A NOTE ON THE CONDITIONAL CHROMATIC POLYNOMIAL

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In this note we consider a finite graph without loops and multiple edges. The colouring of a graph G in λ colours is the colouring of its vertices in such a way that no two of adjacent vertices have the same colours and the number of used colours does not exceed λ [1, 4]. Two colourings of graph G are called *different* if there exists at least one vertex which changes colour when passing from one colouring to another.

DEFINITION. A function $P(G_0 \subset G, \mu, \lambda)$ is called a *conditional chromatic polynomial* of a graph G if it expresses the number of colourings of G in λ colours provided that the given subgraph $G_0 \subset G$ is coloured in $\mu \leq \lambda$ colours.

We suppose that $P(G_0 \subset G, \mu, \lambda) = 0$ if $\mu > \lambda$ or $G_0 \nsubseteq G$. Note that $P(G_0 \subset G, \lambda, \lambda)$ coincides with usual chromatic polynomial [1, p. 352]. If G_0 and G_1 are two subgraphs of G with the same vertex set, then $P(G_0 \subset G, \mu, \lambda) = P(G_1 \subset G, \mu, \lambda)$.

If F_n is a full (also known as 'complete') graph on n vertices, $k \leq n$, then $P(F_k \subset F_n, \mu, \lambda) = \mu^{(k)}(\lambda - k)^{(n-k)}$, where $\mu^{(k)} = \mu(\mu - 1)(\mu - 2) \ldots (\mu - k + 1)$. In particular $P(F_k \subset F_n, \lambda, \lambda) = \lambda^{(k)}(\lambda - k)^{(n-k)} = \lambda^{(n)}$.

If E_n is empty (also known as 'null' or 'totally disconnected' graph on n vertices) then $P(E_k \subset E_n, \mu, \lambda) = \mu^k \lambda^{n-k}$, in particular $P(E_k \subset E_n, \lambda, \lambda) = \lambda^n$.

Let $G = (X, V)$, $|X| = n$ be a graph and $G_0 = (X_0, V_0)$, $X_0 \subset X_0$ $|X_0| = m$ be an induced subgraph of G. Let also $x, y \in X$ be two non-adjacent vertices in G. We construct the graph G_1 from G by joining x and y by an edge and the graph G_2, obtained from G by contracting x and y into single vertex. Then we can observe that the following equality is true:

$$P(G_0 \subset G, \mu, \lambda) = P(G_0^1 \subset G_1, \mu, \lambda) + P(G_0^2 \subset G_2, \mu, \lambda), \tag{1}$$

where $G_0^1 = G_0^2 = G_0$ if $x, y \notin X_0$, and G_0^1, G_0^2 are the subgraphs induced respectively by X_0, $X_0 \cup \{x, y\}$ otherwise. It is so because $P(G_0 \subset G_2, \mu, \lambda)$ equals the number of colourings for which x and y have the same colour.

If we perform this operation further as much as possible we obtain

$$P(G_0 \subset G, \mu, \lambda) = \sum_{i=1}^{t} P(F_{k_i} \subset F_{n_i}, \mu, \lambda).$$

Since $P(F_k \subset F_n, \mu, \lambda) = P(F_k' \subset F_n, \mu, \lambda)$ for any two k-vertex complete subgraphs F_k, F_k' of F_n, we can write

$$P(G_0 \subset G, \mu, \lambda) = \sum_{i=1}^{m} \sum_{j=i}^{n} \alpha_{ij} P(F_i \subset F_j, \mu, \lambda).$$

In order to clear up the meaning of coefficients α_{ij} we consider one more class of colourings. A colouring of a graph is called an *i-colouring*, if exactly i colours are used, but

two i-colourings are assumed to be distinct if there is a pair of vertices x and y in G which have the same colour for one of these colourings and different colours for the other [4, p. 18]. Let $g_i(G)$ be the number of i-colourings of G. By analogy introduce the number $g_{ij}(G_0 \subset G)$ which equals the number of j-colourings of G provided that subgraph $G_0 \subset G$ is i-coloured $(i \le j)$.

It is evident that

$$g_{ij}(F_k \subset F_n) = \begin{cases} 1, & \text{if } i = k, j = n \\ 0 & \text{otherwise.} \end{cases}$$

Since $g_{kl}(G_0 \subset G) = g_{kl}(G_0^1 \subset G_1) + g_{kl}(G_0^2 \subset G_2)$ as in (1), after repeating this procedure as often as possible we have

$$g_{kl}(G_0 \subset G) = \sum_{i=1}^{m} \sum_{j=i}^{n} \alpha_{ij} g_{kl}(F_i \subset F_j).$$

Consequently, $\alpha_{ij} = g_{ij}(G_0 \subset G)$, $i = 1, \ldots, m; j = i, \ldots, n$.

Now we can state the following

THEOREM. *For any graph* $G = (X, Y)$, $|X| = n$ *and its subgraph* $G_0 = (X_0, V_0)$, $X_0 \subset X$, $|X_0| = m$, *the conditional chromatic polynomial has the following form*:

$$P(G_0 \subset G, \mu, \lambda) = \sum_{i=1}^{m} \sum_{j=i}^{n} g_{ij}(G_0 \subset G)\mu^{(i)}(\lambda - i)^{(j-i)}.$$

One can derive many properties of $P(G_0 \subset G, \mu, \lambda)$ which are the generalization of usual properties of $P(G, \lambda)$ [2, 3]. For example when considering the case $E_k \subset E_n$ we obtain an "algebraic" result:

COROLLARY [3]. *The coefficients in the expansion of* $\mu^m \lambda^{n-m}$ $(1 \le m \le n)$ *into the powers of form* $\mu^{(i)}(\lambda - i)^{(j-i)}$ *are the generalizations of Stirling's numbers and are calculated by the formula*:

$$V_{ij}^{nm} = \begin{cases} \begin{rcases} S(i, n), & \text{if } i = j, j = 1, 2, \ldots, n \\ 0 & \text{otherwise} \end{rcases} \text{if } m = n \\ jV_{ij}^{m,n-1} + V_{i,j-1}^{m,n-1}, i = 1, \ldots, m; j = 1, \ldots, n, & \text{if } m < n, \\ 0 & \text{if } i > m \text{ or } j > n \text{ or } m > n \text{ or } i > j, \end{cases}$$

where $S(i, n)$ *are the Stirling numbers of the second kind.*

Proof. $V_{ij}^{mn} = g_{ij}(E_m \subset E_n)$. If $m < n$ consider any $x \in E_n \setminus E_m$. If there exists $y \in E_n$, $y \ne x$ which is coloured in the same colour, for example j, then the number of such colourings is $jg_{ij}(E_m \subset E_{n-1})$, otherwise the number of colourings will be $g_{ij-1}(E_m \subset E_{n-1})$. $\square$

The initial fragment of V_{ij}^{mn}-numbers for $m = 1, 2, 3, 4$, $n = 1, 2, 3, 4$ is shown in the

following table:

n \ m	1	2	3	4
1	1000	—	—	—
2	1100 0000	1000 0100	—	—
3	1310 0000 0000	1100 0210 0000	1000 0300 0010	—
4	1761 0000 0000 0000	1310 0451 0000 0000	1100 0630 0031 0000	1000 0700 0060 0001

The conditional chromatic polynomial λ greatly extends the class of polynomials which may be considered as chromatic. For example polynomial $3\lambda^4$ may be interpreted as $P(E_1 \subset E_5, 3, \lambda)$ and so on. Finally, if $\gamma(G_0)$ is chromatic number of G_0, then the conditional chromatic number $\gamma_{G_0}(G)$ is the smallest integer λ such that $P(G_0 \subset G, \gamma(G_0), \lambda) \neq 0$ (cf. [4, p. 261]).

OPEN PROBLEMS

1. Find a necessary and sufficient condition for polynomial of two variables to be the conditional chromatic polynomial of some graph.

2. Find the meaning of coefficients and roots of the conditional chromatic polynomial.

3. Find, for a graph G, all subgraphs G_i and G_j such that

$$P(G_i \subset G, \mu, \lambda) = P(G_j \subset G, \mu, \lambda).$$

4. When does the equality $P(G_1 \subset G_2, \mu, \lambda) = P(G_3 \subset G_4, \mu, \lambda)$ imply isomorphism of graphs G_1 and G_3 or G_2 and G_4 respectively?

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