A THEOREM ON ZARISKI RINGS

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Given a Noetherian ring A with unit element and an ideal $\mathfrak{m}$ of A such that

$$\bigcap_{n=1}^{\infty} \mathfrak{m}^n = (0),$$

we may topologize A by adopting $\{\mathfrak{m}^n; n = 1, 2, \ldots\}$ as a fundamental system of neighborhoods of 0. This topologized ring is usually referred to as an $\mathfrak{m}$-adic ring, and is called a Zariski ring if its ideals are all closed. An $\mathfrak{m}$-adic ring is a Zariski ring if and only if $\mathfrak{m}$ is contained in its Jacobson radical (i.e., the intersection of all its maximal ideals).

Let A be an $\mathfrak{m}$-adic Zariski ring, and let us denote its completion by $\hat{A}$. Let $\mathfrak{a}, \mathfrak{b}$ be arbitrary ideals of A; then a theorem due to P. Samuel (**4**, p. 158, Theorem 4; **1**, p. 54, Theorem 1) says:

$$(\mathfrak{a} \cap \mathfrak{b})\hat{A} = \mathfrak{a}\hat{A} \cap \mathfrak{b}\hat{A}.$$

In this note, we shall derive two consequences of this theorem. We shall first state the following theorem, which seems not to be found in the literature, at least not in its full generality, though it is known in some special cases and has been found useful.

THEOREM 1. *Let $\mathfrak{a}$ be an arbitrary ideal of A, then if $\mathfrak{a}\hat{A}$ is generated by r elements, so also is $\mathfrak{a}$. In particular, if $\mathfrak{a}\hat{A}$ is principal, so is $\mathfrak{a}$.*

Proof. Suppose that $\mathfrak{a}\hat{A} = (\hat{a}_1, \ldots, \hat{a}_r)$, and take integer ρ (**5**, p. 47, Lemma 3) such that $\mathfrak{m}\mathfrak{a} \supseteq \mathfrak{m}^\rho \cap \mathfrak{a}$. Choose elements a_i $(i = 1, 2, \ldots, r)$ in $\mathfrak{a}$ such that $a_i \equiv \hat{a}_i(\mathfrak{m}^\rho \hat{A})$, and set $\mathfrak{a}' = (a_1, \ldots, a_r)A$. Then

$$a_i - \hat{a}_i \in \mathfrak{m}^\rho \hat{A} \cap \mathfrak{a}\hat{A} = (\mathfrak{m}^\rho \cap \mathfrak{a})\hat{A} \subseteq \mathfrak{m}\mathfrak{a}\hat{A},$$

whence $\hat{a}_i \in \mathfrak{a}'\hat{A} + \mathfrak{m}\mathfrak{a}\hat{A}$, $\mathfrak{a}\hat{A} \subseteq (\mathfrak{a}' + \mathfrak{m}\mathfrak{a})\hat{A}$. Hence $\mathfrak{a} \subseteq \mathfrak{a}' + \mathfrak{m}\mathfrak{a}$, and it follows that $\mathfrak{a} \subseteq \mathfrak{a}' + \mathfrak{m}^\nu \mathfrak{a} \subseteq \mathfrak{a}' + \mathfrak{m}^\nu$ for any positive integer ν. Finally, $\mathfrak{a} = \mathfrak{a}' = (a_1, \ldots, a_r)$.

THEOREM 2. (i) *Let $\mathfrak{z}$ be a prime ideal of A and $\hat{\mathfrak{z}}$ an isolated prime divisor of $\mathfrak{z}\hat{A}$, then* rank $\mathfrak{z} \geqq$ rank $\hat{\mathfrak{z}}$. (ii) *Let $\mathfrak{a}$ be an ideal of A, then* rank $\mathfrak{a} \geqq$ rank $\mathfrak{a}\hat{A}$. (iii) *In addition, if A is a local ring, then* dim $\mathfrak{a}$ = dim $\mathfrak{a}\hat{A}$.

Proof. (i) Put rank $\mathfrak{z} = r$, and take an ideal $\mathfrak{b}$ of A which is generated by r elements and has $\mathfrak{z}$ as one of its isolated prime divisors (**3**, p. 61, Theorem 8). Let $\mathfrak{b} = \mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_s$ be an irredundant representation of $\mathfrak{b}$ as an intersection

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of primary ideals and let $\mathfrak{z}_i$ $(i = 1, 2, \ldots s)$ be the prime divisors of $\mathfrak{q}_i$; we assume $\mathfrak{z}_1 = \mathfrak{z}$. Then

$$\mathfrak{b}\hat{A} = \mathfrak{q}_1 \hat{A} \cap \ldots \cap \mathfrak{q}_s \hat{A}.$$

Let $\hat{\mathfrak{z}}_i$ be a prime divisor of $\mathfrak{q}_i \hat{A}$; then, as is well known, $\hat{\mathfrak{z}}_i \cap A = \mathfrak{z}_i$; and $\mathfrak{z}_i \hat{A}$ and $\mathfrak{q}_i \hat{A}$ have the same isolated prime divisors. From these observations it follows that $\hat{\mathfrak{z}}$ is an isolated prime divisor of $\mathfrak{b}\hat{A}$. Thus our claim is substantiated by Krull's Primidealkettensatz (**3**, p. 60, Theorem 7).

(ii) Let $\mathfrak{z}$ be an isolated prime divisor of $\mathfrak{a}$ such that rank $\mathfrak{a}$ = rank $\mathfrak{z}$; and let $\hat{\mathfrak{z}}$ be an isolated prime divisor of $\mathfrak{z}\hat{A}$. Then it follows from the above observation that $\hat{\mathfrak{z}}$ is an isolated prime divisor of $\mathfrak{a}\hat{A}$, and

$$\text{rank } \mathfrak{z} \geqq \text{rank } \hat{\mathfrak{z}} \geqq \text{rank } \mathfrak{a}\hat{A}.$$

(iii) We have only to recall that $\hat{A}/\mathfrak{a}\hat{A}$ may be regarded as the completion of $A/\mathfrak{a}$; hence $\hat{A}/\mathfrak{a}\hat{A}$ and $A/\mathfrak{a}$ have the same dimension.

Next we shall consider the following two properties of a local ring A.

(α) For any prime ideal $\mathfrak{z}$ of A, rank $\mathfrak{z}$+dim $\mathfrak{z}$ = dim A.

(β) All the maximal chains of prime ideals of A have the same length.

Now (β) obviously implies (α), but the writer does not know whether conversely (α) implies (β). However, *in the case that A is complete,* (α) *and* (β) *are equivalent, and moreover A has the property* (α) *if and only if all the minimal prime ideals of A have the same dimension.* These facts are familiar and readily seen by the fundamental structure theorem for complete local rings due to I. S. Cohen.

COROLLARY. *If $\hat{A}$ has the property* (α), *so also has A.*

Proof. Let $\mathfrak{z}$ be a prime ideal of A, then

$$\dim A \geqq \text{rank } \mathfrak{z} + \dim \mathfrak{z} \geqq \text{rank } \mathfrak{z}\hat{A} + \dim \mathfrak{z}\hat{A} = \dim \hat{A} = \dim A.$$

This noteworthy fact was found by M. Nishi (**2**).

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