempty

subset of $L^2(0, T; U) \times L^2(Q)$. We denote by $\| \cdot \|_X$ the norm of a Banach space X. Now we assume the following conditions concerning (1.1)–(1.2).

HYPOTHESIS (Hyp.b). $b : Q \times \mathbb{R}^2$, $b(x, t, \eta, \xi) \to \mathbb{R}$ is a locally bounded function satisfying the following conditions:

(i) b is continuous in η uniformly with respect to ξ, that is, there exists $\epsilon_0 > 0$ such that for all $(x, t, \eta, \xi) \in Q \times \mathbb{R}^2$ and for all $\delta > 0$, there exists $\gamma = \gamma(\delta, x, t, \eta, \xi) > 0$ such that $|b(x, t, \eta, \xi) - b(x, t, \eta', \xi')| < \delta$ if $|\eta - \eta'| < \gamma$ and $|\xi - \xi'| < \epsilon_0$.

(ii) $(x, t) \to b(x, t, \eta, \xi)$ is continuous on Q for all $\eta \in \mathbb{R}$ and a.e. $\xi \in \mathbb{R}$.

(iii) $(x, t, \xi) \to b(x, t, \eta, \xi)$ is measurable in $Q \times \mathbb{R}$ for all $\eta \in \mathbb{R}$.

(iv) $|b(x, t, \eta, \xi)| \leq \mu_0(x, t) + \mu_1(1 + |\eta| + |\xi|)$, for all $(x, t, \eta, \xi) \in Q \times \mathbb{R}^2$ with a nonnegative function $\mu_0 \in L^2(Q)$ and a positive constant μ_1.

The multi-valued function $\varphi : Q \times \mathbb{R}^2 \to 2^\mathbb{R}$ is obtained by filling in jumps of a function $b(x, t, \eta, \cdot) : \mathbb{R} \to \mathbb{R}$ by means of the functions $\underline{b}_\epsilon, \overline{b}_\epsilon, \underline{b}, \overline{b} : \mathbb{R} \to \mathbb{R}$ as follows:

$$\underline{b}_\epsilon(x, t, \eta, \xi) = \operatorname*{ess\,inf}_{|s-\xi|\leq\epsilon} b(x, t, \eta, s), \qquad \overline{b}_\epsilon(x, t, \eta, \xi) = \operatorname*{ess\,sup}_{|s-\xi|\leq\epsilon} b(x, t, \eta, s),$$

$$\underline{b}(x, t, \eta, \xi) = \lim_{\epsilon\to 0^+} \underline{b}_\epsilon(x, t, \eta, \xi), \qquad \overline{b}(x, t, \eta, \xi) = \lim_{\epsilon\to 0^+} \overline{b}_\epsilon(x, t, \eta, \xi),$$

$$\varphi(x, t, \eta, \xi) = [\underline{b}(x, t, \eta, \xi), \overline{b}(x, t, \eta, \xi)].$$

REMARK 2.1. Let $j : Q \times \mathbb{R}^2 \to \mathbb{R}$ be a locally Lipschitz continuous function with respect to the last variable obtained from b by integration, that is,

$$j(x, t, \eta, \xi) = \int_0^\xi b(x, t, \eta, \tau)d\tau.$$

Then the following relation holds [6]:

$$\varphi(x, t, \eta, \xi) = \partial j(x, t, \eta, \xi),$$

where ∂ denotes the generalised gradient of Clarke (see for example [1] for the definition of and the relevant results for Clarke's generalised gradient).

We shall need a regularisation of b defined by

$$b^m(x, t, \eta, \xi) = m \int_{-\infty}^\infty b(x, t, \eta, \xi - \tau)\rho(m\tau)\,d\tau,$$

where $\rho \in C_0^\infty((-1, 1))$, $\rho \geq 0$ and $\int_{-1}^1 \rho(\tau)\,d\tau = 1$.

REMARK 2.2. It is easy to show that $b^m(x, t, \eta, \xi)$ is continuous in t for all $m \in \mathbb{N}$ and that $\underline{b}_\epsilon, \overline{b}_\epsilon, \underline{b}, \overline{b}, b^m$ satisfy the same condition (Hyp.b) (iv) with possibly different constants if b satisfies (Hyp.b) (iv). So, in the remainder of this paper, we denote different constants by the same symbol as original constants.

HYPOTHESIS (Hyp.B). $B : L^2(0, T; U) \rightarrow L^2(0, T; L^2(\Omega))$ is a bounded linear operator.

HYPOTHESIS (Hyp.U). $\mathscr{U}_{ad}$ is a closed convex subset of $L^2(0, T; U)$ and $\mathscr{W}_{ad}$ is a compact subset of $L^2(Q)$.

HYPOTHESIS (Hyp.g). $g : L^2(\Omega) \rightarrow \mathbb{R}$ is proper, convex and continuous. Moreover there exists $k_1 > 0$ and $k_2 \in \mathbb{R}$ such that $g(y) \geq k_1\|y\| + k_2$, for all $y \in L^2(\Omega)$.

HYPOTHESIS (Hyp.h). $h : U \times L^2(\Omega) \rightarrow \bar{\mathbb{R}}$ is a proper, convex and lower semi-continuous functional satisfying $h(u, v) \geq k_3(\|u\|_U^2 + \|v\|^2) + k_4$, for all $(u, v) \in U \times L^2(\Omega)$ and for some $k_3 > 0$ and $k_4 \in \mathbb{R}$.

For details on the definition of convexity and lower semicontinuity of functionals and the relevant results, we refer the readers to [3, 5].

DEFINITION. Given $(u, v) \in L^2(0, T; U) \times L^2(Q)$, $y_0 \in H_0^1(\Omega)$ and $y_1 \in L^2(\Omega)$, y is said to be *a solution* of (1.1) if $y \in L^\infty(0, T; H_0^1(\Omega))$, $y' \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H_0^1(\Omega))$, $y'' \in L^2(0, T; H^{-1}(\Omega))$, there exists $\Xi \in L^2(0, T; L^2(\Omega))$ and the following identities hold:

$$\int_0^t \langle y''(s), w \rangle \, ds + \alpha \int_0^t (\nabla y'(s), \nabla w) \, ds + \beta \int_0^t (\nabla y(s), \nabla w) \, ds \qquad (2.1)$$

$$+ \int_0^t (\Xi(s), w) \, ds = \int_0^t (Bu(s), w) \, ds, \quad \forall t \in [0, T], \forall w \in H_0^1(\Omega),$$

$$\Xi(x, t) \in \varphi(x, t, v(x, t), y'(x, t)) \quad \text{a.e. } (x, t) \in Q, \qquad (2.2)$$

$$y(0) = y_0, \quad y'(0) = y_1. \qquad (2.3)$$

3. Existence results for hemivariational inequalities

In this section we are going to show the existence of solutions to the problem (1.1) using the Faedo-Galerkin approximation.

THEOREM 3.1. *Assume that* (Hyp.b) *and* (Hyp.B) *hold. Let* $(u, v) \in L^2(0, T; U) \times L^2(Q)$ *and* $(y_0, y_1) \in H_0^1(\Omega) \times L^2(\Omega)$. *Then the problem* (1.1) *has a solution.*

PROOF. We represent by $\{w_j\}_{j \geq 1}$ a basis in $H_0^1(\Omega)$ which is orthogonal in $L^2(\Omega)$. Let V_m be the space generated by $w_1, w_2, \ldots, w_m$. We may choose (y_{0m}) and (y_{1m}) in V_m such that $y_{0m} \to y_0$ in $H_0^1(\Omega)$ and $y_{1m} \to y_1$ in $L^2(\Omega)$. Let $y_m(t) = \sum_{j=1}^m g_{jm}(t)w_j$ be the solution to the Cauchy problem:

$$(y_m''(t), w) + \alpha(\nabla y_m'(t), \nabla w) + \beta(\nabla y_m(t), \nabla w) + (b^m(t, v(t), y_m'(t)), w) \quad (3.1)$$
$$= (Bu(t), w), \quad \forall w \in V_m,$$

$$y_m(0) = y_{0m}, \quad y_m'(0) = y_{1m}. \quad (3.2)$$

By standard differential equation methods, we can prove the existence of a solution to (3.1)–(3.2) on some interval $[0, t_m)$. This solution can then be extended to the closed interval $[0, T]$ using the *a priori* estimates below.

Step 1: *A priori estimates.*   Replacing w by $y_m'(t)$ in (3.1), we get

$$\frac{1}{2}\frac{d}{dt}\|y_m'(t)\|^2 + \alpha\|\nabla y_m'(t)\|^2 + \frac{\beta}{2}\frac{d}{dt}\|\nabla y_m(t)\|^2$$
$$= -(b^m(t, v(t), y_m'(t)), y_m'(t)) + (Bu(t), y_m'(t)). \quad (3.3)$$

By (Hyp.b) (iv), there exists $c_1 > 0$ such that

$$\int_0^t \|b^m(s, v(s), y_m'(s))\|^2\,ds$$
$$= \int_0^t \int_\Omega |b^m(x, s, v(x, s), y_m'(x, s))|^2\,dx\,ds$$
$$\leq 2\|\mu_0\|_{L^2(Q)}^2 + 2\mu_1^2 \int_0^t \int_\Omega (1 + |v(x, s)| + |y_m'(x, s)|)^2\,dx\,ds$$
$$\leq c_1 + 2\mu_1^2 \int_0^t \|y_m'(s)\|^2\,ds \quad (3.4)$$

and hence

$$\left| \int_0^t (b^m(s, v(s), y_m'(s)), y_m'(s))\,ds \right|$$
$$\leq \left(\int_0^t \|b^m(s, v(s), y_m'(s))\|^2\,ds \right)^{1/2} \left(\int_0^t \|y_m'(s)\|^2\,ds \right)^{1/2}$$
$$\leq \frac{1}{2}\left\{ c_1 + (2\mu_1^2 + 1) \int_0^t \|y_m'(s)\|^2\,ds \right\}. \quad (3.5)$$

By (3.3) and (3.5), we have

$$\frac{1}{2}\|y_m'(t)\|^2 + \frac{\beta}{2}\|\nabla y_m(t)\|^2 + \alpha \int_0^t \|\nabla y_m'(s)\|^2\,ds$$

$$\leq \frac{1}{2}\|y_{1m}\|^2 + \frac{\beta}{2}\|\nabla y_{0m}\|^2 + \frac{1}{2}\int_0^t \|Bu(s)\|^2\,ds + \frac{1}{2}\int_0^t \|y_m'(s)\|^2\,ds$$

$$+ \frac{1}{2}\left\{c_1 + (2\mu_1^2 + 1)\int_0^t \|y_m'(s)\|^2\,ds\right\}$$

$$\leq C + C\int_0^t \|y_m'(s)\|^2\,ds. \tag{3.6}$$

Here and in what follows we use C to denote a generic constant independent of m. Gronwall's inequality implies that

$$\|y_m'(t)\|^2 + \|\nabla y_m(t)\|^2 + \int_0^t \|\nabla y_m'(s)\|^2\,ds \leq C. \tag{3.7}$$

From (3.4) and (3.7) we also get

$$\int_0^t \|b^m(s, v(s), y_m'(s))\|^2\,ds \leq C. \tag{3.8}$$

So we can extend the solutions $y_m(t)$ to the whole interval $[0, T]$. Finally we will obtain an estimate for y_m''. From (3.1), we get for all $w \in V_m$,

$$|\langle y_m''(t), w\rangle| = \big| - \alpha(\nabla y_m'(t), \nabla w) - \beta(\nabla y_m(t), \nabla w)$$
$$- (b^m(t, v(t), y_m'(t)), w) + (Bu(t), w)\big|. \tag{3.9}$$

So, by a density argument, we have from (3.7)–(3.9) that

$$(y_m'') \text{ is bounded in } L^2(0, T; H^{-1}(\Omega)). \tag{3.10}$$

Step 2: *Passage to the limit.*   From the *a priori* estimates (3.7), (3.8) and (3.10), we have subsequences (in what follows we denote subsequences by the same symbols as original sequences) such that

$$\begin{cases} y_m \to y & \text{weakly* in } L^\infty(0, T; H_0^1(\Omega)), \\ y_m' \to y' & \text{weakly in } L^2(0, T; H_0^1(\Omega)) \text{ and} \\ & \text{weakly* in } L^\infty(0, T; L^2(\Omega)), \\ y_m'' \to y'' & \text{weakly in } L^2(0, T; H^{-1}(\Omega)), \\ b^m(v, y_m') \to \Xi & \text{weakly in } L^2(0, T; L^2(\Omega)). \end{cases} \tag{3.11}$$

Now we can take the limit $m \to \infty$ in (3.1). Therefore we obtain

$$\langle y''(t), w\rangle + \alpha(\nabla y'(t), \nabla w) + \beta(\nabla y(t), \nabla w) + (\Xi(t), w)$$
$$= (Bu(t), w), \quad \forall w \in H_0^1(\Omega). \tag{3.12}$$

Step 3: *y is a solution of* (1.1).   Let $\phi \in C^1([0, T])$ with $\phi(T) = 0$. By replacing v by $\phi(t)w_j$ in (3.1) and integrating by parts the result over $(0, T)$, we have

$$-\int_0^T (y_m'(t), \phi'(t)w_j)\, dt + \alpha \int_0^T (\nabla y_m'(t), \phi(t)\nabla w_j)\, dt + \langle y_{1m}, \phi(0)w_j \rangle$$

$$+ \beta \int_0^T (\nabla y_m(t), \phi(t)\nabla w_j)\, dt + \int_0^T (b^m(t, v(t), y_m'(t)), \phi(t)w_j)\, dt$$

$$= \int_0^T (Bu(t), \phi(t)w_j)\, dt. \tag{3.13}$$

Similarly from (3.12) we get

$$-\int_0^T (y'(t), \phi'(t)w_j)\, dt + \alpha \int_0^T (\nabla y'(t), \phi(t)\nabla w_j)\, dt + \langle y'(0), \phi(0)w_j \rangle$$

$$+ \beta \int_0^T (\nabla y(t), \phi(t)\nabla w_j)\, dt + \int_0^T (\Xi(t), \phi(t)w_j)\, dt$$

$$= \int_0^T (Bu(t), \phi(t)w_j)\, dt. \tag{3.14}$$

Comparing (3.13) and (3.14) we infer that

$$\lim_{m\to\infty} \langle y_{1m} - y'(0), w_j \rangle = 0, \quad j = 1, 2, \ldots.$$

This implies that $y_{1m} \to y'(0)$ weakly in $H^{-1}(\Omega)$. By the uniqueness of the limit, $y'(0) = y_1$. Analogously, taking $\phi \in C^2([0, T])$ with $\phi(T) = \phi'(T) = 0$, we can obtain that $y(0) = y_0$. Next we will show that $\Xi(x, t) \in \varphi(x, t, v(x, t), y'(x, t))$ a.e. $(x, t) \in Q$. By (3.11) and the Aubin-Lions compactness lemma [7], we get

$$y_m' \to y' \quad \text{strongly in } L^2(0, T; L^2(\Omega))$$

and hence

$$y_m'(x, t) \to y'(x, t) \quad \text{a.e. } (x, t) \in Q.$$

Let $\eta > 0$. Using the theorems of Lusin and Egoroff, we can choose a subset $\omega \subset Q$ such that $\text{meas}(\omega) < \eta$, $y' \in L^\infty(Q \setminus \omega)$ and $y_m' \to y'$ uniformly on $Q \setminus \omega$. Thus, for each $\epsilon > 0$, there is an $N > 2/\epsilon$ such that $|y_m'(x, t) - y'(x, t)| < \epsilon/2$, for all $(x, t) \in Q \setminus \omega$ and $m > N$. Then, if $|y_m'(x, t) - s| < 1/m$, we have $|y'(x, t) - s| < \epsilon$ for all $m > N$ and $(x, t) \in Q \setminus \omega$. Therefore we have

$$\underline{b}_\epsilon(x, t, v(x, t), y'(x, t)) \leq b^m(x, t, v(x, t), y_m'(x, t))$$

$$\leq \overline{b}_\epsilon(x, t, v(x, t), y'(x, t)),$$

for all $m > N$ and $(x, t) \in Q \setminus \omega$. Let $\phi \in L^2(Q)$, $\phi \geq 0$. Then

$$\int_{Q \setminus \omega} \underline{b}_\epsilon(x, t, v(x, t), y'(x, t)) \phi(x, t) \, dx \, dt$$

$$\leq \int_{Q \setminus \omega} b^m(x, t, v(x, t), y'_m(x, t)) \phi(x, t) \, dx \, dt$$

$$\leq \int_{Q \setminus \omega} \overline{b}_\epsilon(x, t, v(x, t), y'(x, t)) \phi(x, t) \, dx \, dt. \tag{3.15}$$

Letting $m \to \infty$ in (3.15) and using (3.11), we obtain

$$\int_{Q \setminus \omega} \underline{b}_\epsilon(x, t, v(x, t), y'(x, t)) \phi(x, t) \, dx \, dt$$

$$\leq \int_{Q \setminus \omega} \Xi(x, t) \phi(x, t) \, dx \, dt$$

$$\leq \int_{Q \setminus \omega} \overline{b}_\epsilon(x, t, v(x, t), y'(x, t)) \phi(x, t) \, dx \, dt. \tag{3.16}$$

Letting $\epsilon \to 0^+$ in (3.16), we infer that $\Xi(x, t) \in \varphi(x, t, v(x, t), y'(x, t))$ a.e. in $Q \setminus \omega$, and letting $\eta \to 0^+$ we get $\Xi(x, t) \in \varphi(x, t, v(x, t), y'(x, t))$ a.e. in Q. Therefore the proof of Theorem 3.1 is complete. $\qquad \square$

4. Existence of the solutions of the optimal control problem

We denote by $\mathscr{S}(u, v)$ the set of all solutions of the problem (1.1) for a given $(u, v) \in \mathscr{U}_{ad} \times \mathscr{W}_{ad}$. Theorem 3.1 implies that $\mathscr{S}(u, v) \neq \emptyset$ for any $(u, v) \in \mathscr{U}_{ad} \times \mathscr{W}_{ad}$. Let us consider the following optimal control problem (P):

$$\text{Minimise } \{ J(y, u, v) : (u, v) \in \mathscr{U}_{ad} \times \mathscr{W}_{ad}, \ y \in \mathscr{S}(u, v) \}. \tag{P}$$

For our purpose we need the following proposition.

PROPOSITION 4.1. *For a given* $(u, v) \in \mathscr{U}_{ad} \times \mathscr{W}_{ad}$, *the following estimate holds*:

$$\sup_{y \in \mathscr{S}(u,v)} \left\{ \|y\|_{L^\infty(0,T;H_0^1(\Omega))} + \|y'\|_{L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega))} + \|y''\|_{L^2(0,T;H^{-1}(\Omega))} \right\}$$

$$\leq c(y_0, y_1) + C \left(\|v\|_{L^2(Q)}^2 + \|u\|_{L^2(0,T;U)}^2 \right),$$

where $C > 0$ *and* $c(y_0, y_1)$ *is a positive constant depending on the initial data* y_0 *and* y_1.

PROOF. Let $y \in \mathscr{S}(u, v)$, then y satisfies (2.1)–(2.3). Replacing w by $y'(s)$ in (2.1) and using Young's inequality, we get

$$\frac{1}{2}\|y'(t)\|^2 + \frac{\beta}{2}\|\nabla y(t)\|^2 + \alpha \int_0^t \|\nabla y'(s)\|^2 \, ds$$

$$\leq c(y_0, y_1) + \frac{1}{2}\int_0^t \|Bu(s)\|^2 \, ds + \frac{1}{2}\int_0^t \|\Xi(s)\|^2 \, ds + \int_0^t \|y'(s)\|^2 \, ds. \quad (4.1)$$

By the assumption on b (see (Hyp.b) (iv)) and Remark 2.2, we can easily show that

$$\int_0^t \|\Xi(s)\|^2 \, ds \leq C + C \left\{ \int_0^t (\|y'(s)\|^2 + \|v(s)\|^2) \, ds \right\}. \quad (4.2)$$

By the Gronwall lemma, we get from (4.1)–(4.2) that

$$\|y'(t)\|^2 + \|\nabla y(t)\|^2 + \int_0^t \|\nabla y'(s)\|^2 \, ds$$

$$\leq c(y_0, y_1) + C \left\{ \int_0^t (\|v(s)\|^2 + \|Bu(s)\|^2) \, ds \right\}. \quad (4.3)$$

Moreover, we have from (2.1) that for all $w \in H_0^1(\Omega)$ and $t \in [0, T]$

$$\int_0^t \langle y''(s), w \rangle \, ds = -\alpha \int_0^t (\nabla y'(s), \nabla w) \, ds - \beta \int_0^t (\nabla y(s), \nabla w) \, ds$$

$$- \int_0^t (\Xi(s), w) \, ds + \int_0^t (Bu(s), w) \, ds.$$

By (4.2) and (4.3), we infer that

$$\|y''\|_{L^2(0,T;H^{-1}(\Omega))}^2 \leq c(y_0, y_1) + C \left\{ \int_0^t (\|v(s)\|^2 + \|Bu(s)\|^2) \, ds \right\}. \quad (4.4)$$

Since B is a bounded linear operator, (4.3) and (4.4) complete the proof of Proposition 4.1.   □

THEOREM 4.2. *Assume that the conditions of Theorem 3.1, (Hyp.U), (Hyp.g) and (Hyp.h) hold. Then the optimal control problem (P) has at least one solution.*

PROOF. Let $d = \inf\{J(y, u, v) \mid (u, v) \in \mathscr{U}_{ad} \times \mathscr{W}_{ad}, \ y \in \mathscr{S}(u, v)\}$. By the assumptions on g and h, it is clear that $d > -\infty$. Let $(y_n, u_n, v_n) \in \mathscr{S}(u_n, v_n) \times \mathscr{U}_{ad} \times \mathscr{W}_{ad}$ be a minimising sequence, that is,

$$\int_0^t \langle y_n''(s), w \rangle \, ds + \alpha \int_0^t (\nabla y_n'(s), \nabla w) \, ds$$

$$+ \beta \int_0^t (\nabla y_n(s), \nabla w) \, ds + \int_0^t (\Xi_n(s), w) \, ds$$

$$= \int_0^t (Bu_n(s), w) \, ds, \quad \forall t \in [0, T], \forall w \in H_0^1(\Omega), \tag{4.5}$$

$$\Xi_n(x, t) \in \varphi(x, t, v_n(x, t), y_n'(x, t)) \quad \text{a.e.} \ (x, t) \in Q, \tag{4.6}$$

$$y_n(0) = y_0, \quad y_n'(0) = y_1 \tag{4.7}$$

and

$$d \le J(y_n, u_n, v_n) \le d + 1/n, \quad n = 1, 2, 3, \ldots. \tag{4.8}$$

From ((Hyp.h)), (u_n, v_n) is bounded in $\mathcal{U}_{ad} \times \mathcal{W}_{ad} \subset L^2(0, T; U) \times L^2(Q)$. Accordingly a subsequence can be determined such that

$$u_n \rightarrow u^* \quad \text{weakly in } L^2(0, T; U). \tag{4.9}$$

By (Hyp.U), $\mathcal{U}_{ad}$ is weakly closed, and hence $u^* \in \mathcal{U}_{ad}$. Also, since $\mathcal{W}_{ad}$ is compact in $L^2(Q)$ and (v_n) is bounded in $\mathcal{W}_{ad}$, we infer that

$$v_n \rightarrow v^* \quad \text{strongly in } L^2(Q) \quad \text{and} \quad v^* \in \mathcal{W}_{ad}. \tag{4.10}$$

Therefore, by Proposition 4.1, we get

$$\begin{cases} (y_n) & \text{is bounded in } L^\infty(0, T; H_0^1(\Omega)), \\ (y_n') & \text{is bounded in } L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H_0^1(\Omega)), \\ (y_n'') & \text{is bounded in } L^2(0, T; H^{-1}(\Omega)). \end{cases} \tag{4.11}$$

This together with the fact that $\int_0^t \| \Xi_n(s) \|^2 ds \le C + C \int_0^t (\| y_n'(s) \|^2 + \| v_n(s) \|^2) \, ds$ implies that (Ξ_n) is bounded in $L^2(0, T; L^2(\Omega))$. Therefore we get, along subsequences, that

$$y_n \rightarrow y^* \quad \text{weakly* in } L^\infty(0, T; H_0^1(\Omega)), \tag{4.12}$$

$$y_n' \rightarrow y^{*\prime} \quad \text{weakly* in } L^\infty(0, T; L^2(\Omega)) \text{ and weakly in } L^2(0, T; H_0^1(\Omega)), \tag{4.13}$$

$$y_n'' \rightarrow y^{*\prime\prime} \quad \text{weakly in } L^2(0, T; H^{-1}(\Omega)), \tag{4.14}$$

$$\Xi_n \rightarrow \Xi^* \quad \text{weakly in } L^2(0, T; L^2(\Omega)). \tag{4.15}$$

Therefore, using (4.9)–(4.15) and letting $m \rightarrow \infty$ in (4.5), we conclude that

$$\int_0^t \langle y^{*\prime\prime}(s), w \rangle \, ds + \alpha \int_0^t (\nabla y^{*\prime}(s), \nabla w) \, ds + \beta \int_0^t (\nabla y^*(s), \nabla w) \, ds$$

$$+ \int_0^t (\Xi^*(s), w) \, ds = \int_0^t (Bu^*(s), w) \, ds, \quad \forall t \in [0, T], \forall w \in H_0^1(\Omega).$$

To show that $y^* \in \mathscr{S}(u^*, v^*)$, it is sufficient to show that

$$\Xi^*(x, t) \in \varphi(x, t, v^*(x, t), y^{*'}(x, t)) \quad \text{a.e. } (x, t) \in Q.$$

Indeed, by (4.13), (4.14) and the Aubin-Lions compactness lemma, we get $y'_n \to y^{*'}$ strongly in $L^2(0, T; L^2(\Omega))$ and hence $y'_n(x, t) \to y^{*'}(x, t)$ a.e. $(x, t) \in Q$. By the theorems of Lusin and Egoroff, for a given $\eta > 0$ we can choose a subset $\omega \subset Q$ such that $\text{meas}(\omega) < \eta$ and $y'_n \to y^{*'}$ uniformly on $Q \setminus \omega$. Thus, for each $\epsilon > 0$, there is a positive integer N such that $|y'_n(x, t) - y^{*'}(x, t)| < \epsilon/2$, for all $(x, t) \in Q \setminus \omega$ and $n > N$. On the other hand, (4.6) implies that

$$\int_{Q \setminus \omega} \underline{b}_{\epsilon/2}(x, t, v_n(x, t), y'_n(x, t)) \phi(x, t) \, dx \, dt$$

$$\leq \int_{Q \setminus \omega} \Xi_n(x, t) \phi(x, t) \, dx \, dt$$

$$\leq \int_{Q \setminus \omega} \overline{b}_{\epsilon/2}(x, t, v_n(x, t), y'_n(x, t)) \phi(x, t) \, dx \, dt, \qquad (4.16)$$

for any $\phi \in L^2(Q)$ with $\phi \geq 0$.

Noting that, for $n > N$,

$$\underline{b}_{\epsilon/2}(x, t, v_n(x, t), y'_n(x, t)) = \operatorname*{ess\,inf}_{|s - y'_n| \leq \epsilon/2} b(x, t, v_n(x, t), s)$$

$$\geq \operatorname*{ess\,inf}_{|s - y^{*'}| \leq \epsilon} b(x, t, v_n(x, t), s)$$

$$= \underline{b}_\epsilon(x, t, v_n(x, t), y^{*'}(x, t))$$

and

$$\overline{b}_{\epsilon/2}(x, t, v_n(x, t), y'_n(x, t)) = \operatorname*{ess\,sup}_{|s - y'_n| \leq \epsilon/2} b(x, t, v_n(x, t), s)$$

$$\leq \operatorname*{ess\,sup}_{|s - y^{*'}| \leq \epsilon} b(x, t, v_n(x, t), s)$$

$$= \overline{b}_\epsilon(x, t, v_n(x, t), y^{*'}(x, t)),$$

we get from (4.16) that

$$\int_{Q \setminus \omega} \underline{b}_\epsilon(x, t, v_n(x, t), y^{*'}(x, t)) \phi(x, t) \, dx \, dt$$

$$\leq \int_{Q \setminus \omega} \Xi_n(x, t) \phi(x, t) \, dx \, dt$$

$$\leq \int_{Q \setminus \omega} \overline{b}_\epsilon(x, t, v_n(x, t), y^{*'}(x, t)) \phi(x, t) \, dx \, dt. \qquad (4.17)$$

Letting $n \to \infty$ in (4.17) and using (4.10) and (Hyp.b) (i), we conclude that

$$\int_{Q \backslash \omega} \underline{b}_\epsilon(x, t, v^*(x, t), y^{*\prime}(x, t)) \phi(x, t) \, dx \, dt$$

$$\leq \int_{Q \backslash \omega} \Xi^*(x, t) \phi(x, t) \, dx \, dt$$

$$\leq \int_{Q \backslash \omega} \overline{b}_\epsilon(x, t, v^*(x, t), y^{*\prime}(x, t)) \phi(x, t) \, dx \, dt. \tag{4.18}$$

Letting $\epsilon \to 0^+$ in (4.18), we infer that $\Xi^*(x, t) \in \varphi(x, t, v^*(x, t), y^{*\prime}(x, t))$ a.e. in $Q \setminus \omega$, and letting $\eta \to 0^+$ we get $\Xi^*(x, t) \in \varphi(x, t, v^*(x, t), y^{*\prime}(x, t))$ a.e. in Q. Hence $(y^*, u^*, v^*) \in \mathcal{S}(u^*, v^*) \times \mathcal{U}_{ad} \times \mathcal{W}_{ad}$ is an admissible pair for problem (P). Taking the limit $n \to \infty$ in (4.8) and using the lower semicontinuity of J, we conclude that

$$d \leq J(y^*, u^*, v^*) \leq \lim_{n \to \infty} J(y_n, u_n.v_n) \leq d.$$

Thus (y^*, u^*, v^*) is a solution of the optimal control problem (P). □

Acknowledgements

References

[1] V. Barbu, *Analysis and control of nonlinear infinite-dimensional systems* (Academic Press, New York, NY, 1993).

[2] M. Bergounioux, "Optimal control of problems governed by abstract elliptic variational inequalities with state constraints", *SIAM J. Control Optim.* **36** (1998) 273–289.

[3] L. Cesari, *Optimization — theory and applications* (Springer, Berlin, 1983).

[4] G. Duvaut and J. L. Lions, *Inequalities in mechanics and physics* (Springer, Berlin, 1976).

[5] I. Ekeland and R. Temam, *Convex analysis and variational problems* (North-Holland, Amsterdam, 1976).

[6] J. Haslinger and P. D. Panagiotopoulos, "Optimal control of systems governed by hemivariational inequalities. Existence and approximation results", *Nonlinear Anal.* **24** (1995) 105–119.

[7] J. L. Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaires* (Dunod-Gauthier Villars, Paris, 1969).

[8] J. L. Lions, *Optimal control of systems governed by partial differential equations* (Springer, Berlin, 1971).

[9] M. Miettinen, "A parabolic hemivariational inequality", *Nonlinear Anal.* **26** (1996) 725–734.

[10] M. Miettinen and P. D. Panagiotopoulos, "On parabolic hemivariational inequalities and applications", *Nonlinear Anal.* **35** (1999) 885–915.

[11] S. Migórski and A. Ochal, "Optimal control of parabolic hemivariational inequalities", *J. Global Optim.* **17** (2000) 285–300.

[12] P. D. Panagiotopoulos, *Inequality problems in mechanics and applications. Convex and nonconvex energy functions* (Birkhäuser, Basel, 1985).

[13] P. D. Panagiotopoulos, "Coercive and semicoercive hemivariational inequalities", *Nonlinear Anal.* **16** (1991) 209–231.

[14] P. D. Panagiotopoulos, *Hemivariational inequalities and applications in mechanics and engineering* (Springer, New York, 1993).

[15] P. D. Panagiotopoulos, "Modelling of nonconvex nonsmooth energy problems. Dynamic hemivariational inequalities with impact effects", *J. Comput. Appl. Math.* **63** (1995) 123–138.

[16] J. Y. Park, H. M. Kim and S. H. Park, "On weak solutions for hyperbolic differential inclusion with discontinuous nonlinearities", *Nonlinear Anal.* **55** (2003) 103–113.

[17] J. Y. Park and S. H. Park, "Solutions for a hyperbolic system with boundary differential inclusion and nonlinear second-order boundary damping", *Electron. J. Differential Equations*, **80** (2003) 1–7.

[18] J. Rauch, "Discontinuous semilinear differential equations and multiple valued maps", *Proc. Amer. Math. Soc.* **64** (1977) 277–282.

[19] G. Wang, "Optimal control of parabolic differential equations with two point boundary state constraints", *SIAM J. Control Optim.* **38** (2000) 1639–1654.