BULL. AUSTRAL. MATH. SOC. 90C29, 49J40, 47H05
VOL. 68 (2003) [295–302]

VECTOR QUASI-EQUILIBRIUM PROBLEMS

ABDUL KHALIQ AND SONAM KRISHAN

In this paper we establish existence theorems for vector quasi-equilibrium problems in Hausdorff topological vector spaces both under compactness and noncompactness assumptions.

1. INTRODUCTION

Quasi-equilibria constitute an extension of Nash equilibria, which are of fundamental importance in the theory of noncooperative games. By scalar *equilibrium problem*, Blum and Oettli [**3**] mean the problem of finding

$$x^* \in K \text{ such that } f(x^*, y) \geqslant 0 \text{ for all } y \in K, \tag{1}$$

where K is a given set and $f : K \times K \to \overline{\mathbb{R}}$ is a given bifunction. This problem include as special cases, variational inequality problems, fixed point problems, optimisation problems and complementarity problems, and has many applications in economics, mathematical physics, game theory and operations research. Recently, Ansari [**1**], Tan and Tinh [**23**], Kazmi [**12**], Konnov and Yao [**17**], Khaliq [**13**], Oettli and Sehlager [**21**], Ansari, Schaible and Yao [**2**] and Giannessi [**10**] generalised problem (1) to vector valued bifunctions. Also Noor and Oettli [**20**], Cubiotti [**5**] and Ding [**6, 7**], have studied quasi-equilibrium problems. Such types of problems include many optimisation problems, Nash equilibrium problems, quasi-variational inequalities, quasi-complementarity problems and others as special cases.

In this note we introduce and establish the existence of solutions of *vector quasi-equilibrium problems* which unifies and generalises the corresponding results mentioned above.

Received 27th February, 2003
The first author was supported by University Grants Commision, India, project No.DAA/2001/4562.

2. Preliminaries

Let X be a Hausdorff topological vector space and K be a non-empty convex subset of X. We shall denote by 2^K the family of all subsets of K, $\text{int}_X K$ the interior of K in X, $\text{cl}_X K$ the closure of K in X, and $\text{co}(K)$ the convex hull of K. Let Y be an ordered Hausdorff topological vector space and $C : K \to 2^Y$ be a multifunction such that for each $x \in K, C(x)$ is a closed convex cone with $\text{int}\, C(x) \neq \phi$, where $\text{int}\, C(x)$ denotes the interior of $C(x)$. It is clear that for each $x \in K$ the cone $C(x)$ can define on Y a partial order $\preceq_{C_x}$ by $y \preceq_{C_x} z$ if and only if $z - y \in C(x)$. We shall write $y \prec_{C_x} z$ if $z - y \in \text{int}\, C(x)$ in the case $\text{int}\, C(x) \neq \phi$. Given a continuous multifunction $A : K \to 2^K$ and a bifunction $f : K \times K \to Y$, then we consider the following *vector quasi-equilibrium problem*.

Find $x^* \in K$ such that

$$x^* \in \text{cl}_K A(x^*) \text{ and } f(x^*, y) \notin -\text{int}_Y C(x^*) \text{ for all } y \in A(x^*). \tag{2}$$

When $A(x) = K$ for each $x \in K$, problem (2) was considered by Ansari [**1**]. When $A(x) = K$ and $C(x) = P$ for each $x \in K$, where P is a convex cone in Y, then problem (2) was considered by Tan and Tinh [**22**] and Kazmi [**12**]. When $C(x) = \mathbb{R}_+$ for each $x \in K$ and $Y = \mathbb{R}$, problem (2) was studied by Lin and Park [**19**]. When $A(x) = K$, $C(x) = \mathbb{R}_+$ for each $x \in K$ and $Y = \mathbb{R}$, problem (2) was studied by Blum and Oettli [**3**], Konnove and Schaible [**16**] and Kalmoun [**11**].

We shall use the following definitions. Let $S, T : K \to 2^Y$ be multifunctions, then the multifunctions $\text{clS}, \text{coS}, S \cap T : K \to 2^X$ are defined as $(\text{clS})(x) = \text{clS}\,(x)$, $(\text{coS})(x) = \text{coS}\,(x)$ and $(S \cap T)(x) = S(x) \cap T(x)$, for each $x \in K$.

Let K be a closed and convex subset of X and $C : K \to 2^Y$ be a multifunction such that for each $x \in K, C(x)$ is closed, convex, solid and pointed cone in Y. A multifunction $g : K \to 2^Y$ is said to be C_x*-convex* if for each $x, y \in K$ and $\lambda \in [0, 1]$,

$$g\big(\lambda y + (1 - \lambda)x\big) \preceq_{C_x} \lambda g(y) + (1 - \lambda)g(x).$$

Let $T : X \to 2^Y$ be a multifunction. Then T is said to be *upper semicontinuous* on X if for each $x \in X$ and each open set U in Y containing $T(x)$, there exists an open neighbourhood V of x in X such that $T(y) \subseteq U$, for each $y \in V$.

The graph of T, denoted by $G(T)$, is

$$G(T) = \big\{(x, y) \in X \times Y : x \in X, y \in T(x)\big\}.$$

The inverse of T, denoted by T^{-1} is a multifunction from $R(T)$, the range of T, to X defined by

$$x \in T^{-1}(y) \text{ if and only if } y \in T(x).$$

Let X be a topological space and K a subset of X such that $K = \bigcup_{n=1}^{\infty} K_n$, where $\{K_n\}_{n=1}^{\infty}$ is an increasing sequence of nonempty compact sets in the sense that $K_n \subseteq K_{n+1}$ for all $n \in \mathbb{N}$. A sequence $\{x_n\}_{n=1}^{\infty}$ in K is said to be *escaping sequence* ([4]) from K (relative to $\{K_n\}_{n=1}^{\infty}$) if for each n there is an M such that $k \geqslant M$, $x_k \notin K_n$.

The following result is a special case of [**8**, Theorem 2].

THEOREM A. *Let K be a nonempty compact convex subset of a Hausdorff topological vector space X. Suppose that $A, \mathrm{cl}_X A, P : K \to 2^K$ are multifunctions such that for each $x \in K, A(x)$ is nonempty convex set, for each $y \in K, A^{-1}(y)$ is open set in K, $\mathrm{cl}_X A$ is upper semicontinuous, for each $x \in K, x \notin \mathrm{coP}(x)$ and for each $y \in K, P^{-1}(y)$ is open in K. Then there exists $x^* \in K$ such that $x^* \in \mathrm{cl}_K A(x^*)$ and $A(x^*) \cap P(x^*) = \phi$.*

The following result is a special case of [**9**, Theorem 2].

THEOREM B. *Let K be a nonempty convex subset of a locally convex Hausdorff topological vector space X, and D be a nonempty compact subset of K. Suppose that $A, P : K \to 2^D$ and $\mathrm{cl}_X A : K \to 2^K$ are multifunctions such that for each $x \in K, A(x)$ is a nonempty convex set, for each $y \in D, A^{-1}(y)$ is an open set in K, $\mathrm{cl}_X A$ is upper semicontinuous, for each $x \in K$ $x \notin \mathrm{coP}(x)$, and for each $y \in D, P^{-1}(y)$ is open in K. Then there exists $x^* \in K$ such that $x^* \in \mathrm{cl}_K A(x^*)$ and $A(x^*) \cap P(x^*) = \phi$.*

3. Existence results in compact sets

In this section, we prove an equilibrium existence theorem in a compact setting.

THEOREM 1. *Let K be a nonempty compact convex subset of a Hausdorff topological vector space X and Y be an ordered Hausdorff topological vector space. Let $f : K \times K \to Y$ be a bifunction. Let $C : K \to 2^Y$ and $A : K \to 2^K$ be multifunctions. Assume that*

- 1^o *for each $x \in K$, $f(x, x) = 0$,*
- 2^o *f is C_x – convex and continuous in the second argument,*
- 3^o *the mapping $W : K \to 2^Y$ defined by $W(x) = Y \setminus (-\mathrm{int}_Y C(x))$ for each $x \in K$, has a closed graph in $K \times Y$,*
- 4^o *for each $x \in K$, $C(x)$ is closed, convex and pointed cone in Y such that $\mathrm{int}_Y C(x)$ is nonempty.*
- 5^o *for each $x \in K$, $A(x)$ is nonempty convex and for each $y \in K$, $A^{-1}(y)$ is open in K. Also $\mathrm{cl}_K A : K \to 2^K$ is upper semicontinuous.*

Then there exists $x^* \in K$ such that

$$x^* \in \mathrm{cl}_K A(x^*) \text{ and } f(x^*, y) \notin -\mathrm{int}_Y C(x^*) \text{ for all } y \in A(x^*).$$

PROOF: We define the multifunction $P : K \to 2^K$ by

$$P(x) = \{y \in K : f(x,y) \in -\operatorname{int}_Y C(x)\}, \text{ for each } x \in K.$$

We show first that $x \notin \operatorname{coP}(x)$, for each $x \in K$. Suppose that $x \in \operatorname{coP}(x)$, for some $x \in K$. Then there exists $x_o \in K$ such that $x_o \in \operatorname{coP}(x_o)$. This implies that x_o can be expressed as

$$x_o = \sum_{i \in I} \lambda_i y_i, \text{ with } \lambda_i \geqslant 0, \quad \sum_{i \in I} \lambda_i = 1, \ i = 1, \dots, n,$$

where $\{y_i : i \in \mathbb{N}\}$ be a finite subset of K, $I \subset \mathbb{N}$ be arbitrary nonempty subset where $\mathbb{N}$ denotes the set of natural numbers. This follows

$$f(x_o, y_i) \in -\operatorname{int}_Y C(x_o) \text{ for all } i = 1, \dots, n.$$

Hence

$$\sum_{i \in I} \lambda_i f(x_o, y_i) \in -\operatorname{int}_Y C(x_o). \tag{3}$$

By assumptions 1^o and 2^o we have

$$0 = f(x_o, x_o) \preceq_{C_{x_o}} \sum_{i \in I} \lambda_i f(x_o, y_i).$$

Hence

$$\sum_{i \in I} \lambda_i f(x_o y_i) \in C(x_o). \tag{4}$$

Combining (3) and (4) yeilds

$$\sum_{i \in I} \lambda_i f(x_o, y_i) \in \{-\operatorname{int}_Y C(x_o)\} \cap C(x_o) = \phi, \tag{5}$$

which is a contradiction. It remains to show that $P^{-1}(y)$ is open in K, which is equivalent to showing that $[P^{-1}(y)]^c = K \setminus P^{-1}(y)$ is closed. Indeed we have

$$P^{-1}(y) = \{x \in K : y \in P(x)\} = \{x \in K : f(y,x) \in -\operatorname{int}_Y C(x)\},$$
$$[P^{-1}(y)]^c = \{x \in K : f(y,x) \notin -\operatorname{int}_Y C(x)\}.$$

Let $u \in \overline{[P^{-1}(y)]^c}$, the closure of $[P^{-1}(y)]^c$ in K. We claim that $u \in [P^{-1}(y)]^c$. Indeed, let $\{x_\lambda\}_{\lambda \in \Lambda}$ be a net in $[P^{-1}(y)]^c$ such that $\{x_\lambda\}$ converges to u. Then we have $f(y, x_\lambda) \notin -\operatorname{int}_Y C(x_\lambda)$ for each $y \in K$, that is, $f(y, x_\lambda) \in W(x_\lambda)$ for all $\lambda \in \Lambda$. Since W has a closed graph in $K \times Y$ and f is continuous in the second argument, we have $f(y, u) \in W(u)$, that is, $f(y, u) \notin -\operatorname{int}_Y C(u)$. Hence $u \in \left[P^{-1}(y)\right]^c$. From assumption 5^o, it follows that all the hypothesis of Theorem A are satisfied. Hence there exists $x^* \in K$ such that

$$x^* \in \operatorname{cl}_K A(x^*) \text{ and } A(x^*) \cap P(x^*) = \phi.$$

This implies that there exists $x^* \in K$ such that

$$x^* \in \operatorname{cl}_K A(x^*) \text{ and } f(x^*, y) \notin -\operatorname{int}_Y C(x^*)$$

for all $y \in A(x^*)$. ∎

As an application of Theorem 1, we have the following results.

Corollary 1. *If for each $x, y \in K, f(x, y) = \Phi(y) - \Phi(x)$, for some function $\Phi : K \to Y$, then we can show that there exists $x^* \in K$ such that*

(6) $$x^* \in \operatorname{cl}_K A(x^*) \text{ and } \Phi(x^*) \not\preceq_{C_x} \Phi(y) \text{ for all } y \in A(x^*).$$

Corollary 2. *Let $L(X, Y)$ be the space of all linear continuous operators from X to Y. If for each $x, y \in K, f(x, y) = \langle T(x), y - x\rangle$, where $T : K \to L(X, Y)$ is a function and $\langle T(x), y\rangle$ denotes the evaluation of the linear operator $T(x)$ at y, then we can show that there exists $x^* \in K$ such that*

(7) $$x^* \in \operatorname{cl}_K A(x^*) \text{ and } \langle T(x^*), y - x^*\rangle \notin -\operatorname{int}_Y C(x^*)$$

for all $y \in A(x^*)$.

Remark 1. Theorem 1 improves and generalises [**16**, Theorem 3.1], [**18**, Lemma 2.1], [**19**, Theorem 4], [**15**, Theorem 1], [**14**, Theorem 2.1] and many other results in the literature.

4. Existence results in noncompact sets

In this section, we prove an equilibrium existence theorem in a noncompact setting.

Theorem 2. *Let K be a nonempty subset of a Hausdorff topological vector space X, and Y be an ordered Hausdorff topological vector space. Let $K = \bigcup_{n=1}^{\infty} K_n$ where $\{K_n\}_{n=1}^{\infty}$ is an increasing sequence of nonempty, compact and convex subsets of K. Let $f : K \times K \to Y$ be a bifunction. Let $C : K \to 2^Y$ and $A : K \to 2^K$ be the multifunctions. Assume that*

1^o *for each $x \in K$, $f(x, x) = 0$,*

https://doi.org/10.1017/S0004972700037679 Published online by Cambridge University Press

2° *f is C_x − convex and continuous in the second argument,*

3° *the mapping $W : K \to 2^Y$ defined by $W(x) = Y \setminus (-\operatorname{int}_Y C(x))$ for each $x \in K$, has a closed graph in $K \times Y$,*

4° *for each $x \in K$, $C(x)$ is closed, convex and pointed cone in Y such that $\operatorname{int}_Y C(x)$ is nonempty,*

5° *for each $x \in K$, $A(x)$ is nonempty convex and for each $y \in K$, $A^{-1}(y)$ is open in K. Also $\operatorname{cl}_K A : K \to 2^K$ is upper semicontinuous,*

6° *for each sequence $\{x_n\}_{n=1}^{\infty}$ in K with $x_n \in K_n$, $n \in \mathbb{N}$ which is escaping from K relative to $\{K_n\}_{n=1}^{\infty}$, there exists $m \in \mathbb{N}$ and $y_m \in K_m \cap A(x_m)$ such that for each $x_m \in \operatorname{cl}_K A(x_m)$,*

$$f(x_m, y_m) \in -\operatorname{int}_Y C(x_m).$$

Then there exists $x^* \in K$ such that

$$x^* \in \operatorname{cl}_K A(x^*) \text{ and } f(x^*, y) \notin -\operatorname{int}_Y C(x^*) \text{ for all } y \in A(x^*).$$

Proof: Since for each $n \in \mathbb{N}$, K_n is compact and convex set in X, Theorem 1 shows for all $n \in \mathbb{N}$, there exists $x_n \in K_n$ such that

$$x_n \in \operatorname{cl}_K A(x_n) \text{ and } f(x_n, z) \notin -\operatorname{int}_Y C(x_n) \tag{8}$$

for all $z \in A(x_n)$. Suppose that the sequence $\{x_n\}_{n=1}^{\infty}$ is escaping from K relative to $\{K_n\}_{n=1}^{\infty}$. By assumption 6°, there exists $m \in \mathbb{N}$ and $z_m \in K_m \cap A(x_m)$ such that for each $x_m \in \operatorname{cl}_K A(x_m)$,

$$f(x_m, z_m) \in -\operatorname{int}_Y C(x_m),$$

which contradicts (8). Hence $\{x_n\}_{n=1}^{\infty}$ is not an escaping sequence from K relative to $\{K_n\}_{n=1}^{\infty}$. Thus using arguments similar to those used by Qun [22] in proving [**22**, Theorem 2] there exists $r \in \mathbb{N}$ and $x^* \in K_r$ such that $x_n \to x^*$ and $f(x^*, y) \in W(x^*)$. Since $\operatorname{cl}_K A :\to 2^K$ is upper semicontinuous with compact values, there exists $x^* \in K$ such that

$$x^* \in \operatorname{cl}_K A(x^*) \text{ and } f(x^*, y) \notin -\operatorname{int}_Y C(x^*)$$

for all $y \in A(x^*)$. ∎

Theorem 3. *Let K be a nonempty convex subset of a locally convex Hausdorff topological vector space X, and D be a nonempty compact subset of K. Let Y be an ordered Hausdorff topological vector space. Let $f : K \times K \to Y$ be a bifunction. Let $C : K \to 2^Y$ and $A : K \to 2^D$ be the multifunctions. Assume that*

1° *for each $x \in K$, $f(x, x) = 0$,*

2° *f is C_x – convex and continuous in the second argument,*

3° *the mapping $W : K \to 2^Y$ defined by $W(x) = Y \setminus \big(-\operatorname{int}_Y C(x)\big)$ for each $x \in K$, has a closed graph in $K \times Y$,*

4° *for each $x \in K$, $C(x)$ is closed, convex and pointed cone in Y such that $\operatorname{int}_Y C(x)$ is nonempty.*

5° *for each $x \in K$, $A(x)$ is nonempty convex and for each $y \in K$, $A^{-1}(y)$ is open in K. Also $\operatorname{cl}_K A : K \to 2^D$ is upper semicontinuous.*

Then there exists $x^* \in K$ such that

$$x^* \in \operatorname{cl}_K A(x^*) \text{ and } f(x^*, y) \notin -\operatorname{int}_Y C(x^*) \text{ for all } y \in A(x^*).$$

PROOF: We define the multifunction $P : K \to 2^K$ by

$$P(x) = \big\{y \in D : f(x,y) \in -\operatorname{int}_Y C(x)\big\} \text{ for all } x \in K.$$

Then by using the same argument which we have used in proving Theorem 1, we have $x \notin \operatorname{coP}(x)$ for each $x \in K$ and $P^{-1}(y)$ is open for each $y \in D$. Thus all the conditions of Theorem B are satisfied. Hence there exists $x^* \in K$ such that

$$x^* \in \operatorname{cl}_K A(x^*) \text{ and } A(x^*) \cap P(x^*) = \phi.$$

Which implies that there exists $x^* \in K$ such that

$$x^* \in \operatorname{cl}_K A(x^*) \text{ and } f(x^*, y) \notin -\operatorname{int}_Y C(x^*)$$

for all $y \in A(x^*)$. ∎

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Post Graduate Department of Mathematics
University of Jammu
Jammu and Kashmir - 180 006
India

https://doi.org/10.1017/S0004972700037679 Published online by Cambridge University Press