Can. J. Math., Vol. XXV, No. 4, 1973, pp. 806–811

# SPECTRAL PROPERTIES FOR INVERTIBLE MEASURE PRESERVING TRANSFORMATIONS

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**1. Introduction.** An invertible measure preserving transformation $T$ on the unit interval $I$ generates a unitary operator $U$ on the space $L^2(I)$ of Lebesque square integrable functions given by $(Uf)(x) = f(Tx)$ for all $f$ in $L^2(I)$ and $x$ in $I$. By definition

$$(f, g) = \int_0^1 f\bar{g}dx$$

for all $f, g$ in $L^2(I)$, the bar denoting complex conjugation. By the spectral theorem (Halmos [**2**]) there exists a spectral measure $E$ on the Borel subsets of the unit circle $C$ in the complex plane such that for all integers $k$,

$$U^k = \int_C z^k E(dz)$$

in the sense of strong convergence. By means of the spectral measure we define the resolution of the identity $E_t$ on $[0, 2\pi)$ such that $E_t = E(\{e^{is}: 0 \leqq s < t\})$. Then (Prohorov and Rozanov [**5**]) in the sense of strong convergence,

$$E_t = \lim_{n\to\infty} \sum_{-n}^{n} \frac{(e^{ijt} - 1)}{2\pi ij} U^{-j} + \tfrac{1}{2}(E(\{1\}) - E(\{e^{it}\})) \tag{1}$$

where

$$E(\{z\}) = \lim_{n\to\infty} \frac{1}{2n+1} \sum_{-n}^{n} z^j U^{-j}$$

for all $z$ in $C$, $(e^{i0t} - 1)/i0$ denoting the value $t$ for convenience of notation in this paper. It shall be shown that the operator $E$ can be extended to the space of Lebesque integrable functions $L^1(I)$ and that for all $f, g$ in $L^2(I)$ we have $E_tf \cdot E_sg = E_{t+s}(fE_sg + gE_tf - fg) + E_t(fg - gE_tf) + E_s(fg - fE_sg)$. From this will follow that for any $f, g$ in $L^2(I)$ and $h$ in the space $L^\infty(I)$ of essentially bounded functions, $(E_tf \cdot E_sg, h)$, which we use to denote

$$\int_0^1 E_tf \cdot E_sg \cdot \bar{h}dx,$$

Received April 6, 1972. This paper was the outcome of research subsidized by the National Research Council of Canada leading to my Ph.D. thesis which included some analogous results for flows (to appear in the Indiana University Mathematics Journal with much longer proofs than given here for the simple operator).



https://doi.org/10.4153/CJM-1973-082-1 Published online by Cambridge University Press

is a function of bounded variation on the square $[0, 2\pi) \times [0, 2\pi)$, and that

$$E(B)(fg) = \int_C E(Bz^{-1})f \cdot E(dz)g$$

which is a generalization of a result due to Koopman [**4**] and proved by Foias [**1**] ($B$ is any Borel subset of $C$). Finally for ergodic transformations it is shown that for almost all $x$ in $I$,

$$\left(\int_B \overline{E(dz)f} \cdot E(dz)g\right)(x) = (E(B)g, f),$$

which is an extension of a result obtained by Sinai [**6**].

**2. The multiplicative property.** Let $U$ be extended to $L^1(I)$ by writing $(Uf)(x) = f(Tx)$ for all $f$ in $L^1(I)$. Clearly by the measure preserving property of $T$, $Uf$ lies in $L^1(I)$ and $|(U^jf, h)| \leqq \|f\|_1 \|h\|_\infty$ for all $h$ in $L^\infty(I)$, the norms being the usual ones on $L^1(I)$ and $L^\infty(I)$ respectively. Thus

$$\lim_{n\to\infty} \sum_{-n}^{n} \frac{(e^{ijt} - 1)}{2\pi ij} (U^{-j}f, h)$$

is a square integrable function with respect to $t$ which we shall denote by $(E_t f, h)$ and it is easily seen that this is equal to $(f, E_t h)$. In a similar manner $E(\{z\})$ can be extended to $L^1(I)$.

Suppose now that $f$, $g$ are in $L^2(I)$ and $h$ is a function in $L^\infty(I)$. In the case of unitary operators induced by invertible measure preserving transformations we have the multiplicative property $U(fg) = (Uf)(Ug)$. In fact by a theorem of von Neumann (Halmos [**3**]) this is a necessary and sufficient condition for $U$ to be induced by such an operator. Note that

$$\frac{(e^{ijt} - 1)(e^{iks} - 1)}{2\pi ij 2\pi ik} (U^{-j}f \cdot U^{-k}g, h) = \frac{(e^{ijt} - 1)(e^{iks} - 1)}{2\pi ij 2\pi i(k - j)} (U^{-j}(fU^{j-k}g), h) + \frac{(e^{ijt} - 1)(e^{iks} - 1)}{2\pi ik 2\pi i(j - k)} (U^{-k}(gU^{k-j}f), h).$$

Using this in conjunction with equation (1) we get

$$E_t f \cdot E_s g = (E_{t+s} - E_s)(f(E_s g - g')) + E_t(fg') + (E_{t+s} - E_t)(g(E_t f - f')) + E_s(f'g)$$

where $f'$, $g'$ are functions in $L^2(I)$. Setting $t$ and $s$ equal to $2\pi$ we get

$$fg = f'g + fg'$$

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and thus

$$(2)\quad E_tf\cdot E_sg = E_{t+s}(fE_sg+gE_tf-fg)+E_t(fg-gE_tf)+E_s(fg-fE_sg).$$

Now $U$ maps real functions into real ones and hence $\overline{(1-E_{\tilde{t}})\bar{f}} = E_tf$, $\tilde{t} = 2\pi - t$, as can be seen by equation (1). Hence equation (2) implies that if $t' < t$ and $s' < s$ then

$$(3)\quad (E_t - E_{t'})f\cdot(E_s - E_{s'})g = (E_{t+s} - E_{t'+s'}) \times ((E_t - E_{t'})f\cdot(E_s - E_{s'})g));$$

in other words, if $B'$ and $B''$ are two Borel subsets of $C$ corresponding to intervals then

$$(4)\quad E(B')f\cdot E(B'')g = E(B'\cdot B'')(E(B')f\cdot E(B'')g).$$

Later we shall see that this holds for all Borel subsets of $C$.

For any Borel subset $B$ of $C$, if $||\cdot||_2$ represents the $L^2(I)$ norm then

$$(5)\quad \begin{aligned}|(f, E(B)h)| &\leqq ||f||_2||E(B)h||_2\\ &\leqq ||f||_2||h||_2\\ &\leqq ||f||_2||h||_\infty,\end{aligned}$$

from which follows the existence of some constant (depending only on $h$) such that for all sequences $\{B_j\}$ of disjoint Borel subsets of $C$ the essential supremum of $\Sigma|E(B_j)h|$ is bounded by it, for otherwise there exists a sequence $\{B_j\}$ such that either the real or imaginary part of $\Sigma E(B_j)h$ is much larger than $||h||_\infty$ which contradicts (5) above for appropriate choices of $f$ in $L^2(I)$. Using this along with (4) we shall now show that for given $f, g$ in $L^2(I)$ and $h$ in $L^\infty(I)$, $(E_tf\cdot E_sg, h)$ is of bounded variation on $[0, 2\pi)\times[0, 2\pi)$. For this it is sufficient to show the existence of a constant $K$ such that for any partition $\{B_j\}$ of $C$ into intervals of the same length we have $\Sigma_j\Sigma_k|(E(B_j)f\cdot E(B_k)g, h)| \leqq K$.

$$\begin{aligned}\sum_j\sum_k |(E(B_j)f\cdot E(B_k)g, h)| &= \sum_j\sum_k |(E(B_j)f\cdot E(B_k)g, \sum_m E(B_m)h)|\\ &\leqq \sum_j\sum_k\sum_m |(E(B_j)f\cdot E(B_k)g, E(B_m)h)|\\ &\leqq \sum_j\sum_k\sum_m |(E(B_j\cdot B_k)(E(B_j)f\\ &\qquad\cdot E(B_k)g), E(B_m)h)|.\end{aligned}$$

But for fixed $j$ and $m$, $B_m$ intersects $B_j\cdot B_k$ for at most two $k$, $m_j'$ and $m_j''$ say, for which $\{B_{m_j'}\}$ is a disjoint sequence as is $\{B_{m_j''}\}$. Hence.

$$\sum_j \sum_k |(E(B_j)fE(B_k)g, h)| \leqq \sum_m \sum_j |(E(B_j)f \cdot E(B_{m_j'})g, E(B_m)h)|$$

$$+ \sum_m \sum_j |(E(B_j)fE(B_{m_j''})g, E(B_m)h)|$$

$$\leqq \sum_m \left( \sum_j |E(B_j)f \cdot E(B_{m_j'})g|, |E(B_m)h| \right)$$

$$+ \sum_m \left( \sum_j |E(B_j)f \cdot E(B_{m_j''})g|, |E(B_m)h| \right)$$

$$\leqq \sum_m \left( \left( \sum_j |E(B_j)f|^2 \right)^{\frac{1}{2}} \left( \sum_j |E(B_{m_j'})g|^2 \right)^{\frac{1}{2}}, |E(B_m)h| \right)$$

$$+ \sum_m \left( \left( \sum_j |E(B_j)f|^2 \right)^{\frac{1}{2}} \left( \sum_j |E(B_{m_j''})g|^2 \right)^{\frac{1}{2}}, |E(B_m)h| \right)$$

$$\leqq \sum_m \left( \left( \sum_j |E(B_j)f|^2 \right)^{\frac{1}{2}} \left( \sum_j |E(B_j)g|^2 \right)^{\frac{1}{2}}, |E(B_m)h| \right)$$

$$+ \sum_m \left( \left( \sum_j |E(B_j)f|^2 \right)^{\frac{1}{2}} \left( \sum_j |E(B_j)g|^2 \right)^{\frac{1}{2}}, |E(B_m)h| \right)$$

$$\leqq 2 \left\| \left( \sum_j |E(B_j)f|^2 \right)^{\frac{1}{2}} \left( \sum_j |E(B_j)g|^2 \right)^{\frac{1}{2}} \right\|_1 \left\| \sum_m |E(B_m)h| \right\|_\infty$$

$$\leqq 2||f||_2||g||_2 \left\| \sum_m |E(B_m)h| \right\|_\infty .$$

Thus for $K$ one need but choose the upper bound of $||\Sigma|E(B_m)h|\,||_\infty$ over all partitions $\{B_j\}$ of $C$ (which is finite by above) times $||f||_2||g||_2$.

As a consequence of bounded variation for any Borel subset $A$ of $C \times C$,

$$\int\int_A E(du)f \cdot E(dv)g$$

converges weakly to a function in $L^1(I)$ for all $f, g$ in $L^1(I)$, equation (4) holds for all Borel subsets $B'$ and $B''$ of $C$, and for all $f$ in $L^1(I)$, $E_tf$ defined by equation (1) converges weakly to a function in $L^1(I)$ for all $t$. To prove the last statement, express $(E_t(f^{\frac{1}{2}}f^{\frac{1}{2}}), h)$ for $h$ in $L^\infty(I)$ by (1), replace $U^jf$ by $(U^jf^{\frac{1}{2}})(U^jf^{\frac{1}{2}})$ for all integers $j$, use the spectral theorem, and then use Lebesque's dominated convergence theorem.

THEOREM 1. *A unitary operator with spectral measure $E$ is induced by an invertible measure preserving transformation if and only if for all $f, g$ in $L^2(I)$ and any Borel subset $B$ of $C$*

$$E(B)(fg) = \int\int_{uv \in B} E(du)f \cdot E(dv)g. \tag{6}$$

*Proof.* If the unitary operator induced by an invertible measure preserving transformation then we get (4) and as a consequence (6), since by (4)

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$$E(B)(fg) = E(B)\left(\int\int E(du)f \cdot E(dv)g\right)$$
$$= \int\int_{uv\in B} E(du)f \cdot E(dv)g.$$

By von Neumann's theorem (introduced above) to prove the converse it is sufficient to show that for all $f$, $g$ in $L^2(I)$ such that $fg$ also lies in $L^2(I)$ we have $U(fg) = (Uf)(Ug)$. As shown by Foias [**1**], equation (6) implies

$$U(fg) = \int zE(dz)(fg)$$
$$= \int zE(dz)\left(\int\int E(du)f \cdot E(dv)g\right)$$
$$= \int\int uvE(du)f \cdot E(dv)g$$
$$= \left(\int uE(du)f\right)\left(\int vE(dv)g\right)$$
$$= (Uf)(Ug).$$

**3. Ergodic transformations.** Define the integral $\int \overline{E(du)f} \cdot E(du)g$, $f$, $g$ in $L^2(I)$, to be the limit in $L^1(I)$ of sums $\Sigma_j \overline{E(B_j)f} \cdot E(B_j)g$ as the mesh of the finite partition $\{B_j\}$ of $C$ tends to zero. By (6) this is equivalent to $E(\{1\})(\bar{f}g)$ (recall that for any Borel subset $B$ of $C$, since $U$ maps real functions into real ones we have $E(\bar{B})\bar{f} = \overline{E(B)f}$). By Birkhoff's ergodic theorem (Halmos [**3**]) a transformation is ergodic if and only if the function $E(\{1\})(\bar{f}g)$ is a constant almost everywhere for all $f$, $g$ in $L^2(I)$. Clearly we can extend the integral above to the case

$$\int_B \overline{E(du)f} \cdot E(du)g,$$

$B$ a Borel subset of $C$.

THEOREM 2. *If an invertible measure preserving transformation on the unit interval $I$ with spectral measure $E$ is ergodic, then for any Borel subset $B$ of $C$ and functions $f$, $g$ in $L^2(I)$ we have for almost all $x$ in $I$*

$$\left(\int_B \overline{E(du)f} \cdot E(du)g\right)(x) = (E(B)g, f). \tag{7}$$

*In particular, for almost all $x$ in $I$*

$$\left(\int_B |E(du)f|^2\right)(x) = (E(B)f, f). \tag{8}$$

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*Proof*. Using the fact that $E(\{1\})(fg)$ is a constant almost everywhere we obtain

$$\begin{aligned}\int_B \overline{E(du)f} \cdot E(du)g &= \int \overline{E(du)f} \cdot E(du)E(B)g \\ &= E(\{1\})(\bar{f}E(B)g) \\ &= \int_I E(\{1\})(\bar{f}E(B)g)dx \\ &= \int_I \left( \int_B \overline{E(du)f} \cdot E(du)g\right)dx \\ &= \int_B (E(du)g, E(du)f) \\ &= (E(B)g, f).\end{aligned}$$

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