BULL. AUSTRAL. MATH. SOC. 20M20, 20M30
VOL. 35 (1987) 373-378

# THE MINIMAL FAITHFUL DEGREE OF A FUNDAMENTAL INVERSE SEMIGROUP

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This paper shows that the smallest size of a set for which a finite fundamental inverse semigroup can be faithfully represented by partial transformations of that set is the number of join irreducible elements of its semilattice of idempotents.

## 1. Introduction

In faithfully representing a given semigroup by partial transformations of a set, it is natural to ask what is the least size of a set for which this is possible. This question, among others, was posed by Schein in [7, Problem 45] .

Here, the problem is solved for finite fundamental inverse semigroups. The special case of a semilattice is considered, from which the general case follows by applying a theorem of Munn, which describes fundamental inverse semigroups in terms of principal ideal isomorphisms of semilattices.

## 2. Preliminaries

Standard terminology and basic results relating to inverse semigroups, and semilattices in particular, as given by Howie in [1], will be

Received 19 May 1986.

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https://doi.org/10.1017/S0004972700013356 Published online by Cambridge University Press

assumed. The following result, due to Munn [3, 4], will be used in Section 4. Recall that if $E$ is a semilattice, the Munn semigroup $T_E$ consists of all isomorphisms between principal ideals of $E$.

THEOREM 1 [1, 4.10]. *An inverse semigroup $S$ with semilattice of idempotents $E$ is fundamental if and only if it is isomorphic to a full inverse subsemigroup of $T_E$.*

Hereafter all semigroups will be assumed to be finite. Define the minimal faithful degree $\mu(S)$ of a semigroup $S$ to be the least non-negative integer $n$ such that $S$ can be embedded in $PT_X$, the semigroup of partial transformations of a set $X$, where $X$ contains $n$ elements. Note that if $T$ is a subsemigroup of $S$ then $\mu(T) \leq \mu(S)$. If $X$ is a set, let $\mathrm{id}_X$ denote the identity mapping on $X$.

The following result follows from more general results about inverse semigroups (see for example [5, IV.5.9], [2, II.8.4],[8] or [6]), though a proof is included for completeness.

PROPOSITION 2. *Let $E$ be a semilattice and $\phi : E \to PT_X$ a faithful representation. Then there is a faithful representation $\bar{\phi} : E \to PT_X$ such that, for all $e \in E$,*

$$e\bar{\phi} = id_Y \quad \textit{for some} \quad Y \subseteq X .$$

Proof. Define $\bar{\phi} : E \to PT_X$ by, for $e \in E$,

$$e\bar{\phi} = e\phi \Big|_{\mathrm{range}(e\phi)} .$$

It is routine to verify that $\bar{\phi}$ is a faithful representation. Moreover, if $x \in \mathrm{domain}(e\bar{\phi})$ then $x \in \mathrm{range}(e\phi)$, so, for some $y \in X$, $x = y(e\phi)$, which yields $x(e\bar{\phi}) = [y(e\phi)](e\bar{\phi}) = [y(e\phi)](e\phi) = y(e\phi) = x$, that is, $e\bar{\phi} = \mathrm{id}_{\mathrm{domain}(e\bar{\phi})}$. This completes the proof.

If $E$ is a semilattice, the symbols $\wedge$ and $\vee$ will be used to denote infimum and supremum respectively, when they exist, with respect to the partial order of $E$. Thus $\wedge E$ is the least element of $E$, which exists since $E$ is finite, which will be denoted by $0$, the zero of $E$. Note also that $0 = \vee\emptyset$. Call an element $e$ of $E$ join

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<u>irreducible</u> if $e \neq 0$ and, for $f,g \in E$,

$$e = f \vee g \quad \text{implies} \quad e = f \quad \text{or} \quad e = g .$$

Note that by this definition $0$ is not join irreducible. Note also that, since $E$ is finite, any non-zero element of $E$ can be expressed as the join of a set of join irreducible elements. If $e \in E$, define

$$\overline{e} = \{f \in E \mid f \leq e \text{ and } f \text{ is join irreducible}\} .$$

LEMMA 3. *If $E$ is a semilattice then, for $e \in E$, $e = \vee\overline{e}$.*

Proof. Let $e \in E$. If $e = 0$ then $\overline{e} = \emptyset$ and indeed $e = \vee\overline{e}$. If $e \neq 0$ then write $e$ as the join of join irreducible elements $e_1, \ldots, e_n$:

$$e = \vee\{e_1, \ldots, e_n\} .$$

Certainly $e$ is an upper bound for $\overline{e}$. Suppose also $f$ is an upper bound for $\overline{e}$, so in particular $f$ is an upper bound for $\{e_1, \ldots, e_n\}$. Hence $e \leq f$, which shows $e$ is the least upper bound for $\overline{e}$, which completes the proof.

LEMMA 4. *Let $E$ be the semilattice which is the subsemigroup of $PT_X$ consisting of all partial transformations of a set $X$ which are identity mappings on their respective domains. Then, in $E$, for $Y, Z \subseteq X$,*

(i) $id_Y \leq id_Z$ *if and only if* $Y \subseteq Z$ ;

(ii) $id_{Y \cap Z} = id_Y \wedge id_Z$ ;

(iii) $id_{Y \cup Z} = id_Y \vee id_Z$ .

Proof. These follow immediately from the fact that $id_Y id_Z = id_{Y \cap Z}$.

LEMMA 5. *Let $(P, \leq)$ be any non-empty finite partially ordered set with $n$ elements. Then there is a listing $P = \{p_1, \ldots, p_n\}$ such that, whenever $1 \leq i < j \leq n$,*

$$p_j \nleq p_i .$$

Proof. Define the listing inductively. Let $p_1$ be any minimal element of $P$, which exists by finiteness. Assume $p_1, \ldots, p_k$ have

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been chosen, where $k < n$ . Choose $p_{k+1}$ to be any minimal element of $P\backslash\{p_1,\ldots,p_k\}$ . The minimality of $p_{k+1}$ ensures that the condition of the lemma holds.

## 3. Semilattices.

**THEOREM 6.** *Let $E$ be a finite semilattice with $n$ join irreducible elements. Then $\mu(E) = n$ .*

**Proof.** If $E = \{0\}$ then $n = 0$ and $\mu(E) = 0$ , so the statement of the theorem holds. Suppose then $E$ contains a non-zero element, so $n \geq 1$ .

Let $X$ be the set of all join irreducible elements of $E$ . Define a mapping $\phi : E \to PT_X$ by, for $e \in E$ ,

$$e\phi = id_{\bar{e}} .$$

For $e,f \in E$ , $\overline{ef} = \bar{e} \cap \bar{f}$ , so

$$id_{\overline{ef}} = id_{\bar{e}\cap\bar{f}} = id_{\bar{e}}\, id_{\bar{f}} ,$$

which shows $\phi$ is homomorphic. Also, if $\bar{e} = \bar{f}$ then, by Lemma 3, $e = \vee\bar{e} = \vee\bar{f} = f$ , so $\phi$ is one-one. Hence $\phi$ is a faithful representation, so $\mu(E) \leq n$ .

Suppose now $\psi : E \to PT_Y$ is a faithful representation where $Y$ is a set with $m$ elements. It will be shown that $m \geq n$ . Suppose to the contrary that $m < n$ .

By Proposition 2, it may be assumed that for each $e \in E$ , $e\psi$ is the identity mapping on some subset of $Y$ . Let $X = \{x_1,\ldots,x_n\}$ , so that for $i = 1$ to $n$ there is a subset $X_i$ of $Y$ for which $x_i\psi = id_{X_i}$ .

By Lemma 5, it may further be assumed that $\{x_1,\ldots,x_n\}$ has been listed so that for $1 \leq i < j \leq n$ , $x_j \nleq x_i$ , so by Lemma 4, $X_j \nsubseteq X_i$ .

Put $Y_i = X_1 \cup \ldots \cup X_i$ , for $i = 1$ to $n$ . Thus $Y_1 \subseteq \ldots \subseteq Y_n$ so $|Y_1| \leq \ldots \leq |Y_n| \leq m < n$ .

If $n = 1$ then $Y_1 = \emptyset$ , so $x_1\psi = id_{\emptyset} = 0\psi$ , which contradicts the fact that $\psi$ is faithful. Hence $n > 1$ , and thus $Y_{k+1} = Y_k$ for some

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$k$ , where $1 \le k \le n - 1$ , which means that $X_{k+1} \subseteq X_1 \cup \ldots \cup X_k$ .

Put $X'_i = X_i \cap X_{k+1}$ for $i = 1$ to $k$ , so

$$X_{k+1} = X_{k+1} \cap (X_1 \cup \ldots \cup X_k)$$
$$= X'_1 \cup \ldots \cup X'_k .$$

Thus, by Lemma 4, and since $\psi$ is a faithful representation,

$$x_{k+1}\psi = id_{X_{k+1}} = id_{X'_1} \vee \ldots \vee id_{X'_k}$$
$$= \left(id_{X_1} id_{X_{k+1}}\right) \vee \ldots \vee \left(id_{X_k} id_{X_{k+1}}\right)$$
$$= [(x_1x_{k+1}) \vee \ldots \vee (x_kx_{k+1})]\psi ,$$

so

$$x_{k+1} = (x_1x_{k+1}) \vee \ldots \vee (x_kx_{k+1}) .$$

But $x_{k+1}$ is join irreducible, so $x_{k+1} = x_jx_{k+1}$ , for some $j$ where $1 \le j \le k$ . Thus $x_{k+1} \le x_j$ , so

$$id_{X_{k+1}} = x_{k+1}\psi \le x_j\psi = id_{X_j} .$$

By Lemma 4, $X_{k+1} \subseteq X_j$ , which contradicts the fact that $X_{k+1} \not\subseteq X_j$ . This shows $m \ge n$ , so $\mu(E) \ge n$ .

Hence $\mu(S) = n$ , which completes the proof.

## 4. Fundamental inverse semigroups.

THEOREM 7. *Let $S$ be a finite fundamental inverse semigroup with semilattice of idempotents $E$ . Then*

$$\mu(S) = \mu(T_E) = n$$

*where $n$ is the number of join irreducible elements of $E$ .*

Proof. Clearly $\mu(E) \le \mu(S) \le \mu(T_E)$ . By Theorem 6, it remains to prove $\mu(T_E) \le n$ .

Let $X = \{x_1, \ldots, x_n\}$ be the set of join irreducible elements of $E$.

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Define $\phi : T_E \to PT_X$ by, for $\alpha \in T_E$,

$$\alpha\phi : x_i \to \begin{cases} x_i\alpha & \text{if } x_i \in \text{domain}(\alpha) \\ \text{undefined otherwise} . \end{cases}$$

Note that an element of a principal ideal of $E$ is join irreducible in that ideal if and only if it is join irreducible in $E$. Hence $\phi$ is well-defined and homomorphic because join irreducible elements are sent to join irreducible elements by semilattice isomorphisms, and one-one because a semilattice isomorphism of a finite semilattice is completely determined by its action on join irreducible elements.

Hence $\phi$ is a faithful representation, so $\mu(T_E) \le n$, which completes the proof.

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