BULL. AUSTRAL. MATH. SOC. 58C20, 46G05
VOL. 51 (1995) [235–248]

THE SUBDIFFERENTIAL OF THE SUM OF TWO FUNCTIONS IN BANACH SPACES II. SECOND ORDER CASE

ROBERT DEVILLE AND EL MAHJOUB EL HADDAD

We prove a formula for the second order subdifferential of the sum of two lower semi continuous functions in finite dimensions. This formula yields an Alexandrov type theorem for continuous functions. We derive from this uniqueness results of viscosity solutions of second order Hamilton-Jacobi equations and singlevaluedness of the associated Hamilton-Jacobi operators. We also provide conterexamples in infinite dimensional Hilbert spaces.

1. INTRODUCTION AND DEFINITIONS

In the second section of this paper, we give a formula for the second order subdifferential (see precise definition below) of the sum of two lower semi continuous functions in finite dimensions. This formula was implicitly proved by Jensen [4], we refer to the survey paper of Crandall, Ishii and Lions [2]. However it has never been stated separately and we feel that it deserves special emphasis.

In the following section, we apply these results to second order Hamilton-Jacobi equations in finite dimensions. We show how the formula of second order subdifferential of the sum of two lower semi continuous functions allows a proof to be given of uniqueness of viscosity solutions of some second order Hamilton-Jacobi equations without the assumption of ellipticity. We also prove a Alexandrov type theorem for continuous functions. As a consequence, we show the singlevaluedness of the second order Hamilton-Jacobi operator associated to a uniformly continuous Hamiltonian.

We shall give also two examples; the first one shows that the formula for the second order subdifferential of the sum of two lower semi continuous functions is not available in $\ell^2(\mathbb{N})$ and the second one shows that this formula is not available in $\ell^2(\Gamma)$ in the Gâteaux sense if Γ is an uncountable set.

Let X be a Banach space and $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a lower semi continuous function. As usual, we denote by $D(f)$ the domain of f :

$$D(f) := \{x \in X\,;\, f(x) < +\infty\}.$$

Received 24th May, 1994

DEFINITION 1.1: Let $x \in D(f)$. We say that f is twice subdifferentiable at x if the set :

$$D^{2,-}f(x) = \{J(\varphi'(x), \varphi''(x))\,;\, \varphi : X \longrightarrow \mathbb{R} \text{ is } C^2 \text{ and } f-\varphi \text{ has a local minimum at } x\}$$

is not empty. And we say that f is twice superdifferentiable at x if the set:

$$D^{2,+}f(x) = \{(\varphi'(x), \varphi''(x))\,;\, \varphi : X \longrightarrow \mathbb{R} \text{ is } C^2 \text{ and } f-\varphi \text{ has a local maximum at } x\}$$

is not empty. (For $x \notin D(f)$, we define $D^{2,-}f(x) = D^{2,+}f(x) = \emptyset$.)

This definition is justified by the following smooth variational principle Theorem 1.2 which is proved in [**3**]:

THEOREM 1.2. *Let X be a Banach space, $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a lower semi continuous, bounded below function such that $D(f) \neq \emptyset$. Assume that there exists a C^2-bump function b on X such that b' is Lipschitz continuous. Then for every $\varepsilon > 0$, there exists a C^2-function g on X such that :*

(a) *$f + g$ has a strong minimum at some point $x_0 \in D(f)$,*

(b) *$\|g\|_\infty < \varepsilon$, $\|g'\|_\infty < \varepsilon$ and $\|g''\|_\infty < \varepsilon$.*

Moreover, we have the following localisation property : there exists a constant $c > 0$ (depending only on the space X) such that whenever $y \in X$ satisfies $f(y) \leqslant \inf\{f(x); x \in X\} + c\varepsilon^3$, then the point x_0 can be chosen such that $\|y - x_0\| < \varepsilon$.

Let us recall that a function $F : X \longrightarrow \mathbb{R}$ attains a strong minimum at $x_0 \in X$ if, by definition, $F(x_0) = \inf\{F(x)\,;\ x \in X\}$ and every minimising sequence (y_n) in X (that is, $\lim\limits_{n\to\infty} F(y_n) = F(x_0)$) converges to x_0.

The norm $\|g''(x)\|$ is the usual norm of a quadratic form on X :

$$\|g''(x)\| = \sup\{\, g''(x)(h,k)\,;\, (h,k) \in X^2\,,\ \|h\| \leqslant 1\,,\ \|k\| \leqslant 1\,\}.$$

Using Theorem 1.2, one can prove that for every lower semi continuous $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$, the set of all points $x \in D(f)$ such that $D^{2,-}f(x) \neq \emptyset$ is dense in $D(f)$. However, it does not seen possible to have a good calculus for second order subdifferentials. Indeed, if we denote by $C^{1,1}(\ell^2(\mathbb{N}))$ the class of all $f : \ell^2(\mathbb{N}) \longrightarrow \mathbb{R}$, whose first derivatives are locally Lipschitz, then on the Hilbert space $\ell^2(\mathbb{N})$ there exists convex $f \in C^{1,1}(\ell^2(\mathbb{N}))$, such that for each $x \in \ell^2(\mathbb{N}), f$ has no second order expansion at x (see the proof of Proposition 4.4(1)). Moreover the formula for the second order subdifferential of the sum of two lower semi continuous functions is not available in this space (see Proposition 4.1). So we are lead to introduce a weak notion of second order subdifferential:

Let $\mathcal{B}(X)$ be the set of a bilinear symmetric continuous forms on X. We say that f is twice Gâteaux differentiable at $x \in D(f)$, if it is the Gâteaux differentiable in a neighbourhoud of x and there exists convex $Q_x \in \mathcal{B}(X)$ such that for all $h, k \in X$

$$\lim_{t \to 0} \frac{\langle f'(x+th) - f'(x), k\rangle}{t} = Q_x(h,k).$$

Using l'Hôpital's rule, we obtain that f is twice Gâteaux differentiable at $x \in D(f)$, then f has a second order expansion in the Gâteaux sense at x that is, there exist $p_x \in X^*$ and $Q_x \in \mathcal{B}(X)$ such that for all $h \in X$

$$\lim_{t \to 0} \frac{f(x+th) - f(x) - t\langle f'(x), h\rangle - t^2 2^{-1} Q_x(h,h)}{t^2} = 0.$$

For convex f a converse also holds (see [**1**]).

Now if X is a separable Banach space with separable dual X^* and if we denote by $C_G^{1,1}(D)$ the class of all $f : D \longrightarrow \mathbb{R}$, D is open, whose first Gâteaux derivatives are locally Lipschitz, then f' is Gâteaux differentiable on a dense subset D_f of D that is, f is is twice Gâteaux differentiable on D_f. This fact is not true in general, for example on the Hilbert space $\ell^2(\Gamma)$ there exists convex $f \in C^{1,1}(\ell^2(\Gamma))$, such that for each $x \in \ell^2(\Gamma), f$ has no second order expansion at x in the Gâteaux sense (see Proposition 4.4(2) for the proof).

DEFINITION 1.3: For $x \in D(f)$, the Gâteaux subdifferential of order two of f at x is the set: $D_G^{2,-} f(x) = \{(\varphi'(x), \varphi''(x)); \varphi : X \longrightarrow \mathbb{R}$ is twice Gâteaux differentiable and $f - \varphi$ has a local minimum at $x\}$.

We say that f is twice Gâteaux subdifferentiable at x if $D_G^{2,-} f(x)$ is not empty.

In a similar way we define the Gâteaux superdifferential of order two of f at x by: $D_G^{2,+} f(x) = \{(\varphi'(x), \varphi''(x)); \varphi : X \longrightarrow \mathbb{R}$ is twice Gâteaux differentiable and $f - \varphi$ has a local maximum at $x\}$.

And f is twice Gâteaux superdifferentiable at x if $D_G^{2,+} f(x)$ is not empty (for $x \notin D(f)$, we define $D_G^{2,-} f(x) = D_G^{2,+} f(x) = \emptyset$.)

It is easy to see that if $D_G^{2,-} f(x) \neq \emptyset$ (respectively $D_G^{2,+} f(x) \neq \emptyset$), then there exist $p_x \in X^*$ and $Q_x \in \mathcal{B}(X)$ such that for all $h \in X$

$$\varliminf_{t \to 0} \frac{f(x+th) - f(x) - t\langle p_x, h\rangle - t^2 2^{-1} Q_x(h,h)}{t^2} \geqslant 0$$

respectively

$$\varlimsup_{t \to 0} \frac{f(x+th) - f(x) - t\langle p_x, h\rangle - (t^2/2) Q_x(h,h)}{t^2} \leqslant 0.$$

In section 4 (Proposition 4.2), we shall see that the formula for the second order subdifferential of the sum of two lower semi continuous functions is not available in $\ell^2(\Gamma)$ in the Gâteaux sense. So, in this paper we shall be interested in the formula and its applications in the finite dimensional case.

2. The second order subdifferential of the sum of two lower semi continuous functions in finite dimensions

In this section, we shall prove the following :

Theorem 2.1. *Let u_1, u_2 be two real valued lower semi continuous functions defined on $\mathbb{R}^n$. Suppose that x_0 and $(p, Q) \in D^{2,-}(u_1+u_2)(x_0)$ are given. Then, for every $\varepsilon > 0$, there exist $x_1, x_2 \in \mathbb{R}^n$, there exist $(p_1, Q_1) \in D^{2,-}u_1(x_1)$ and $(p_2, Q_2) \in D^{2,-}u_2(x_2)$ such that:*

(i) $\|x_1 - x_0\| < \varepsilon$ and $\|x_2 - x_0\| < \varepsilon$.

(ii) $|u_1(x_1) - u_1(x_0)| < \varepsilon$ and $|u_2(x_2) - u_2(x_0)| < \varepsilon$.

(iii) $\|p_1 + p_2 - p\| < \varepsilon$ and $\|Q_1 + Q_2 - Q\| < \varepsilon$.

Alexandrov's theorem states that convex functions on $\mathbb{R}^n$ have a second order expansion almost everywhere. The following result can be considered as a weak version of Alexandrov's theorem for continuous functions.

Corollary 2.2. *Let u be a continuous function defined on $\mathbb{R}^n$. Then for every $x \in \mathbb{R}^n$ and for every $\varepsilon > 0$, there exist $x_1, x_2 \in \mathbb{R}^n$, $(p^-, Q^-) \in D^{2,-}u(x_1)$ and $(p^+, Q^+) \in D^{2,+}u(x_2)$ such that :*

(i) $\|x_1 - x\| < \varepsilon$ and $\|x_2 - x\| < \varepsilon$.

(ii) $|u(x_1) - u(x)| < \varepsilon$ and $|u(x_2) - u(x)| < \varepsilon$.

(iii) $\|p^- - p^+\| < \varepsilon$ and $\|Q^- - Q^+\| < \varepsilon$.

In order to prove this result, it is enough to apply Theorem 2.1, with $u_1 = u$ and $u_2 = -u$, and to observe that $D^{2,-}(-u)(x_2) = -D^{2,+}u(x_2)$.

Remark 2.3.

(1) We have an analogous result for the second order superdifferential of the sum of two upper semi continuous functions in finite dimensions.

(2) In Theorem 2.1, we can replace $\mathbb{R}^n$ by an open subset of $\mathbb{R}^n$.

Proof of Theorem 2.1: In this proof we use the following result of Crandall, Ishii and Lions [**2**, Theorem 3.2]:

Theorem. *Let $\mathcal{O}_i$ be a locally compact subset of $\mathbb{R}^{n_i}$ for $i = 1, 2$, $\mathcal{O} = \mathcal{O}_1 \times \mathcal{O}_2$, let u_i be lower semi continuous on $\mathcal{O}_i$, and let φ be twice continuously differentiable in a neighbourhood of $\mathcal{O}$. Set*

$$w(x) = u_1(x_1) + u_2(x_2) \text{ for } x = (x_1, x_2) \in \mathcal{O}$$

and suppose $\widehat{x} = (\widehat{x_1}, \widehat{x_2}) \in \mathcal{O}$ is a local minimum of $w - \varphi$ relative to $\mathcal{O}$. Then for each $\varepsilon > 0$, there exists $X_i \in S(n_i)$ such that $\big(D_{x_i}\varphi(\widehat{x}), X_i\big) \in \overline{D}^{2,-}u_i(\widehat{x_i})$ for $i = 1, 2$

https://doi.org/10.1017/S0004972700014076 Published online by Cambridge University Press

and the block diagonal matrix with entries X_i satisfies

$$-\left(\frac{1}{\varepsilon}+\|A\|\right)I \geqslant \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix} \geqslant A+\varepsilon A^2$$

where $A = D^2\varphi(\widehat{x}) \in S(n)$, $n = n_1 + n_2$, $S(n)$ is the set of symmetric $n \times n$ matrices and $\overline{D}^{2,-}u_i(\widehat{x}_i) = \{(p,Q) \in \mathbb{R}^{n_i} \times S(n_i)$; *there exists* $(x_n, p_n, Q_n) \in \mathcal{O}_i \times \mathbb{R}^{n_i} \times S(n_i); (p_n, Q_n) \in D^{2,-}u_i(x_n)$ *and* $\lim_{n\to\infty}(x_n, u_i(x_n), p_n, Q_n) = (x_i, u_i(x_i), p, Q)\Big\}$.

The proof proceeds in two steps.

STEP 1. Here we prove that if $u_1 + u_2$ has a strict local minimum at x_0, then for every $\varepsilon > 0$, there exist $x_1, x_2 \in \mathbb{R}^n$, there exist $(p_1, Q_1) \in D^{2,-}u_1(x_1)$ and $(p_2, Q_2) \in D^{2,-}u_2(x_2)$ such that:

(i) $\|x_1 - x_0\| < \varepsilon$ and $\|x_2 - x_0\| < \varepsilon$.

(ii) $|u_1(x_1) - u_1(x_0)| < \varepsilon$ and $|u_2(x_2) - u_2(x_0)| < \varepsilon$.

(iii) $\|p_1 + p_2\| < \varepsilon$ and $\|Q_1 + Q_2\| < \varepsilon$.

Let us fix $\varepsilon > 0$ and suppose that $u_1 + u_2$ has a strict local minimum at x_0, so there exists $r > 0$ such that

$$(u_1+u_2)(x_0) < (u_1+u_2)(x) \text{ for } x \text{ in } \overline{B(x_0,r)} \setminus \{x_0\}.$$

Set $\mathcal{K} = \overline{B(x_0,r)}$. For $\alpha > 0$, consider the function

$$w_\alpha(x,y) = u_1(x) + u_2(y) + \frac{\alpha}{2}\|x-y\|^2 \text{ for } (x,y) \in \mathcal{K}\times\mathcal{K}.$$

We shall need the following elementary lemma (see Crandall, Ishii and Lions [**2**, Lemma 3.1] for a proof).

LEMMA. *Let $\mathcal{O}$ be a subset of $\mathbb{R}^n$ and u_1, u_2 be two lower semi continuous functions on $\mathcal{O}$. Set*

$$M_\alpha = \inf_{\mathcal{O}\times\mathcal{O}}\left(u_1(x)+u_2(y)+\frac{\alpha}{2}\|x-y\|^2\right),$$

for $\alpha > 0$. Let $M_\alpha > -\infty$ for large α and (x_α, y_α) be such $\lim_{\alpha\to\infty}\big(M_\alpha - \big(u_1(x_\alpha)+u_2(y_\alpha) + (\alpha/2)\|x_\alpha - y_\alpha\|^2\big)\big) = 0$. *Then the following holds :*

(i) $\lim_{\alpha\to\infty} \alpha\|x_\alpha - y_\alpha\|^2 = 0$ *and*

(ii) $\lim_{\alpha\to\infty} M_\alpha = u_1(\widetilde{x}) + u_2(\widetilde{x}) = \inf_{\mathcal{O}}(u_1(x)+u_2(x))$

whenever $\widetilde{x} \in \mathcal{O}$ is a limit point of x_α as $\alpha \to \infty$.

Since $\mathcal{K}$ is compact, the function w_α has a minimum on $\mathcal{K}\times\mathcal{K}$ at some point (x_α, y_α) satisfying the following conditions:

(i′) $\lim_{\alpha\to\infty} \alpha\,\|x_\alpha - y_\alpha\|^2 = 0$

(ii′) $\lim_{\alpha\to\infty} w_\alpha(x_\alpha, y_\alpha) = u_1(\overline{x}) + u_2(\overline{x}) = \min\{u_1(x)+u_2(x); x \in \mathcal{K}\}$. Here $\overline{x} \in \mathcal{K}$ is a limit point of x_α as $\alpha \to \infty$.

From (i′) we deduce that $\lim\limits_{\alpha\to\infty} y_\alpha = \overline{x}$ and since x_0 is a strict minimum of u_1+u_2 on $\mathcal{K}$, by (ii′) we have $\overline{x} = x_0$. Let η be a real positive number. By (ii′) if α is large enough, we have $w_\alpha(x_\alpha, y_\alpha) \leqslant u_1(x_0) + u_2(x_0) + \eta$, which yields $u_1(x_\alpha) + u_2(y_\alpha) \leqslant u_1(x_0) + u_2(x_0) + \eta$ and hence $u_1(x_\alpha) \leqslant u_1(x_0) + u_2(x_0) - u_2(y_\alpha) + \eta$. Since u_2 is lower semi continuous, $y_\alpha \to x_0$ as $\alpha \to \infty$ and η is arbitrary, we have $\overline{\lim\limits_{\alpha\to\infty}}\, u_1(x_\alpha) \leqslant u_1(x_0)$. The function u_1 is lower semi continuous, so $\lim\limits_{\alpha\to\infty} u_1(x_\alpha) = u_1(x_0)$. Similarly, we prove that $\lim\limits_{\alpha\to\infty} u_2(y_\alpha) = u_2(x_0)$. Consequently, there exists $M > 0$, such that for $\alpha \geqslant M$, we have

(1) $\|x_\alpha - x_0\| < \varepsilon/2$ and $\|y_\alpha - x_0\| < \varepsilon/2$.

(2) $|u_1(x_\alpha) - u_1(x_0)| < \varepsilon/2$ and $|u_2(y_\alpha) - u_2(x_0)| < \varepsilon/2$.

On the other hand, if we set $\overline{p}_1 = -\alpha(x_\alpha - y_\alpha)$, $\overline{p}_2 = \alpha(x_\alpha - y_\alpha)$ and $\overline{Q}_1 = \overline{Q}_2 = -\alpha I$ where I is the identity matrix of $\mathbb{R}^n$, then $(\overline{p}_1, \overline{Q}_1) \in D^{2,-}(u_1)(x_\alpha)$ and $(\overline{p}_2, \overline{Q}_2) \in D^{2,-}(u_2)(y_\alpha)$. Unfortunately, $\overline{Q}_1 + \overline{Q}_2 = -2\alpha I < 0$. From now on, we fix $\alpha \geqslant M$. We apply the previous theorem with $n_1 = n_2 = n$, $w = w_\alpha$, $\widehat{x} = x_\alpha$ and $\widehat{y} = y_\alpha$ on $\mathcal{O}_i = \mathcal{K}$ for $i = 1, 2$. This yields the existence of two points x_1, x_2 in $\mathbb{R}^n$, $\left(\widetilde{p}_1, \widetilde{Q}_1\right)$ in $D^{2,-}(u_1)(x_1)$, $\left(\widetilde{p}_2, \widetilde{Q}_2\right)$ in $D^{2,-}(u_2)(x_2)$ and X_1, X_2 in $S(n)$ such that :

(3) $\|x_1 - x_\alpha\| < \varepsilon/2$, $\|x_2 - y_\alpha\| < \varepsilon/2$.

(4) $|u_1(x_1) - u_1(x_\alpha)| < \varepsilon/2$, $|u_2(x_2) - u_2(y_\alpha)| < \varepsilon/2$.

(5) $\|\widetilde{p}_i - \overline{p}_i\| < \varepsilon/2$ for $i = 1, 2$, $\left\|\widetilde{Q}_i - X_i\right\| < \varepsilon/2$ for $i = 1, 2$

and

$$\begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix} \geqslant A + \varepsilon A^2 \text{ where } A = \begin{pmatrix} -\alpha I & \alpha I \\ \alpha I & -\alpha I \end{pmatrix}.$$

Thus

$$A + \varepsilon A^2 = \alpha \begin{pmatrix} -I & I \\ I & -I \end{pmatrix} + 2\varepsilon\alpha^2 \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}.$$

Let x be in $\mathbb{R}^n$. Since the right hand side annihilates the vectors ${}^t(x, x)$ (where ${}^tZ$ denotes the transpose of the matrix Z), we have

$$(x, x) \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix} {}^t(x, x) \geqslant 0$$

for all $x \in \mathbb{R}^n$. So $X_1 + X_2 \geqslant 0$. Finally, if we set $p_i = \widetilde{p}_i$ for $i = 1, 2$, $Q_1 = \widetilde{Q}_1$ and $Q_2 = \widetilde{Q}_2 - (X_1 + X_2)$ we claim that x_i, p_i, Q_i, $i = 1, 2$ satisfy the conditions (i), (ii) and (iii) of Theorem 2.1 with $p = 0$ and $Q = 0$. Indeed, since $Q_2 \geqslant \widetilde{Q}_2, (p_i, Q_i) \in D^{2,-}u_i(x_i)$,

$$\|p_1 + p_2\| = \|\widetilde{p}_1 - \overline{p}_1 + \overline{p}_1 - \overline{p}_2 + \overline{p}_2 - \widetilde{p}_2\| \leqslant \|\widetilde{p}_1 - \overline{p}_1 + \overline{p}_2 - \widetilde{p}_2\| < \varepsilon$$

https://doi.org/10.1017/S0004972700014076 Published online by Cambridge University Press

and

$$\begin{aligned} Q_1 + Q_2 &= \widetilde{Q}_1 + \widetilde{Q}_2 - (X_1 + X_2) \\ &= \widetilde{Q}_1 - X_1 + \widetilde{Q}_2 - X_2. \end{aligned}$$

Thus

$$\|Q_1 + Q_2\| \leqslant \left\|\widetilde{Q}_1 - X_1\right\| + \left\|\widetilde{Q}_2 - X_2\right\| < \varepsilon.$$

Moreover

$$\begin{aligned} |u_1(x_1) - u_1(x_0)| &= |u_1(x_1) - u_1(x_\alpha)| + |u_1(x_\alpha) - u_1(x_0)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \\ |u_2(x_2) - u_2(x_0)| &= |u_2(x_2) - u_2(y_\alpha)| + |u_2(y_\alpha) - u_2(x_0)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

and

$$\begin{aligned} \|x_1 - x_0\| &\leqslant \|x_1 - x_\alpha\| + \|x_\alpha - x_0\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \\ \|x_2 - x_0\| &\leqslant \|x_2 - y_\alpha\| + \|y_\alpha - x_0\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

STEP 2. Let x_0 be in $\mathbb{R}^n$, (p, Q) be in $D^{2,-}(u_1 + u_2)(x_0)$ and $0 < \varepsilon < 1$ be fixed. By the definition of the second order subdifferential, there exists a C^2-function $\varphi : \mathbb{R}^n \longrightarrow \mathbb{R}$ such that $u_1 + u_2 - \varphi$ has a local minimum at x_0 with $p = \varphi'(x_0)$ and $Q = \varphi''(x_0)$. Set $\widetilde{u}_1 = u_1 - \varphi - (\varepsilon/4)\,\|. - x_0\|^2$. The functions $\widetilde{u}_1$ and u_2 satisfy the conditions of Step 1, so there exist $x_1, x_2 \in \mathbb{R}^n$, there exist $\left(\widetilde{p}_1, \widetilde{Q}_1\right) \in D^{2,-}(\widetilde{u}_1)(x_1)$ and $(p_2, Q_2) \in D^{2,-}(u_2)(x_2)$ such that :

(i″) $\|x_1 - x_0\| < \varepsilon$ and $\|x_2 - x_0\| < \varepsilon/2$.

(ii″) $|\widetilde{u}_1(x_1) - \widetilde{u}_1(x_0)| < \varepsilon/2$ and $|u_2(x_2) - u_2(x_0)| < \varepsilon/2$.

(iii″) $\|\widetilde{p}_1 + p_2\| < \varepsilon/2$ and $\left\|\widetilde{Q}_1 + Q_2\right\| < \varepsilon/2$.

But $\widetilde{p}_1 = p_1 - \varphi'(x_0) - \varepsilon/2(x_1 - x_0)$, $\widetilde{Q}_1 = Q_1 - \varphi''(x_0) - \varepsilon I$ and it is clear that $x_1, x_2, (p_1, Q_1), (p_2, Q_2)$ satisfy the conditions (i), (ii) and (iii) of Theorem 2.1. ∎

3. Applications to viscosity solutions of second order Hamilton-Jacobi equations

Let $H : \mathbb{R}^n \times \mathbb{R}^n \times S(n) \longrightarrow \mathbb{R}$ be a uniformly continuous function. In this section, we are interested in the uniqueness of the viscosity solution $u : \mathbb{R}^n \longrightarrow \mathbb{R}$ of the equation

$$u + H\big(x, Du, D^2u\big) = 0. \tag{3.1}$$

https://doi.org/10.1017/S0004972700014076 Published online by Cambridge University Press

DEFINITION 3.1: A function $u : \mathbb{R}^n \longrightarrow \mathbb{R}$ is a viscosity subsolution of (3.1) if

(1) u is upper semi continuous on $\mathbb{R}^n$.

(2) For every x in $\mathbb{R}^n$ and for every $(p, Q) \in D^{2,+}u(x)$:

$$u(x) + H(x, p, Q) \leqslant 0.$$

The function u is a viscosity supersolution of (3.1) if

(3) u is lower semi continuous on $\mathbb{R}^n$.

(4) For every x in $\mathbb{R}^n$ and for every $(p, Q) \in D^{2,-}u(x)$:

$$u(x) + H(x, p, Q) \geqslant 0.$$

Finally u is a viscosity solution of (3.1) if u is both a viscosity subsolution and a viscosity supersolution of (3.1).

For the uniqueness of viscosity solutions of (3.1), we have the following result where no assumption of ellipticity is assumed.

PROPOSITION 3.2. *Suppose that the Hamiltonian H is uniformly continuous from $\mathbb{R}^n \times \mathbb{R}^n \times S(n)$ into $\mathbb{R}$. Let u, v be two real valued functions defined on $\mathbb{R}^n$, with u bounded above and v bounded below. If u is a viscosity subsolution of (3.1) and v is a viscosity supersolution of (3.1), then*

$$u \leqslant v.$$

REMARK 3.3. (1) Let us mention that, if the Hamiltonian H is degenerate elliptic, then a classical solution of (3.1) is a viscosity solution of (3.1), (see [**2**]). This fact is no longer true if H is not degenerate elliptic, for example on $\mathbb{R}$ the function $x \longrightarrow x^2 - 2$ is a classical solution of the equation $u + u'' - x^2 = 0$ but is not a viscosity solution.

(2) Observe that when H is not degenerate elliptic, Perron's method for proving the existence of a viscosity solution of Hamilton-Jacobi equations no longuer works. Indeed this method uses the fact that classical solutions of (3.1) are viscosity solutions of (3.1).

(3) Using the techniques of Crandall, Lions and Ishii [**2**] one can prove a general assertion that u and v are not bounded but satisfy some growth condition at infinity.

PROOF OF PROPOSITION 3.2: Let us fix $\varepsilon > 0$. Since the function $u - v$ is upper semi continuous and bounded above, by Theorem 1.2 applied to $f = -(u - v)$ there exist $x_0 \in \mathbb{R}^n$ and $(p, Q) \in D^{2,+}(u - v)(x_0)$ such that $\|p\| < \varepsilon$, $\|Q\| < \varepsilon$ and $(u - v)(x_0) > \sup(u - v) - \varepsilon$. Now by applying Theorem 2.1 with $u_1 = v$ and $u_2 = -u$, there exist x_1, x_2 in $\mathbb{R}^n$, $(p_1, Q_1) \in D^{2,-}v(x_1)$ and $(p_2, Q_2) \in D^{2,+}u(x_2)$ satisfying

(1) $\|x_1 - x_0\| < \varepsilon$ and $\|x_2 - x_0\| < \varepsilon$.

https://doi.org/10.1017/S0004972700014076 Published online by Cambridge University Press

(2) $|v(x_1) - v(x_0)| < \varepsilon$ and $|u(x_2) - u(x_0)| < \varepsilon$.

(3) $\|p_2 - p_1 - p\| < \varepsilon$ and $\|Q_2 - Q_1 - Q\| < \varepsilon$.

The function u is a viscosity subsolution of (3.1), so

$$u(x_2) + H(x_2, p_2, Q_2) \leqslant 0.$$

The function v is a viscosity supersolution of (3.1), so

$$v(x_1) + H(x_1, p_1, Q_1) \geqslant 0.$$

Consequently

$$\begin{aligned}\sup(u - v) &\leqslant (u - v)(x_0) + \varepsilon \\ &< u(x_2) - v(x_1) + 3\varepsilon \\ &< H(x_1, p_1, Q_1) - H(x_2, P_2, Q_2) + 3\varepsilon.\end{aligned}$$

Moreover,

$$\begin{aligned}\|x_1 - x_2\| &\leqslant \|x_1 - x_0\| + \|x_0 - x_2\| < 2\varepsilon, \\ \|p_2 - p_1\| &\leqslant \|p_2 - p_1 - p\| + \|p\| < 2\varepsilon\end{aligned}$$

and

$$\|Q_2 - Q_1\| \leqslant \|Q_2 - Q_1 - Q\| + \|Q\| < 2\varepsilon.$$

Using the uniform continuity of H and sending ε to zero, we get:

$$\sup(u - v) \leqslant 0.$$

$\square$

We conclude this section by studying the problem of singlevaluedness of the second order Hamilton-Jacobi operator. Assume that the map $H : \mathbb{R}^n \times \mathbb{R}^n \times S(n) \longrightarrow \mathbb{R}$ is uniformly continuous. We denote by $C(\mathbb{R}^n)$ the space of continuous functions defined on $\mathbb{R}^n$. We define the operator $A_H : C(\mathbb{R}^n) \longrightarrow C(\mathbb{R}^n)$ by :

$$A_H u = \{f \in C(\mathbb{R}^n); f(x) = H\big(x, u(x), Du(x), D^2u(x)\big) \text{ in the viscosity sense }\}.$$

We prove the following :

Theorem 3.4. *For every uniformly continuous function H defined on $\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times S(n)$, the associated second order Hamilton-Jacobi operator A_H is singlevalued.*

Proof: Let us assume that $H\big(x, u(x), Du(x), D^2u(x)\big) = f(x)$ and $H\big(x, u(x), Du(x), D^2u(x)\big) = g(x)$ for $x \in \mathbb{R}^n$ in the viscosty sense, where f and g

are two continuous real valued functions defined on $\mathbb{R}^n$. We want to prove that $f = g$. Let $x \in \mathbb{R}^n$ and $\varepsilon > 0$ be fixed. By Corollary 2.2, there exist $(p^-, Q^-) \in D^{2,-}u(x_1)$ and $(p^+, Q^+) \in D^{2,+}u(x_2)$ such that :

(1) $\|x_1 - x\| < \varepsilon$ and $\|x_2 - x\| < \varepsilon$.
(2) $|u(x_1) - u(x)| < \varepsilon$ and $|u(x_2) - u(x)| < \varepsilon$.
(3) $\|p^- - p^+\| < \varepsilon$ and $\|Q^- - Q^+\| < \varepsilon$.

Since u is a viscosity subsolution of $H(x, u, Du, D^2u) = f$, we have

$$H(x_1, u(x_1), p^+, Q^+) \leqslant f(x_1).$$

Since u is a viscosity supersolution of $H(x, u, Du, D^2u) = g$, we have

$$H(x_2, u(x_2), p^-, Q^-) \geqslant g(x_2).$$

Consequently

$$g(x_2) - f(x_1) \leqslant H(x_2, u(x_2), p^-, Q^-) - H(x_1, u(x_1), p^+, Q^+).$$

Using (1), (2) and (3) the continuity of the functions f and g at x and the uniform continuity of H, we obtain, as ε goes to 0 :

$$g(x) - f(x) \leqslant 0$$

Similarly, $g(x) - f(x) \geqslant 0$. This is true for all $x \in \mathbb{R}^n$, so $f = g$. ∎

4. Infinite dimensional case

Our first example will prove that the formula for the second order subdifferential of the sum of two lower semi continuous functions is not available in $\ell^2(\mathbb{N})$. For this let $\|.\|$ be the norm induced by the scalar product $\langle , \rangle$ in $\ell^2(\mathbb{N})$ and (e_n) its canonical basis.

Proposition 4.1. *On $\ell^2(\mathbb{N})$, there exist $C^{1,1}$ and convex u_1, u_2 from $\ell^2(\mathbb{N})$ into $\mathbb{R}$, satisfying : for all $x \in \ell^2(\mathbb{N})$ there exist $(p, Q) \in D^{2,-}(u_1 + u_2)(x)$ such that for all $x_1, x_2 \in \ell^2(\mathbb{N})$ and for all $(p_i, Q_i) \in D^{2,-}u_i(x_i)$, $i = 1, 2$,*

$$\|Q_1 + Q_2 - Q\| \geqslant 2.$$

We now observe that in non separable Hilbert spaces the formula of the second order subdifferential of the sum is not true even in the Gâteaux sense.

Let Γ be an uncountable set, and consider the Hilbert space $\ell^2(\Gamma)$ equiped with the Euclidian norm $\|.\|^2 = \langle .,. \rangle$ and (e_γ) its canonical basis.

PROPOSITION 4.2. *On $\ell^2(\Gamma)$, there exist $C^{1,1}$ and convex u_1, u_2 from $\ell^2(\Gamma)$ into satisfying : for all $x \in \ell^2(\Gamma)$ there exists $(p,Q) \in D_G^{2,-}(u_1+u_2)(x)$ such that for all $x_1, x_2 \in \ell^2(\Gamma)$ and for all $(p_i,Q_i) \in D_G^{2,-}u_i(x_i)$, $i=1,2$*

$$\|Q_1+Q_2-Q\| \geqslant 2.$$

Before proving Propositons 4.1 and 4.2, let us mention the following open problem.

PROBLEM. Is it possible to obtain a formula for the second order Gâteaux subdifferential of the sum of two uniformly continuous or even $C^{1,1}$ functions on a separable Hilbert space?

PROOF OF PROPOSITION 4.1: For $x=(x_n) \in \ell^2(\mathbb{N})$, set $u_1(x) = \sum_n (x_n^+)^2$ and $u_2(x) = \sum_n (x_n^-)^2$. Let us observe that $u = u_1 + u_2 = \|.\|^2$ which is twice differentiable on $\ell^2(\mathbb{N})$. Thus $D^{2,-}u(x) = \{(2x,Q); Q \geqslant 2\mathcal{I}\}$ for $x \in \ell^2(\mathbb{N})$. In order to prove that u_1 and u_2 do not satisfy Theorem 2.1, we need the following:

LEMMA 4.3. *If $(p,Q) \in D^{2,-}u_i(x)$; $i=1,2$ and if $p_n = \langle p, e_n\rangle$, $q_n = \langle Qe_n, e_n\rangle$, then*

(1) $p_n = 2x_n^+$
(2) $\overline{\lim_{n\to\infty}}\, q_n \leqslant 0$.

Moreover if $x_n = 0$, then $p_n = 0$ and $q_n \leqslant 0$.

PROOF: We only prove the lemma for i = 1, because $u_2(x) = u_1(-x)$. Since $(p,Q) \in D^{2,-}u_1(x)$, we have

$$\underline{\lim}_{\|h\|\to 0} \frac{u_1(x+h) - u_1(x) - \langle p,h\rangle}{\|h\|} \geqslant 0. \tag{4.1}$$

If $x_n > 0$, from (4.1) with $h = te_n$ and $h = -te_n$; $t > 0$, we have

$$\underline{\lim}_{t\to 0+} \frac{u_1(x+tx_ne_n) - u_1(x) - tx_np_n}{tx_n} \geqslant 0$$

and

$$\underline{\lim}_{t\to 0+} \frac{u_1(x-tx_ne_n) - u_1(x) + tx_np_n}{tx_n} \geqslant 0.$$

These yield $\underline{\lim}_{t\to 0+} (t+2)x_n - p_n \geqslant 0$ and $\underline{\lim}_{t\to 0+} (t-2)x_n + p_n \geqslant 0$. Thus $p_n = 2x_n$.

Now if $x_n \leqslant 0$, since $u_1(x-te_n) - u_1(x) = u_1(x+te_n) - u_1(x) = 0$ for $x_n < 0$ and t small enough and $u_1(x+te_n) - u_1(x) = t^2$ for $x_n = 0$, using (4.1) with $h = te_n$, we obtain $\underline{\lim}_{t\to 0+} (t^2 - tp_n)/t \geqslant 0$ and $\liminf_{t\to 0+} (tp_n)/t \geqslant 0$. Thus $p_n = 0$.

Let $k \in \mathbb{N}^*$ be fixed. Since $\lim\limits_{n\to\infty} x_n = 0$ and $(p,Q) \in D^{2,-}u_1(x)$, we have

$$\text{(4.2)} \qquad \varliminf_{\substack{n\to\infty \\ n/x_n>0}} \frac{u_1(x - kx_n e_n) - u_1(x) + kx_n p_n - 2^{-1}k^2x_n^2 q_n}{k^2x_n^2} \geqslant 0$$

and

$$\text{(4.3)} \qquad \varliminf_{\substack{n\to\infty \\ n/x_n>0}} \frac{u_1(x + kx_n e_n) - u_1(x) - kx_n p_n - 2^{-1}k^2x_n^2 q_n}{k^2x_n^2} \geqslant 0.$$

But

$$u_1(x + kx_n e_n) - u_1(x) = \begin{cases} 0 & \text{if } x_n \leqslant 0 \\ (k^2 + 2k)x_n^2 & \text{if } x_n > 0 \end{cases}$$

and

$$u_1(x - kx_n e_n) - u_1(x) = \begin{cases} (k-1)^2 x_n^2 & \text{if } x_n \leqslant 0 \\ -x_n^2 & \text{if } x_n > 0. \end{cases}$$

By (4.2), we have $\varliminf\limits_{\substack{n\to\infty \\ n/x_n>0}} \left(-x_n^2 + 2kx_n^2 - 2^{-1}k^2x_n^2q_n\right)/\left(k^2x_n^2\right) \geqslant 0$, that is, $\varliminf\limits_{\substack{n\to\infty \\ n/x_n>0}} -\left(1/k^{-2}\right) + 2/k - 1/2q_n \geqslant 0$ and a simple calculation yields $\varlimsup\limits_{\substack{n\to\infty \\ n/x_n>0}} q_n \leqslant 2k^{-2} - 4k^{-1}$. Consequently, by sending k to ∞, we obtain $\varlimsup\limits_{\substack{n\to\infty \\ n/x_n>0}} q_n \leqslant 0$. Now (4.3) implies $\varliminf\limits_{\substack{n\to\infty \\ n/x_n<0}} \left(-kx_n^2 - 2^{-1}k^2x_n^2q_n\right)/\left(k^2x_n^2\right) \geqslant 0$, that is, $\varliminf\limits_{\substack{n\to\infty \\ n/x_n>0}} -k^{-1} - 2^{-1}q_n \geqslant 0$. Thus $\varlimsup\limits_{\substack{n\to\infty \\ n/x_n>0}} q_n \leqslant -2k^{-1} \leqslant 0$.

If $x_n = 0$, since $p_n = 0$ by the definition of subdifferential of order two, we have $\varliminf\limits_{t\to 0} \left(u_1(x - te_n) - u_1(x) - t^2 2^{-1} q_n\right)/t^2 \geqslant 0$. But $u_1(x - te_n) - u_1(x) = 0$, so $q_n \leqslant 0$.

End of the proof of Proposition 4.1: let $x_1, x_2 \in \ell^2(\mathbb{N})$ and $(p_i, Q_i) \in D^{2,-}u_i(x_i)$; $i = 1,2$. From Lemma 4.2, if $p_n^i = \langle p_i, e_n\rangle$, and $q_n^i = \langle Q_i e_n, e_n\rangle$, then

(1) $p_n^1 = 2x_n^+$, $p_n^2 = 2x_n^-$ and
(2) $\varlimsup\limits_{n\to\infty} q_n^i \leqslant 0$.

By (1), we observe that u_1 and u_2 are $C^{1,1}$ on $\ell^2(\mathbb{N})$, and since $2\mathcal{I}(e_n, e_n) = 2$ for $n \in \mathbb{N}$, by (2), if $Q = 2\mathcal{I}$, we have

$$\|Q_1 + Q_2 - Q\| \geqslant \varliminf_{n\to\infty} 2 - \left(q_n^1 + q_n^2\right) \geqslant 2.$$

$\square$

https://doi.org/10.1017/S0004972700014076 Published online by Cambridge University Press

Proof of Proposition 4.2.: Let u_1, u_2 from $\ell^2(\Gamma)$ into $\mathbb{R}$ be defined by :

$$u_1\big((x_\gamma)\big) = \sum_\gamma \left(x_\gamma^+\right)^2 \text{ and } u_2\big((x_\gamma)\big) = \sum_\gamma \left(x_\gamma^-\right)^2.$$

By reasoning analogous to that in Lemma 4.3 ($x_n = 0$ case), we get that if $x_1, x_2 \in \ell^2(\Gamma)$, $(p_i, q_i) \in D_G^{2,-} u_i(x_i)$ and if $\gamma \notin \operatorname{supp}(x_1) \cup \operatorname{supp}(x_2)$, then $Q_i(e_\gamma, e_\gamma) \leqslant 0$ and $\langle p_i, e_\gamma\rangle = 0$. ∎

Finally we observe that in infinite dimensional Hilbert spaces a convex and $C^{1,1}$-function can be nowhere twice differentiable. This is shown by the following :

Proposition 4.4. *(1) On $\ell^2(\mathbb{N})$, the function defined by $g\big((x_n)\big) = \sum\limits_n \left(x_n^+\right)^2$ is $C^{1,1}$ and convex but nowhere twice differentiable.*

(2) On $\ell^2(\Gamma)$, the function defined by $f\big((x_\gamma)\big) = \sum\limits_\gamma \left(x_\gamma^+\right)^2$ is $C^{1,1}$ and convex but nowhere twice Gâteaux differentiable.

Proof: (2) Assume on the contrary that f is twice Gâteaux differentiable at some point $x = (x_\gamma) \in \ell^2(\Gamma)$, so there exist $p_x \in \ell^2(\Gamma)$ and $Q_x \in \mathcal{B}\big(\ell^2(\Gamma)\big)$ such that for all $h \in \ell^2(\Gamma)$

$$\lim_{t \to 0} \frac{f(x+th) - f(x) - t\langle p_x, h\rangle - \big(t^2/2\big) Q_x(h,h)}{t^2} = 0. \tag{4.4}$$

Since Γ is an uncountable set, there exists $\gamma \in \Gamma$ such that $\gamma \notin \operatorname{supp}(x)$. According to Lemma 4.3 $\langle p_x, e_\gamma\rangle = 0$ and $Q_x(e_\gamma, e_\gamma) \leqslant 0$. But f is convex, so $\langle p_x, e_\gamma\rangle = Q_x(e_\gamma, e_\gamma) = 0$ and if we put $h = e_\gamma$ in (4.4), we obtain a contradiction.

(1) Assume on the contrary that g is twice differentiable at some point $x = (x_n) \in \ell^2(\mathbb{N})$. By the proof of (2), $x_n \neq 0$ for all $n \in \mathbb{N}$ and by the definition of the differentiability of order two, there exist $p \in \ell^2(\mathbb{N})$ and $Q \in \mathcal{B}\big(\ell^2(\mathbb{N})\big)$ such that

$$\frac{\lim\limits_{\|h\| \to 0} g(x+h) - g(x) - \langle p, h\rangle - Q(h,h)/2}{\|h\|^2} = 0. \tag{4.5}$$

From Lemma 4.3 and the convexity of g, $p_n = \langle p, e_n\rangle = 2x_n^+$ and $\lim\limits_{n \to \infty} \langle Qe_n, e_n\rangle = 0$. Finally if we put $h = -3x_n e_n$ in (4.5), we obtain a contradiction. ∎

References

[1] J.M. Borwein and M. Fabian, *On generic second-order Gâteaux differentiability*, Combinatorics and Optimization (University of Waterloo, 1991).

[2] M.G. Crandall, H. Ishii and P.L. Lions, 'User's guide to viscosity solutions of second order fully non linear partial differential equations', *Bull. Amer. Math. Soc.* **27** (1992), 1–67.

[3] R. Deville, G. Godefroy and V. Zizler, 'A smooth variational principle with applications to Hamilton-Jacobi equations in infinite dimensions', *J. Funct. Anal.* **111** (1993), 197–212.

[4] R. Jensen, 'The maximum principle for viscosity solutions of fully non linear second order partial differential equation', *Arch. Rational Mech. Anal.* **101** (1988), 1–27.

Université Bordeaux1
Laboratoire de Mathématiques
351, cours del la Libération
33400 Talence
France

Université de Franche-Comté
Laboratoire de Mathématique
16, route de Gray
25000 Besançon
France