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A TOPOLOGICAL CHARACTERIZATION OF GLEASON PARTS OF REAL FUNCTION ALGEBRAS

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ABSTRACT. It is well-known that a topological space is a Gleason part of some complex function algebra if and only if it is completely regular and σ-compact. In the present paper, a Gleason part of a real function algebra is characterized as a completely regular σ-compact topological space which admits an involutoric homeomorphism.

Introduction. Garnett [1] has characterized a Gleason part of a complex function algebra as a completely regular σ-compact topological space. In [2], the authors have introduced real function algebras and their Gleason parts. The purpose of the present note is to characterize a Gleason part of a real function algebra as a completely regular σ-compact topological space admitting an involutoric homeomorphism.

Preliminaries and notations. Let X be a compact Hausdorff space, and denote by $C(X)$ the complex Banach algebra of all complex-valued continuous functions on X, with the supremum norm.

Let $\tau: X \to X$ be an involutoric homeomorphism and $\sigma: C(X) \to C(X)$ be defined by

$$\sigma(f)(x) = \overline{f(\tau(x))}, \qquad f \in C(X), \qquad x \in X.$$

A *real function algebra A on* (X, τ) is a uniformly closed real subalgebra of $C(X)$ such that

(i) $\sigma(f) = f$ for all $f \in A$,

(ii) $1 \in A$, and

(iii) A separates the points of X, i.e., for $x_1 \neq x_2$ in X, there is $f \in A$ with $f(x_1) \neq f(x_2)$.

Let A be a real function algebra on (X, τ) and let Φ_A denote the set of all non-zero real-linear complex-valued homomorphisms of A. For $x \in X$, let $\phi_x(f) = f(x)$, $f \in A$. Then $\phi_x \in \Phi_A$, and we identify x with ϕ_x. For $\phi \in \Phi_A$, let $\tau_0(\phi)(f) = \overline{\phi(f)}$, $f \in A$. Then it is easy to see that the natural involutoric

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homeomorphism $\tau_0 : \Phi_A \to \Phi_A$ extends $\tau : X \to X$, i.e., $\tau_0(\phi_x) = \phi_{\tau(x)}$ for all $x \in X$.

It is proved in [2] that for $\phi, \psi \in \Phi_A$, the relation

$$\sup\{|\phi(f) - \psi(f)|\, |\phi(f) - \tau_0(\psi)(f)| : f \in A, \|f\| < 1\} < 4$$

is an equivalence relation on Φ_A. The equivalence classes under this relation are called the *Gleason parts of* A.

Let $B = \{f + ig : f, g \in A\}$. Then it is easy to see that $\|f + ig\| = \|f - ig\|$ for $f, g \in A$, and hence B is a complex function algebra on X. B is called the *complexification of* A. Let Φ_B denote the set of all non-zero complex-linear complex-valued homomorphisms of B and consider $\alpha : \Phi_A \to \Phi_B$, defined by

$$\alpha(\phi)(f + ig) = \phi(f) + i\phi(g).$$

Then α is a bijection and $\alpha(\phi)|_A = \phi$ for all $\phi \in \Phi_A$.

LEMMA. *Let B be any complex function algebra on X and $\tau : X \to X$ an involutoric homeomorphism such that $\sigma(B) = B$. If we let $A = \{f \in B : \sigma(f) = f\}$, then A is a real function algebra on (X, τ) and B is the complexification of A. Moreover, if $P \subset \Phi_B$ is a Gleason part of B, then $\alpha^{-1}(P) \cup \tau_0(\alpha^{-1}(P))$ is a Gleason part of A.*

Proof. Since B is a complex function algebra, it is clear that A is a uniformly closed real subalgebra of $C(X)$ and that $1 \in A$. To show that A separates the points of X, consider $x_1 \neq x_2$ in X. Since $\sigma(B) = B$, we note that $\sigma(h) \in B$ for every $h \in B$, and since

$$\sigma(h + \sigma(h)) = h + \sigma(h)$$
$$\sigma(h\sigma(h)) = h\sigma(h),$$

we conclude that $h + \sigma(h)$, $h\sigma(h) \in A$. First assume that $x_2 \neq \tau(x_1)$. Find $h \in B$ such that $h(x_1) = h(\tau(x_1)) = 1$ and $h(x_2) = 0$. Then $(h\sigma(h))(x_1) = 1$ and $(h\sigma(h))(x_2) = 0$. Next, let $x_2 = \tau(x_1)$. Find $h \in B$ such that $h(x_1) = i$ and $h(x_2) = 0$. Then $(h + \sigma(h))(x_1) = i$ and $(h + \sigma(h))(x_2) = -i$. Hence x_1 and x_2 are separated by A. Thus, A is a real function algebra on (X, τ). Also, since for every $h \in B$,

$$h = \frac{h + \sigma(h)}{2} + i\left(\frac{h - \sigma(h)}{2i}\right),$$

where $(h + \sigma(h))/2$, $(h - \sigma(h))/2i \in A$, it follows that B is the complexification of A.

Next, if $\phi \in \Phi_A$ and Q is the Gleason part of A containing ϕ, then it is proved in [2], Lemma 2.1 that $\alpha(Q)$ is the union of the Gleason parts of B containing $\alpha(\phi)$ or $\alpha(\tau_0(\phi))$. Now, let $P \subset \Phi_B$ be a Gleason part of B. Let $\alpha(\phi) \in P$ for some $\phi \in \Phi_A$ and let $\bar{P}$ be the Gleason part of B containing

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$\alpha(\tau_0(\phi))$. For any $\psi \in \Phi_A$, $\|\alpha(\phi)-\alpha(\psi)\|<2$ iff $\|\alpha(\tau_0(\phi))-\alpha(\tau_0(\psi))\|<2$. Hence $\alpha^{-1}(\bar{P})=\tau_0(\alpha^{-1}(P))$. Thus,

$$\begin{aligned}\alpha^{-1}(P\cup\bar{P})&=\alpha^{-1}(P)\cup\alpha^{-1}(\bar{P})\\&=\alpha^{-1}(P)\cup\tau_0(\alpha^{-1}(P))\end{aligned}$$

is a Gleason part of A. Q.E.D.

PROPOSITION. *Let V be a complex function algebra on Φ_V and $\tau':\Phi_V\to\Phi_V$ be an involutoric homeomorphism such that $\sigma'(V)=V$. Let P' be a Gleason part of V and S' a hull-kernel closed subset of Φ_V such that $\tau'(S')=S'$ and $\tau'(P'\cap S')=P'\cap S'$. Then there exists a real function algebra A, a Gleason part Q of A and a homeomorphism F of $P'\cap S'$ onto Q such that $F\circ\tau'=\tau_0\circ F$.*

Proof. Let U be the complex function algebra on the torus $T^2=\{(e^{i\theta_1}, e^{i\theta_2}):0\le\theta_1, \theta_2<2\pi\}$ generated by the functions $z_1^n z_2^m$, where n and m are integers with $n+m\sqrt{2}\ge 0$. Let ψ_0 be the point in Φ_U represented by the Haar measure on the torus T^2. Let

$$J=\{z=(z_1, z_2)\in T^2\colon \operatorname{Re} z_1, \operatorname{Re} z_2\le 0\}$$

and

$$X=(J\times\Phi_V)\cup(\Phi_U\times S').$$

Then X is a compact subset of $\Phi_U\times\Phi_V$. Let $U\otimes V$ be the complex function algebra on $\Phi_U\times\Phi_V$ generated by all functions of the form $(f\otimes g)(z,\psi)=f(z)g(\psi)$, where $f\in U$ and $g\in V$. Let B denote the uniform closure of $\{h|_X\colon h\in U\otimes V\}$ in $C(X)$. It is shown in the course of the proof of Theorem 1 of [1] that B is a complex function algebra on X, $\Phi_B=X$ and $P=\{\psi_0\}\times\{P'\cap S'\}$ is a Gleason part of B.

Define $\tau: X\to X$ as follows:

$$\tau(z,\psi)=(\bar{z}, \tau'(\psi)), \qquad z\in\Phi_U, \qquad \psi\in\Phi_V,$$

where $\bar{z}=(\bar{z}_1, \bar{z}_2)$ for $z=(z_1, z_2)$. Then τ is an involutoric homeomorphism on X. Moreover, if $f\in U$ and $g\in V$, then for all $z\in\Phi_U$ and $\psi\in\Phi_V$,

$$\begin{aligned}\sigma((f\otimes g)|_X)(z,\psi)&=\overline{(f\otimes g)|_X(\tau(z,\psi))}\\&=\overline{(f\otimes g)(\bar{z},\tau'(\psi))}\\&=\overline{f(\bar{z})g(\tau'(\psi))}.\end{aligned}$$

Now, if we let $f^*(z)=\overline{f(\bar{z})}$, $z\in\Phi_U$, then clearly $f^*\in U$, and if we let $g^*(\psi)=\overline{g(\tau'(\psi))}$, $\psi\in\Phi_V$, then by our assumption, $g^*\in V$. Hence $\sigma((f\otimes g)|_X)\in B$. It follows that $\sigma(B)=B$.

By the previous lemma, we see that $A=\{f\in B\colon \sigma(f)=f\}$ is a real function algebra on (X,τ), B is the complexification of A and $Q=\alpha^{-1}(P)\cup\tau_0(\alpha^{-1}(P))$

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is a Gleason part of A, where $\tau_0(\phi)=\bar{\phi}$ is the natural involutoric homeomorphism on Φ_A. Now, since the diagram

$$\begin{tikzcd} \Phi_B \arrow[r, "\tau"] & \Phi_B \\ \Phi_A \arrow[u, "\alpha"] \arrow[r, "\tau_0"] & \Phi_A \arrow[u, "\alpha"'] \end{tikzcd}$$

commutes, and since

$$\begin{aligned}\tau(P)&=\{\bar{\psi}_0\}\times\tau'(P'\cap S')\\&=\{\psi_0\}\times(P'\cap S')\\&=P,\end{aligned}$$

it follows that $\tau_0(\alpha^{-1}(P))=\alpha^{-1}(P)$. Thus,

$$Q=\alpha^{-1}(P) \quad\text{and}\quad \tau_0(Q)=Q.$$

Finally, define $F: P'\cap S'\to Q$ by $F(\psi)=\alpha^{-1}((\psi_0,\psi))$ for $\psi\in P'\cap S'$. Then F is a homeomorphism of $P'\cap S'$ onto Q and the diagram

$$\begin{tikzcd} P'\cap S' \arrow[r, "F"] & Q \\ P'\cap S' \arrow[u, "\tau'"] \arrow[r, "F"] & Q \arrow[u, "\tau_0"'] \end{tikzcd}$$

commutes, i.e., $F\circ\tau'=\tau_0\circ F$. Q.E.D.

REMARK. It was proved in Theorem 1.3 of [2] that if $\phi\in\Phi_A$ for a real function algebra A on (X,τ), then $\psi\in\Phi_A$ belongs to the Gleason part containing ϕ iff either $\|\phi-\psi\|<2$ or $\|\bar{\phi}-\psi\|<2$. Thus, $\bar{\phi}$ always belongs to the Gleason part containing ϕ. Let us consider the following question: Does there exist a Gleason part Q of a real function algebra A and $\phi\in\Phi_A$ such that $Q=\{\phi,\bar{\phi}\}$ but $\bar{\phi}\neq\phi$? The answer is in the affirmative, as the following example shows: Let V be the algebra of all complex-valued continuous functions on the closed unit disk D in the complex plane, which are analytic on the open unit disk D^0. Then $\Phi_V=D$. Let $\tau':\Phi_V\to\Phi_V$ be given by $\tau'(z)=\bar{z}$, $z\in D$. Then for every $f\in V$, $\sigma'(f)(z)=\overline{f(\tau'(z))}=\overline{f(\bar{z})}$, $z\in D$, also belongs to V, i.e., $\sigma(V)=V$. Thus, $A=\{f\in V: f(\bar{z})=\overline{f(z)}\}$ is a real function algebra on (D,τ') and for any θ in $(0,\pi)$, $\{e^{i\theta},e^{-i\theta}\}$ constitutes a Gleason part of A. Note here that $\|e^{i\theta}-e^{-i\theta}\|=2$ for any $\theta\in(0,\pi)$. This raises the following question: Does there exist a Gleason part Q of a real function algebra A and $\phi\in\Phi_A$ such that $Q=\{\phi,\bar{\phi}\}$, $\phi\neq\bar{\phi}$ and $\|\phi-\bar{\phi}\|<2$. The construction in the earlier

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proposition can be used to answer this question affirmatively. With V, τ' and σ' as described above, let $P'=D^0$ and $S'=\{i/2, -i/2\}$. Then P' is a Gleason part of V, S' is hull-kernel closed in Φ_V and $\tau'(P'\cap S')=\tau'(S')=S'=P'\cap S'$. Hence by the earlier proposition, there exists a real function algebra A, a Gleason part Q of A and a homeomorphism F of S' onto Q with $F\circ\tau'=\tau_0\circ F$. Let $\phi=F(i/2)$. Then $\bar\phi=F(-i/2)$ and $Q=\{\phi, \bar\phi\}$. Moreover, the proof of the earlier proposition shows that $P=\alpha(Q)$ is a Gleason part of the complexification B of A. Hence $\|\alpha(\phi)-\alpha(\bar\phi)\|<2$, which in turn implies that $\|\phi-\bar\phi\|<2$, by Theorem 2.2(a) of [2].

THEOREM. *Let A be a real function algebra on (X,τ) and $Q\subset\Phi_A$ a Gleason part of A. Then Q is a completely regular σ-compact topological space and $\tau_0: Q\to Q$.*

Conversely, let K be a completely regular, σ-compact topological space and let $\tau: K\to K$ be an involutoric homeomorphism. Then there is a real function algebra A, a Gleason part $Q\subset\Phi_A$ and a homeomorphism F of K onto Q such that $F\circ\tau=\tau_0\circ F$.

Proof. If $Q\subset\Phi_A$ is a Gleason part of a real function algebra A, then Q is completely regular since it is a subset of the compact Hausdorff space Φ_A. Now, by Theorem 1.3 of [2],

$$\psi\in Q \quad \text{iff} \quad \|\phi-\psi\|<2 \quad \text{or} \quad \|\bar\phi-\psi\|<2.$$

For natural numbers n and m, let $A_n=\{\psi\in\Phi_A: \|\phi-\psi\|\le 2-1/n\}$ and $B_m=\{\psi\in A: \|\bar\phi-\psi\|\le 2-1/m\}$. Then A_n and B_m are closed subsets of Φ_A, and hence they are compact. Since

$$Q=\Big(\bigcup_n A_n\Big)\cup\Big(\bigcup_m B_m\Big),$$

it follows that Q is σ-compact. That τ_0 maps Q into Q is also clear.

Conversely, let K be a completely regular σ-compact topological space, and $\tau: K\to K$ an involutoric homeomorphism. Let

$$K'=\begin{cases} K\cup\{\infty\}, & \text{if } K \text{ is compact,}\\ \beta K, & \text{if } K \text{ is non-compact,}\end{cases}$$

where βK denotes the Stone–Cech compactification of K. First note that τ can be extended as an involutoric homeomorphism of K'.

Let $I=K'-K$. (I will be a singleton set if K is compact.) For $i\in I$, let D_i be the closed unit disk in the complex plane and Y_I the product of D_i's with $i\in I$. Let A_I be the subalgebra of $C(Y_I)$ generated by the coordinate functions z_i, $i\in I$, where $z_i(p)=p_i$. Then $\Phi_{A_I}=Y_I$. Let θ be the origin in Y_I, that is, $z_i(\theta)=0$ for all $i\in I$, and let P_0 be the Gleason part of Φ_{A_I} containing θ. Let

$$V=\{f\in C(K'\times Y_I): f|_{\{x\}\times Y_I}\in A_I \text{ for all } x\in K' \text{ and } f|_{K'\times\{\theta\}} \text{ is constant}\}.$$

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Then $\Phi_V = K' \times Y_I/\approx$, where $\approx$ identifies $K' \times \{\theta\}$ to a point. Also, $P' \equiv \{(x, z) \in \Phi_V, z \in P_0\}$ is a Gleason part of V, since P' is the union of several copies of P_0 whose centres are identified. (See [1], proof of Theorem 2.)

Let $\tau': \Phi_V \to \Phi_V$ be defined by $\tau'(x, z) = (\tau(x), \bar{z})$, $x \in K'$, $z \in Y_I$, where $\bar{z}_i(p) = \bar{p}_i$ whenever $z_i(p) = p_i$. From the definition of V, it is easy to see that for every $f \in V$, the function f^* defined by $f^*(x, z) = \overline{f(\tau(x), \bar{z})}$, $(x, z) \in \Phi_V$, also belongs to V.

Now, we show that there exist a hull-kernel closed set S' in Φ_V such that $\tau'(S') = S'$ and $\tau'(P' \cap S') = P' \cap S'$ and a homeomorphism $G: K \to P' \cap S'$ such that $G \circ \tau = \tau' \circ G$. Then an application of the earlier proposition would complete our proof.

Case (*i*). Let K be compact, so that $K' = K \cup \{\infty\}$. Since $\{\infty\}$ is isolated in K', there is a continuous function $h: K' \to [\frac{1}{2}, 1]$ such that $h^{-1}(1) = \{\infty\}$ and $h(\tau(x)) = h(x)$ for all $x \in K$. Let $S' = \{(x, h(x)) : x \in K'\} \subset \Phi_V$. Then the function $g(x, z) = (h(x) - z)/(3h(x) - z)$ vanishes exactly on S' and belongs to V. Hence S' is hull-kernel closed. Also, $\tau'(S') = S'$ and $\tau'(P' \cap S') = P' \cap S'$ since for $(x, h(x)) \in P' \cap S'$, we have

$$\tau'((x, h(x)) = (\tau(x), \overline{h(x)}) = (\tau(x), h(x)),$$

which is in $P' \cap S'$. Let $G: K \to P' \cap S'$ be defined by $G(x) = (x, h(x))$. Then G is a homeomorphism of K onto $P' \cap S'$ and $G \circ \tau = \tau' \circ G$.

Case (*ii*). Let K be non-compact, so that $K' = \beta K$. Write $K = \bigcup_{n=1}^{\infty} K_n$, where $K_n \subset K_{n+1}$, $\tau(K_n) = K_n$ and each K_n is compact. Then for each $t \in K' - K$, there exists a continuous function $h_t: K' \to [\frac{1}{2}, 1]$ with $h_t(t) = 1$, $h_t(x) \leq 1 - 2^{-n}$ for all $x \in K_n$ and $h_t(\tau(x)) = h_t(x)$ for all $x \in K$. Let $G: \beta K \to \Phi_V$ be defined by $G(x) = (x, H(x))$, where $(H(x))_t = h_t(x)$ for each $t \in I$. Then G is a homeomorphism of βK onto $S' = G(\beta K)$, and $G(K) = P' \cap S'$. Also, as in Case (i), it follows that $\tau'(S') = S'$, $\tau'(P' \cap S') = P' \cap S'$ and that $G \circ \tau = \tau' \circ G$. Finally, S' is hull-kernel closed in Φ_V because $S' = \bigcap \{g_t^{-1}(0) : t \in I\}$, where

$$g_t(x, z) = (h_t(x) - z_t)/(3h_t(x) - z_t). \quad \text{Q.E.D.}$$

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