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ISOMETRIC RESULTS ON A MEASURE OF NON-COMPACTNESS FOR OPERATORS ON BANACH SPACES

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For each $\lambda \geq 1$ a class of Banach spaces ϕ_λ is defined. Isometric results are obtained on the equivalence between a measure of non-compactness and the essential norm of a linear operator defined on a ϕ_λ space. Best values of λ for the classical Banach spaces and for spaces with unconditional basis are investigated. For the space c of convergent sequences the non-existence of a λ-unconditional basis with $\lambda < 2$ is deduced.

Recall that a Banach space E is said to be a π_λ space $(1 \leq \lambda < \infty)$ if for every finite dimensional subspace G of E and for each $\epsilon > 0$ there exist a finite-dimensional subspace $H \supseteq G$ and a projection P from E onto H with $||P|| \leq \lambda + \varepsilon$ (see [6]). We need the following dual notion.

DEFINITION 1. Suppose that $1 \leq \lambda < \infty$. A Banach space E will be said to be a ϕ_λ space if for every closed subspace M of finite codimension in E and for each $\varepsilon > 0$ there exist a closed subspace

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$N \subseteq M$ of finite codimension and a projection P from E onto N with $||P|| \leq \lambda + \varepsilon$.

The following proposition is essentially known (see [4]), but we indicate a short proof.

PROPOSITION 2. *(a) If* E^* *is a* π_λ *space then* E *is a* $\phi_{1+\lambda}$ *space.*

(b) If E *is a* ϕ_λ *space then* E^* *is a* $\pi_{1+\lambda}$ *space.*

Proof. (a) Let M be a closed subspace of finite codimension in E and let $\varepsilon > 0$ be given. By a consequence of the principle of local reflexivity (see [6]) there exists a finite dimensional subspace H containing $M\perp = \{f \in E^* : f(x) = 0$ for all $x \in M\}$ and a weak*-continuous projection P from E^* onto H with $||P|| \leq \lambda + \varepsilon$. Then $(I - P^*)|_E$ is a projection whose range is a subspace of finite codimension contained in M and $||I - P^*|| \leq 1 + \lambda + \varepsilon$. So E is a $\phi_{1+\lambda}$ space.

(b) This is a simple duality argument and will be omitted.

Let $T : E \to F$ be a bounded operator between Banach spaces E and F . The essential norm of T , denoted $||T||_e$, is defined by $||T||_e = \inf\{||T + K|| : K : E \to F$ is a compact operator$\}$. Following [7] we define a measure of non-compactness of T , denoted $c(T)$, by $c(T) = \inf\{||T|_M|| : \text{codim } (M) < \infty\}$. The familiar Kuratowski measure of non-compactness, $\gamma(T)$, which is defined by $\gamma(T) = \inf\{r$: the image of the unit ball of E is covered by finitely many balls in F of radius $r\}$, is related to $c(T)$ by the inequalities $\frac{1}{2}c(T) \leq \gamma(T) \leq 2c(T)$ (see [7]) .

PROPOSITION 3. *Suppose that* E *is a* ϕ_λ *space and that* $T : E \to F$ *is a bounded operator. Then* $||T||_e \leq \lambda c(T)$ *; in particular,* $||T||_e \leq \lambda ||T^*||_e$.

Proof. Suppose that $K : E \to F$ is any compact operator and let $\varepsilon > 0$ be given. Then $c(K) = 0$ and so there exists a closed subspace L of finite codimension such that $||K|_L|| < \varepsilon$. Let M be any closed subspace of finite codimension. Since E is a ϕ_λ space there exists a closed subspace $N \subseteq L \cap M$ of finite codimension in E and a

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projection P from E onto N with $||P|| \le \lambda + \varepsilon$. Then $(T + K)P$ is a compact perturbation of T and $||(T + K)P|| \le (\lambda + \varepsilon)(||T|_M|| + \varepsilon)$. Since ε and M are arbitrary it follows that $||T||_e \le \lambda c(T)$. We obviously have $c(T^{**}) \le ||T^{**}||_e \le ||T^*||_e$, while $c(T^{**}) \ge c(T)$ follows easily from the definition of the measure of non-compactness $c(\cdot)$. Combining these inequalities gives $||T||_e \le \lambda||T^*||_e$.

Remark 4. It is not known whether there exists a constant K such that $||T||_e \le K\,||T^*||_e$ for all Banach spaces E and F and operators $T : E \to F$. The unpublished folklore result $a_n(T) \le 3a_n(T^*)$, where $a_n(T)$ denotes the n^{th} approximation number of T, shows that $||T||_e \le 3||T^*||_e$ provided E^* has the approximation property, and so the second statement in Proposition 3 is good only for $\lambda < 3$.

COROLLARY 5. *Suppose that* E *is a classical Banach space and that* $T : E \to F$ *is a bounded operator. Then* $||T||_e \le 2c(T)$ *(and so* $||T||_e \le 2||T^*||_e$*)*.

Proof. E^* is a π_1-space, and so the result follows from Propositions 2 and 3.

Remark 6. The constant 2 is best possible (see Corollary 10(a) below). Results related to Proposition 3 are obtained in [7] under the assumption that F has the compact approximation property but without any assumption on E. In [1] the Banach spaces for which $\gamma(T)$ and $||T||_e$ are equivalent semi-norms are characterized.

Now suppose that E is a Banach space with a Schauder basis $(e_k)_{k=1}^{\infty}$. The basis constant μ is defined by $\mu = \sup\{||P_n|| : n \ge 1\}$, where P_n is the natural projection from E onto $[e_k]_{k=1}^{n}$ (the closed linear span of $e_1, e_2, \ldots, e_n$). The basis is said to be shrinking if $||f|_{[e_k]_{k=n}^{\infty}}|| \to 0$ as $n \to \infty$ for every $f \in E^*$. Further, the basis is said to be λ-unconditional if $||\sum_1^n \pm a_k e_k|| \le \lambda||\sum_1^{\infty} a_k e_k||$ for all $n \ge 1$, for all scalars $(a_k)_{k=1}^{\infty}$, and for all choices of signs. It

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follows from Proposition 2 that if E has a shrinking basis, with basis constant μ , then E is a $\phi_{1+\mu}$ space. We have the following refinement for spaces with a λ-unconditional shrinking basis.

PROPOSITION 7. *Suppose that E has a λ-unconditional shrinking basis. Then E is a ϕ_λ space.*

Proof. For each $x = \sum_1^\infty a_k e_k$ in E we define $|||x||| = \sup||\sum_1^\infty \pm a_k e_k||$, where the supremum is taken over all choices of signs. Since $||x|| \le |||x||| \le \lambda||x||$ it is sufficient to prove the proposition for the norm $|||\cdot|||$, for which $(e_k)_{k=1}^\infty$ is a 1-unconditional basis; so we may assume that $\lambda = 1$. Suppose that $\varepsilon > 0$ and that M is any closed subspace of codimension one in E . We prove the claim that there exists a subspace $N \subset M$ of finite codimension in E which is $(1+\varepsilon)$-complemented in E and which possesses a $(1+\varepsilon)$-unconditional shrinking basis. There exists $f \in E^*$ such that $||f|| = 1$ and $M = \{x \in E : f(x) = 0\}$. We may choose $x \in E$ and a positive integer n_o such that $f(x) = 1$, $||x|| \le 2$, and $x = \sum_{k=1}^{n_o} x_k e_k$. Given $\eta > 0$ there exists $n_1 > n_o$ such that $||f|_{[e_k]_{k=n_1}^\infty}|| \le \eta$. Then for any $m \ge n_1$ and for all scalars $a_{n_1}, \ldots, a_m$. we have

$$(1-2\eta)|||\sum_{n_1}^m a_k e_k||| \le |||\sum_{n_1}^m a_k f_k||| \le (1+2\eta)|||\sum_{n_1}^m a_k e_k||| ,$$

where $f_k = e_k - f(e_k)x$. Let P be the natural projection from E onto $[e_k]_{k=n_1}^\infty$ (which is a contraction because the basis is 1-unconditional); then P is an isomorphism from $[f_k]_{k=n_1}^\infty$ onto $[e_k]_{k=n_1}^\infty$ with $||P||\ ||P^{-1}|| \le \frac{1+2\eta}{1-2\eta}$. Moreover,

$$Q = \left(P|_{[f_k]_{k=n_1}^\infty}\right)^{-1} \circ P$$

is a projection from E onto $[f_k]_{k=n_1}^\infty$ with $||Q|| \le \frac{1+2\eta}{1-2\eta}$. The claim

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now follows by taking η sufficiently small. The general result for a closed subspace M of arbitrary finite codimension is obatined by applying the claim finitely many times and by considering a subspace of codimension one at each stage of the argument.

Remark 8. Say that a Schauder basis $(e_k)_{k=1}^{\infty}$ is λ-bimonotone if $\sup\{||P_n||, ||I-P_n|| : n \geq 1\} \leq \lambda$. Then the proof of Proposition 7 shows that E is a ϕ_λ space if $(e_k)_{k=1}^{\infty}$ is a λ-bimonotone shrinking basis of E.

Let c denote the space of convergent sequences $x = (x_k)_{k=1}^{\infty}$ with the norm $||x|| = \sup|x_k|$, and let c_o be the subspace of sequences which tend to zero; let ℓp $(1 \leq p < \infty)$ denote the space of sequences for which

$$||x||_p = \left(\sum_1^{\infty}|x_k|^p\right)^{1/p} < \infty$$

COROLLARY 9. *The Banach spaces* c_o *and* ℓp $(1 < p < \infty)$ *are* ϕ_1 *spaces.*

COROLLARY 10. *(a)* c *and* ℓ_1 *are* ϕ_2 *spaces but are not* ϕ_λ *spaces for any* $\lambda < 2$.

(b) Let $(e_k)_{k=1}^{\infty}$ *be a* λ*-unconditional basis for* c. *Then* $\lambda \geq 2$; *in particular, the Banach-Mazur distance from* c *to any space with a 1-unconditional basis is at least 2.*

Proof. (a) Let $I : \ell_1 \to c_o$ be the formal identity operator and let $j : c_o \to c$ be the natural inclusion. Let $(e_k)_{k=1}^{\infty}$ be the standard basis of ℓ_1 and define $K : \ell_1 \to c$ by $K(e_k) = u (k \geq 1)$, where u is the sequence which has every term equal to one. Then $||jI - \frac{1}{2}K|| = \frac{1}{2}$, and so $c(I) \leq \frac{1}{2}$; it now follows from Proposition 3 that ℓ_1 is not a ϕ_λ space for any $\lambda < 2$. Let M be any subspace of finite codimension contained in c_o and let P be a projection on c whose range is M.

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Then $P(I - \frac{1}{2}K)$ is a compact perturbation of I, and so $||P(jI - \frac{1}{2}K)|| \geq 1$. It follows that $||P|| \geq 2$, and so c is not a ϕ_λ space for any $\lambda < 2$. The fact that c and ℓ_1 are ϕ_2 spaces is a consequence of Proposition 2.

(b) Any unconditional basis of c_0 (and hence of c) is equivalent to the standard basis (see for example [8, p.71]), and so must be shrinking. The result now follows from (a) and Proposition 7.

Remark 11. C.V. Hutton ([5]) discussed the formal identity from ℓ_1 to c_0 as an example of an operator T with the property that $a_n(T) \neq a_n(T^*)$.

Remark 12. Banach ([2, p.242]) asked whether c and c_0 were almost isometric. Cambern proved in [3] that the Banach-Mazur distance from c to c_0 is 3.

It is very easy to prove, in fact, that it is not possible to imbed an infinite-dimensional $C(K)$ space almost isometrically into c_0, as the following lemma shows.

LEMMA 13. *Let K be an infinite compact Hausdorff space and let $T : C(K) \to c_0$ be a Banach isomorphism onto a subspace of c_0. Then $||T||\ ||T^{-1}|| \geq 2$.*

Proof. We may assume that $||T|| = 1$; let $(e_k^*)_{k=1}^{\infty}$ denote the functionals biorthogonal to the standard basis of c_0. Given $\varepsilon > 0$ there exists n_0 such that $|e_k^*T(1)| < \varepsilon$ for all $k > n_0$, where $1 \in C(K)$ is the constant one function. Select $y \in C(K)$ such that $||y|| = 1$ and $e_k^*(Ty) = 0$ for $1 \leq k \leq n_0$. Then $\max(||x + y||, ||x - y||) = 2$, whereas $\max(||Tx + Ty||, ||Tx - Ty||) \leq 1 + \varepsilon$, and it follows that $||T^{-1}|| \geq 2$.

Remark 14. Say that E has the distortion property (see [9]) if, given $\varepsilon > 0$, a Banach space F will contain a $(1 + \varepsilon)$-isomorphic copy of E whenever E and F are isomorphic. It is well known that c_0

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and ℓ_1 share this property, but since c and c_0 are isomorphic the previous lemma shows (taking $C(K) = c$) that c coes not.

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