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Geometric Waldspurger periods

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Abstract

Let X be a smooth projective curve. We consider the dual reductive pair $H = \mathbb{G}\mathbb{O}_{2m}$, $G = \mathbb{G}\mathbb{S}\mathrm{p}_{2n}$ over X, where H splits on an étale two-sheeted covering $\pi : \tilde{X} \to X$. Let Bun_G (respectively, Bun_H) be the stack of G-torsors (respectively, H-torsors) on X. We study the functors F_G and F_H between the derived categories $\mathrm{D}(\mathrm{Bun}_G)$ and $\mathrm{D}(\mathrm{Bun}_H)$, which are analogs of the classical theta-lifting operators in the framework of the geometric Langlands program. Assume $n = m = 1$ and H nonsplit, that is, $H = \pi_*\mathbb{G}_m$ with $\tilde{X}$ connected. We establish the geometric Langlands functoriality for this pair. Namely, we show that $F_G : \mathrm{D}(\mathrm{Bun}_H) \to \mathrm{D}(\mathrm{Bun}_G)$ commutes with Hecke operators with respect to the corresponding map of Langlands L-groups ${}^LH \to {}^LG$. As an application, we calculate Waldspurger periods of cuspidal automorphic sheaves on $\mathrm{Bun}_{\mathrm{GL}_2}$ and Bessel periods of theta-lifts from $\mathrm{Bun}_{\mathbb{G}\mathbb{O}_4}$ to $\mathrm{Bun}_{\mathbb{G}\mathbb{S}\mathrm{p}_4}$. Based on these calculations, we give three conjectural constructions of certain automorphic sheaves on $\mathrm{Bun}_{\mathbb{G}\mathbb{S}\mathrm{p}_4}$ (one of them makes sense for $\mathcal{D}$-modules only).

1. Introduction and main results

1.0 This paper, which is a sequel to [Lys06a], is a part of two (related) research projects: (i) a geometric version of the Howe correspondence (an analog of the theta-lifting in the framework of the geometric Langlands program); (ii) a geometric Langlands program for $\mathbb{G}\mathbb{S}\mathrm{p}_4$.

We consider only the (unramified) dual reductive pair $(H = \mathbb{G}\mathbb{O}_{2m}, G = \mathbb{G}\mathbb{S}\mathrm{p}_{2n})$ over a smooth projective connected curve X. Let Bun_G (respectively, Bun_H) denote the stack of G-torsors (respectively, H-torsors) on X. Using the theta-sheaf introduced in [Lys06a], we define functors $F_G : \mathrm{D}(\mathrm{Bun}_H) \to \mathrm{D}(\mathrm{Bun}_G)$ and $F_H : \mathrm{D}(\mathrm{Bun}_G) \to \mathrm{D}(\mathrm{Bun}_H)$ between the corresponding derived categories, which are geometric analogs of the theta-lifting operators. Based on classical Howe correspondence (cf., for example, [Ada89, Kud96, MVW87, Ral82]) and our results from [Lys07], we conjecture a precise relation between the theta-lifting functors and Hecke functors on Bun_G and Bun_H (cf. Conjecture 1). For $n = m$ (respectively, for $m = n + 1$) the functor F_G (respectively, F_H) is expected to realize the geometric Langlands functoriality for a morphism of Langlands L-groups $H^L \to G^L$ (respectively, $G^L \to H^L$).

We prove this conjecture for the dual pair $(\mathbb{G}\mathbb{O}_2, \mathrm{GL}_2)$, where $\mathbb{G}\mathbb{O}_2 = \pi_*\mathbb{G}_m$ is a group scheme over X, here $\pi : \tilde{X} \to X$ is a nontrivial étale two-sheeted covering. If $\tilde{E}$ is a rank-one local system on $\tilde{X}$, then this provides a new proof of the geometric Langlands conjecture for $\pi_*\tilde{E}$ independent of the existing proof due to Frenkel, Gaitsgory and Vilonen [FGV02, Gai04].

Let us describe our remaining main results in a form that is less technical than their actual formulation. Assume that the ground field $k = \mathbb{F}_q$ is finite of q elements with q odd. Set $G = \mathrm{GL}_2$. Let E be a rank-two irreducible ℓ-adic local system on X. Write Aut_E for the corresponding automorphic

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sheaf on Bun_G (cf. Definition 8). Let $f_E : \mathrm{Bun}_G(k) \to \bar{\mathbb{Q}}_\ell$ denote the function 'trace of Frobenius' of Aut_E.

Let $\phi : Y \to X$ be a nontrivial étale two-sheeted covering. Write $\operatorname{Pic} Y$ for the Picard stack of Y. Let $\mathcal{J}$ be a rank-one local system on Y equipped with an isomorphism $N(\mathcal{J}) \widetilde{\to} \det E$, where $N(\mathcal{J})$ is the norm of $\mathcal{J}$ (cf. Appendix A.1). Write $f_{\mathcal{J}} : (\operatorname{Pic} Y)(k) \to \bar{\mathbb{Q}}_\ell$ for the corresponding character (the trace of Frobenius of the automorphic local system $A\mathcal{J}$ corresponding to $\mathcal{J}$). The Waldspurger period of f_E is

$$\int_{\mathcal{B} \in (\operatorname{Pic} Y)(k)/(\operatorname{Pic} X)(k)} f_E(\phi_* \mathcal{B}) f_{\mathcal{J}}^{-1}(\mathcal{B})\, d\mathcal{B}$$

(the function that we integrate does not change when $\mathcal{B}$ is tensored by $\phi^* L$, $L \in \operatorname{Pic} X$), here $d\mathcal{B}$ is a Haar measure. A beautiful theorem of Waldspurger says that *the square* of this period is equal (up to an explicit harmless coefficient) to the value of the L-function $L(\phi^* E \otimes \mathcal{J}^{-1}, \frac{1}{2})$ (cf. [Wal85]).

We prove a geometric version of this result. The role of the L-function in geometric setting is played by the complex

$$\bigoplus_{d \geqslant 0} \mathrm{R}\Gamma(Y^{(d)}, (\phi^* E \otimes \mathcal{J}^*)^{(d)})[d]. \tag{1}$$

Here $Y^{(d)}$ is the dth symmetric power of Y and $V^{(d)}$ denotes the dth symmetric power of a local system V on X. The geometric Waldspurger period is

$$\mathrm{R}\Gamma_c(\operatorname{Pic} Y / \operatorname{Pic} X, \phi_1^* \mathrm{Aut}_E \otimes A\mathcal{J}^{-1}), \tag{2}$$

where $\phi_1 : \operatorname{Pic} Y \to \mathrm{Bun}_G$ sends $\mathcal{B}$ to $\phi_* \mathcal{B}$. The sense of the quotient $\operatorname{Pic} Y / \operatorname{Pic} X$ is made precise in § 6.3.3; this stack has two connected components (the degree of $\mathcal{B}$ modulo two), so (2) is naturally $\mathbb{Z}/2\mathbb{Z}$-graded. Our Theorem 5 says that there is a $\mathbb{Z}/2\mathbb{Z}$-graded isomorphism between (1) and the tensor square of (2), the $\mathbb{Z}/2\mathbb{Z}$-grading of (1) is given by the parity of d. If $\phi^* E$ is still irreducible, then (1) is the exterior algebra of the vector space $\mathrm{H}^1(Y, \mathcal{J}^* \otimes \phi^* E)$, which is placed in cohomological degree zero.

In the classical theory of automorphic forms there has been a philosophy that for multiplicity one models of representations the corresponding periods of Hecke eigenforms can be expressed in terms of the L-functions (of the corresponding eigenvalue local system). In addition to the Waldspurger periods, we also consider Bessel periods for $\mathrm{G}\mathbb{S}\mathrm{p}_4$ (see § 6) and generalized Waldspurger periods for GL_4 (see § 7), which all illustrate this phenomenon.

Consider now the dual pair $(G, \tilde{H})$, where $G = \mathrm{G}\mathbb{S}\mathrm{p}_4$ and $\tilde{H}$ is as follows. Let $\mathrm{G}\mathbb{O}_4^0$ be given by the exact sequence $1 \to \mathbb{G}_m \to \mathrm{GL}_2 \times \mathrm{GL}_2 \to \mathrm{G}\mathbb{O}_4^0 \to 1$, where the first map sends $x \in \mathbb{G}_m$ to (x, x^{-1}). Let $\pi : \tilde{X} \to X$ be an étale degree two covering, set $\Sigma = \mathrm{Aut}_X(\tilde{X})$. The group Σ act on this exact sequence permuting the two copies of GL_2. Let $\tilde{H}$ be the group scheme on X, the twisting of $\mathrm{G}\mathbb{O}_4^0$ by the Σ-torsor $\pi : \tilde{X} \to X$. The above exact sequence yields a morphism of stacks $\rho : \mathrm{Bun}_{2,\tilde{X}} \to \mathrm{Bun}_{\tilde{H}}$, here $\mathrm{Bun}_{2,\tilde{X}}$ denotes the stack of rank-two vector bundles on $\tilde{X}$. Write Bun_r for the stack of rank-r vector bundles on X.

Let $\tilde{E}$ be an irreducible rank-two local system on $\tilde{X}$ and let $\mathrm{Aut}_{\tilde{E}}$ be the corresponding automorphic sheaf on $\mathrm{Bun}_{2,\tilde{X}}$ (cf. Definition 8). Assume given a rank-one local system χ on X equipped with an isomorphism $\det \tilde{E} \widetilde{\to} \pi^* \chi$. Then $\mathrm{Aut}_{\tilde{E}}$ descends naturally to a perverse sheaf $K_{\tilde{E},\chi,\tilde{H}}$ on $\mathrm{Bun}_{\tilde{H}}$. Now assume that $\tilde{X}$ is connected. For the theta-lifting functor $F_G : \mathrm{D}(\mathrm{Bun}_{\tilde{H}}) \to \mathrm{D}(\mathrm{Bun}_G)$ our Theorem 6 calculates the Bessel periods of $K := F_G(K_{\tilde{E},\chi,\tilde{H}})$.

At the level of functions the Bessel periods are defined, for example, in [BFF97]. In geometric setting, let $P \subset G$ be the Siegel parabolic, $\nu_P : \mathrm{Bun}_P \to \mathrm{Bun}_G$ be the natural map. The stack Bun_P of P-torsors on X classifies collections: $L \in \mathrm{Bun}_2$, $\mathcal{A} \in \mathrm{Bun}_1$ and an exact sequence $0 \to \mathrm{Sym}^2 L \to ? \to \mathcal{A} \to 0$ on X. Let $\mathcal{S}_P$ be the stack classifying $L \in \mathrm{Bun}_2$, $\mathcal{A} \in \mathrm{Bun}_1$ together

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with a map $\operatorname{Sym}^2 L \to \mathcal{A} \otimes \Omega$. Here Ω is the canonical line bundle on X. So, $\mathcal{S}_P$ and Bun_P are dual (generalized) vector bundles over $\operatorname{Bun}_2 \times \operatorname{Bun}_1$, and one has the Fourier transform functor $\operatorname{Four}_\psi : \mathrm{D}(\operatorname{Bun}_P) \to \mathrm{D}(\operatorname{Bun}_{\mathcal{S}_P})$ between the corresponding derived categories of ℓ-adic sheaves.

Let $\phi : Y \to X$ be a nontrivial étale two-sheeted covering. Let $e : \operatorname{Pic} Y \to \mathcal{S}_P$ be the map sending $\mathcal{B} \in \operatorname{Pic} Y$ to the pair $L = \phi_* \mathcal{B}$, $\mathcal{A} \otimes \Omega = N(\mathcal{B})$ with natural symmetric form $\operatorname{Sym}^2 L \to \mathcal{A} \otimes \Omega$ (cf. § 6.1.1). Let $\mathcal{J}$ be a rank-one local system on Y equipped with $N(\mathcal{J}) \widetilde{\to} \chi$. The complex

$$A\mathcal{J} \otimes e^* \operatorname{Four}_\psi(\nu_P^* K)$$

descends naturally with respect to the map $\operatorname{Pic} Y \to \operatorname{Pic} Y / \operatorname{Pic} X$. The Bessel period of K is the ($\mathbb{Z}/2\mathbb{Z}$-graded) complex

$$\mathrm{R}\Gamma_c(\operatorname{Pic} Y / \operatorname{Pic} X,\ A\mathcal{J} \otimes e^* \operatorname{Four}_\psi(\nu_P^* K)). \tag{3}$$

Our Theorem 6 says that (up to a shift) there is a $\mathbb{Z}/2\mathbb{Z}$-graded isomorphism between

$$\bigoplus_{d \geqslant 0} \mathrm{R}\Gamma(Y^{(d)}, (\mathcal{J} \otimes \phi^*(\pi_* \tilde{E}^*))^{(d)})[d] \tag{4}$$

and the tensor square of (3). The $\mathbb{Z}/2\mathbb{Z}$-grading on (4) is given by the parity of d.

From Conjecture 1 it would follow that K is an automorphic sheaf on Bun_G corresponding to the local system $E_{\check{G}}$ given by the pair $(\pi_* \tilde{E}^*, \chi^{-1})$ with symplectic form $\bigwedge^2 (\pi_* \tilde{E}^*) \to \chi^{-1}$ (cf. also Conjecture 2). Here $\check{G}$ stands for the Langlands dual group. As predicted by the general philosophy on multiplicity one models, the complex (4) makes sense for all $\check{G}$-local systems on X, this allows us to formulate a conjectural answer for the Bessel periods of all automorphic sheaves on Bun_G (cf. Conjectures 4 and 5).

The geometry suggests that one should be able to recover an automorphic sheaf on Bun_G from the knowledge of all of its Bessel periods (including those for ramified two-sheeted coverings $\phi : Y \to X$). To formulate the corresponding conjecture we switch from ℓ-adic sheaves to $\mathcal{D}$-modules (for § 8 only), as it requires the Fourier–Laumon transform, which is not known in ℓ-adic setting.

We also propose one more conjectural construction of automorphic sheaves on Bun_G as theta-lifting from $\mathrm{G}\mathbb{O}_6$ (cf. Conjecture 6).

1.1 General notation

Let k denote an algebraically closed field of characteristic $p > 2$, all of the schemes (or stacks) we consider are defined over k. Fix a prime $\ell \neq p$. For a scheme (or stack) S write $\mathrm{D}(S)$ for the derived category of ℓ-adic étale sheaves on S and $\mathrm{P}(S) \subset \mathrm{D}(S)$ for the category of perverse sheaves.

Fix a nontrivial character $\psi : \mathbb{F}_p \to \bar{\mathbb{Q}}_\ell^*$ and denote by $\mathcal{L}_\psi$ the corresponding Artin–Shreier sheaf on $\mathbb{A}^1$. Since we are working over an algebraically closed field, we systematically ignore Tate twists.

If $V \to S$ and $V^* \to S$ are dual rank-n vector bundles over a stack S, we normalize the Fourier transform $\operatorname{Four}_\psi : \mathrm{D}(V) \to \mathrm{D}(V^*)$ by $\operatorname{Four}_\psi(K) = (p_{V^*})_!(\xi^* \mathcal{L}_\psi \otimes p_V^* K)[n]$, where p_V, p_{V^*} are the projections and $\xi : V \times_S V^* \to \mathbb{A}^1$ is the pairing.

Let X be a smooth projective connected curve. Write Ω for the canonical line bundle on X. For a smooth scheme of finite type S and a locally free $\mathcal{O}_S$-module $\mathcal{L}$ write $\mathcal{L}^\star = \mathcal{L}^* \otimes \Omega_S$, where Ω_S is the canonical line bundle on S. For a morphism of stacks $f : Y \to Z$ we denote by $\operatorname{dim.rel}(f)$ the function of a connected component C of Y given by $\dim C - \dim C'$, where C' is the connected component of Z containing $f(C)$.

Write Bun_k for the stack of rank-k vector bundles on X. For $k = 1$ we also write $\operatorname{Pic} X$ for the Picard stack Bun_1 of X. We have a line bundle $\mathcal{A}_k$ on Bun_k with fibre $\det \mathrm{R}\Gamma(X, V)$ at $V \in \operatorname{Bun}_k$. View it as a $\mathbb{Z}/2\mathbb{Z}$-graded placed in degree $\chi(V) \bmod 2$. Our conventions about $\mathbb{Z}/2\mathbb{Z}$-grading are those of [Lys06a, § 3.1].

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1.2 Other results and ideas of proofs

1.2.1 *Theta-sheaf.* Let G_k denote the sheaf of automorphisms of $\mathcal{O}_X^k \oplus \Omega^k$ preserving the natural symplectic form $\bigwedge^2(\mathcal{O}_X^k \oplus \Omega^k) \to \Omega$. The stack Bun_{G_k} of G_k-bundles on X classifies $M \in \mathrm{Bun}_{2k}$ equipped with a symplectic form $\bigwedge^2 M \to \Omega$. We have a μ_2-gerbe $\widetilde{\mathrm{Bun}}_{G_k} \to \mathrm{Bun}_{G_k}$, where $\widetilde{\mathrm{Bun}}_{G_k}$ is the stack of metaplectic bundles on X. In [Lys06a] we have introduced the theta-sheaf $\mathrm{Aut} = \mathrm{Aut}_g \oplus \mathrm{Aut}_s$ on $\widetilde{\mathrm{Bun}}_{G_k}$ (cf. § 2.1 for precise definitions). We refer to Aut_g (respectively, to Aut_s) as the generic (respectively, special) part of Aut. We write ${}_X\mathrm{Aut} = \mathrm{Aut}$ when we need to express the dependence on X.

Let $P_k \subset G_k$ be the Siegel parabolic preserving the Lagrangian subsheaf $\mathcal{O}_X^k \subset \mathcal{O}_X^k \oplus \Omega^k$. Write $\nu_k : \mathrm{Bun}_{P_k} \to \mathrm{Bun}_{G_k}$ for the projection, where Bun_{P_k} is the stack of P_k-bundles on X. We extend ν_k to a map $\tilde{\nu}_k : \mathrm{Bun}_{P_k} \to \widetilde{\mathrm{Bun}}_{G_k}$ (cf. § 2.1).

The stack Bun_{P_k} classifies $L \in \mathrm{Bun}_k$ together with an exact sequence $0 \to \mathrm{Sym}^2 L \to ? \to \Omega \to 0$ of $\mathcal{O}_X$-modules. Let ${}^0\mathrm{Bun}_{P_k} \subset \mathrm{Bun}_{P_k}$ be the open substack given by $\mathrm{H}^0(X, \mathrm{Sym}^2 L) = 0$.

In [Lys06a, Definition 3] we have introduced the complex $S_{P,\psi}$ on Bun_{P_k} by some explicit construction (cf. § 2.1 for details). It was shown in [Lys06a, Proposition 7] that there is an isomorphism over ${}^0\mathrm{Bun}_{P_k}$

$$\mathfrak{r}_k : S_{P,\psi} \widetilde{\to} \tilde{\nu}_k^* \mathrm{Aut}[\dim.\mathrm{rel}(\nu_k)].$$

We show that $\mathfrak{r}_k$ extends naturally to an isomorphism over Bun_{P_k} (cf. Proposition 1).

Let $\pi : \tilde{X} \to X$ be an étale degree-two covering, $\Sigma = \mathrm{Aut}_X(\tilde{X}) = \{1, \sigma\}$ the automorphisms group of $\tilde{X}$ over X. Let $\mathcal{E}$ be the Σ-anti-invariants in $\pi_*\mathcal{O}$, so $\mathcal{E}$ is equipped with a trivialization $\kappa : \mathcal{E}^2 \widetilde{\to} \mathcal{O}$. Let $\mathcal{E}_0$ be the Σ-anti-invariants in $\pi_*\bar{\mathbb{Q}}_\ell$, it is equipped with $\mathcal{E}_0^2 \widetilde{\to} \bar{\mathbb{Q}}_\ell$. Let g denote the genus of X.

Write $\mathrm{Bun}_{G_n,\tilde{X}}$ for the stack classifying rank-$2n$ vector bundles W on $\tilde{X}$ with symplectic form $\bigwedge^2 W \to \Omega_{\tilde{X}}$. Let $\pi_n : \mathrm{Bun}_{G_n,\tilde{X}} \to \mathrm{Bun}_{G_{2n}}$ be the map sending the above point to π_*W equipped with natural symplectic form $\bigwedge^2(\pi_*W) \to \Omega$. Let $\widetilde{\mathrm{Bun}}_{G_n,\tilde{X}}$ denote the corresponding stack of metaplectic bundles. The map π_n extends to a map $\tilde{\pi}_n : \widetilde{\mathrm{Bun}}_{G_n,\tilde{X}} \to \widetilde{\mathrm{Bun}}_{G_{2n}}$ (cf. § 3.5). We establish a canonical isomorphism

$${}_{\tilde{X}}\mathrm{Aut} \widetilde{\to} \tilde{\pi}_n^* \mathrm{Aut}[\dim.\mathrm{rel}(\tilde{\pi}_n)]$$

preserving the generic and special parts (cf. Proposition 3).

1.2.2 *Theta-lifting functors.* Let $n, m \in \mathbb{N}$ and $G = \mathrm{G}\mathbb{S}\mathrm{p}_{2n}$. Let $\mathbb{H} = \mathrm{G}\mathbb{O}_{2m}^0$ denote the connected component of unity of the split orthogonal similitude group $\mathrm{G}\mathbb{O}_{2m}$ over $\mathrm{Spec}\, k$. Pick a maximal torus and a Borel subgroup $\mathbb{T}_\mathbb{H} \subset \mathbb{B}_\mathbb{H} \subset \mathbb{H}$. We pick an involution $\tilde{\sigma} \in \mathbb{O}_{2m}(k)$ preserving $\mathbb{T}_\mathbb{H}$ and $\mathbb{B}_\mathbb{H}$ such that $\tilde{\sigma} \notin \mathrm{S}\mathbb{O}_{2m}$. So, for $m \geqslant 2$ (and $m \neq 4$) it induces the unique nontrivial automorphism of the Dynkin diagram of $\mathbb{H}$. Consider the corresponding Σ-action on $\mathrm{G}\mathbb{O}_{2m}^0$ by conjugation. Let $\tilde{H}$ be the group scheme on X, the twisting of $\mathrm{G}\mathbb{O}_{2m}^0$ by the Σ-torsor $\pi : \tilde{X} \to X$.

The stack Bun_G of G-torsors on X classifies $M \in \mathrm{Bun}_{2n}, \mathcal{A} \in \mathrm{Bun}_1$ with symplectic form $\bigwedge^2 M \to \mathcal{A}$. The stack $\mathrm{Bun}_{\tilde{H}}$ of $\tilde{H}$-torsors on X classifies $V \in \mathrm{Bun}_{2m}, \mathcal{C} \in \mathrm{Bun}_1$, a nondegenerate symmetric form $\mathrm{Sym}^2 V \to \mathcal{C}$, and a compatible trivialization $\gamma : \mathcal{C}^{-m} \otimes \det V \widetilde{\to} \mathcal{E}$. This means that the composition

$$\mathcal{C}^{-2m} \otimes (\det V)^2 \xrightarrow{\gamma^2} \mathcal{E}^2 \widetilde{\to} \mathcal{O}$$

is the isomorphism induced by $V \widetilde{\to} V^* \otimes \mathcal{C}$.

Let RCov^0 denote the stack classifying a line bundle $\mathcal{U}$ on X together with a trivialization $\mathcal{U}^{\otimes 2} \widetilde{\to} \mathcal{O}$. Its connected components are indexed by $\mathrm{H}^1_{et}(X, \mathbb{Z}/2\mathbb{Z})$.

Let Bun_H be the stack classifying $V \in \mathrm{Bun}_{2m}, \mathcal{C} \in \mathrm{Bun}_1$ and a symmetric form $\mathrm{Sym}^2 V \to \mathcal{C}$ such that the corresponding trivialization $(\mathcal{C}^{-m} \otimes \det V)^2 \widetilde{\to} \mathcal{O}$ lies in the component of RCov^0 given by $(\mathcal{E}, \kappa)$. Note that

$$\mathrm{Bun}_{\tilde{H}} \widetilde{\to} \operatorname{Spec} k \times_{\mathrm{RCov}^0} \mathrm{Bun}_H,$$

where the map $\operatorname{Spec} k \to \mathrm{RCov}^0$ is given by $(\mathcal{E}, \kappa)$. The projection $\rho_H : \mathrm{Bun}_{\tilde{H}} \to \mathrm{Bun}_H$ is a μ_2-torsor.

Write $\mathrm{Bun}_H^d \subset \mathrm{Bun}_H$ for the open substack given by $\deg \mathcal{C} = d$, and similarly for $\mathrm{Bun}_{\tilde{H}}^d$. Set

$$\mathrm{Bun}_{G,H} = \mathrm{Bun}_H \times_{\operatorname{Pic} X} \mathrm{Bun}_G,$$

where the map $\mathrm{Bun}_H \to \operatorname{Pic} X$ sends $(V, \mathcal{C}, \mathrm{Sym}^2 V \to \mathcal{C})$ to $\Omega \otimes \mathcal{C}^{-1}$. The map $\mathrm{Bun}_G \to \operatorname{Pic} X$ sends $(M, \bigwedge^2 M \to \mathcal{A})$ to $\mathcal{A}$. We have an isomorphism $\mathcal{C} \otimes \mathcal{A} \widetilde{\to} \Omega$ for a point of $\mathrm{Bun}_{G,H}$. Let

$$\tau : \mathrm{Bun}_{G,H} \to \mathrm{Bun}_{G_{2nm}}$$

be the map sending a point as above to $V \otimes M$ with symplectic form $\bigwedge^2 (V \otimes M) \to \Omega$. We extend τ to a map $\tilde{\tau} : \mathrm{Bun}_{G,H} \to \widetilde{\mathrm{Bun}}_{G_{2nm}}$ (cf. § 3.2.1). Let $\mathrm{Bun}_{G,\tilde{H}}$ be the stack obtained from $\mathrm{Bun}_{G,H}$ by the base change $\mathrm{Bun}_{\tilde{H}} \to \mathrm{Bun}_H$.

Viewing $\tilde{\tau}^* \mathrm{Aut}$ as a kernel of integral operators, we define functors $F_G : \mathrm{D}(\mathrm{Bun}_H) \to \mathrm{D}(\mathrm{Bun}_G)$ and $F_H : \mathrm{D}(\mathrm{Bun}_G) \to \mathrm{D}(\mathrm{Bun}_H)$, set also $F_{\tilde{H}} = \rho_H^* \circ F_H$ (cf. § 3.2.1).

1.2.3 *The pair* $\mathrm{G}\mathbb{O}_2$, GL_2. Assume $n = m = 1$ and $\tilde{X}$ connected. In this case we prove Conjecture 1 for the functor F_G. To do so, we first prove Theorem 1 saying how the action of Hecke operators on $\tilde{\tau}^* \mathrm{Aut}$ with respect to G is expressed in terms of the similar action with respect to $\tilde{H}$. This is a global geometric analog of a particular case of the theorem of Rallis [Ral82] (cf. also [Lys07]).

We also show that both $\tilde{\tau}^* \mathrm{Aut}_g[\dim.\mathrm{rel}(\tilde{\tau})]$ and $\tilde{\tau}^* \mathrm{Aut}_s[\dim.\mathrm{rel}(\tilde{\tau})]$ are self-dual irreducible perverse sheaves on each connected component of $\mathrm{Bun}_{G,\tilde{H}}$ (cf. Proposition 5), and the functor $F_G : \mathrm{D}(\mathrm{Bun}_{\tilde{H}}) \to \mathrm{D}(\mathrm{Bun}_G)$ commutes with the Verdier duality.

If $\tilde{E}$ is a rank-one local system on $\tilde{X}$, let $K_{\tilde{E}}$ denote the automorphic sheaf on $\mathrm{Bun}_{\tilde{H}} \widetilde{\to} \operatorname{Pic} \tilde{X}$ corresponding to $\tilde{E}$. Then $F_G(K_{\tilde{E}})$ is an automorphic sheaf on Bun_G corresponding to the local system $E = (\pi_* \tilde{E})^*$. We check that (up to a tensoring by a one-dimensional vector space) the sheaf $F_G(K_{\tilde{E}})$ coincides with the perverse sheaf Aut_E constructed via Whittaker models in [FGV02] (cf. Proposition 6).

Theorem 1 also allows us to calculate the following Rankin–Selberg-type convolution (we need it for our proof of Theorem 5). Let E be an irreducible rank-two local system on X, E_1 be a rank-one local system on X. We denote by $\mathrm{Aut}_{E_1 \oplus \bar{\mathbb{Q}}_\ell}$ the corresponding geometric Eisenstein series (cf. § 4.3). Our Theorem 2 provides an explicit calculation of $F_{\tilde{H}}(\mathrm{Aut}_{E_1 \oplus \bar{\mathbb{Q}}_\ell} \otimes \mathrm{Aut}_E)$. The method of its proof is inspired by [Lys02]. We do not know whether this Rankin–Selberg convolution was known before in classical theory of automorphic forms.

1.2.4 *Waldspurger periods.* Let us explain how we calculate the Waldspurger periods (Theorem 5). Mainly, we follow the approach of Waldspurger [Wal85], but there are some new phenomena in geometric settings.

Let $n = 2$, so $G = \mathrm{GL}_2$. Let E be an irreducible rank-two local system on X and let Aut_E be the corresponding automorphic sheaf on Bun_G. Take both $\pi : \tilde{X} \to X$ and $\tilde{H} = \mathrm{G}\mathbb{O}_4^0$ split. Recall the perverse sheaf $K_{\pi^* E, \det E, \tilde{H}}$ on $\mathrm{Bun}_{\tilde{H}}$ from § 1.0. First, we identify $K_{\pi^* E, \det E, \tilde{H}}$ with the theta-lift $F_{\tilde{H}}(\mathrm{Aut}_{E^*})$ from Bun_G (cf. Proposition 8). This is a geometric version of a Theorem of Shimizu (see [Wal85]).

Then we consider the diagram

$$\begin{array}{ccccc} \operatorname{Pic} Y & \xleftarrow{m} & \operatorname{Pic} Y \times \operatorname{Pic} Y & \xrightarrow{\phi_1 \times \phi_1} & \operatorname{Bun}_2 \times \operatorname{Bun}_2 \\ & \nwarrow^{\mathfrak{p}_{R_\phi}} & \downarrow & & \downarrow \\ & & \operatorname{Bun}_{R_\phi} & \xrightarrow{\mathfrak{q}_{R_\phi}} & \operatorname{Bun}_{\tilde{H}} \end{array}$$

where m is the tensor product map (followed by the automorphism sending $\mathcal{B} \in \operatorname{Pic} Y$ to $\mathcal{B}^* \otimes \Omega_Y$), and the vertical arrows are, roughly speaking, the quotients by the action of $\operatorname{Pic} X$, where $\mathcal{L} \in \operatorname{Pic} X$ sends $(L_1, L_2) \in \operatorname{Bun}_2 \times \operatorname{Bun}_2$ to $L_1 \otimes \mathcal{L}, L_2 \otimes \mathcal{L}^*$. Recall that ϕ_1 sends $\mathcal{B} \in \operatorname{Pic} Y$ to $\phi_* \mathcal{B} \in \operatorname{Bun}_2$.

The key step is Theorem 4 that calculates the complex $(\mathfrak{p}_{R_\phi})_! \mathfrak{q}^*_{R_\phi} F_{\tilde{H}}(\operatorname{Aut}_{E^*})$ explicitly in terms of E and $\phi : Y \to X$ (in our actual formulation of Theorem 4 the covering $\pi : \tilde{X} \to X$ may be nonsplit). We derive Theorem 4 from the properties of the theta-lifting between $\mathrm{G}\mathbb{O}_2$ and GL_2 (Proposition 6) combined with our Rankin–Selberg convolution result (Theorem 2).

Let us indicate at this point that the existence of the geometric Waldspurger periods of automorphic sheaves (the fact that condition (C_W) in Definition 10 holds) is a consequence of an intriguing acyclicity result (Theorem 3, § 6.1.2). It says that the Hecke property of a given automorphic sheaf $\mathcal{S}$ on Bun_2 already implies that $\mathcal{S}$ is universally locally acyclic (ULA) over 'the moduli of spectral curves'. This allows us to control perversity of the complexes $\mathcal{K}_E$ in Theorem 5 (and similarly for Bessel periods in Theorem 6).

We also formulate conjectural answers for the geometric Waldspurger periods (Conjecture 3) and the geometric Bessel periods (Conjecture 4) of all automorphic sheaves. In addition, we verify Conjecture 3 for geometric Eisenstein series on Bun_2 (cf. Proposition 10).

1.2.5 *Case* $H = \mathrm{G}\mathbb{O}_6$. Assume $m = 3$ and $\tilde{X}$ split, so $\tilde{H} = \mathrm{G}\mathbb{O}_6^0$. Let $n = 2$, so $G = \mathrm{G}\mathbb{S}\mathrm{p}_4$. Let $E_{\check{G}}$ be a $\check{G}$-local system on X viewed as a pair (E, χ), where E (respectively, χ) is a rank-four (respectively, rank-one) local system on X with symplectic form $\bigwedge^2 E \to \chi$. Assume that E is irreducible. Recall that $\check{G}$ is a subgroup of $\mathrm{G}\mathbb{S}\mathrm{pin}_6$, which is the Langlands dual to $\tilde{H}$. We define the perverse sheaf $K_{E,\chi,\tilde{H}}$ on $\operatorname{Bun}_{\tilde{H}}$ corresponding to a $\mathrm{G}\mathbb{S}\mathrm{pin}_6$-local system (E, χ). We conjecture that

$$F_G(K_{E^*,\chi^*,\tilde{H}}) \tag{5}$$

is an automorphic sheaf on Bun_G corresponding to $E_{\check{G}}$ (cf. Conjecture 6). We show that the geometric Bessel periods of (5) are essentially the generalized Waldspurger periods of $K_{E^*,\chi^*,\tilde{H}}$ (cf. Proposition 11).

2. Theta-sheaf

2.1 Let G_k denote the sheaf of automorphisms of $\mathcal{O}_X^k \oplus \Omega^k$ preserving the natural symplectic form $\bigwedge^2(\mathcal{O}_X^k \oplus \Omega^k) \to \Omega$. The stack Bun_{G_k} of G_k-bundles on X classifies $M \in \operatorname{Bun}_{2k}$ equipped with a symplectic form $\bigwedge^2 M \to \Omega$. Write $\mathcal{A}_{G_k}$ for the line bundle on Bun_{G_k} with fibre $\det \mathrm{R}\Gamma(X, M)$ at M. We view it as a $\mathbb{Z}/2\mathbb{Z}$-graded line bundle (purely of degree zero). Denote by $\widetilde{\operatorname{Bun}}_{G_k} \to \operatorname{Bun}_{G_k}$ the μ_2-gerbe of square roots of $\mathcal{A}_{G_k}$.

Recall the definition of the theta-sheaf Aut on $\widetilde{\operatorname{Bun}}_{G_k}$ from [Lys06a]. Let ${}_i\operatorname{Bun}_{G_k} \subset \operatorname{Bun}_{G_k}$ be the locally closed substack given by $\dim \mathrm{H}^0(X, M) = i$ for $M \in \operatorname{Bun}_{G_k}$. Let ${}_i\widetilde{\operatorname{Bun}}_{G_k}$ denote the preimage of ${}_i\operatorname{Bun}_{G_k}$ in $\widetilde{\operatorname{Bun}}_{G_k}$.

Write ${}_i\mathcal{B}$ for the line bundle on ${}_i\operatorname{Bun}_{G_k}$ whose fibre at $M \in {}_i\operatorname{Bun}_{G_k}$ is $\det \mathrm{H}^0(X, M)$. View ${}_i\mathcal{B}$ as $\mathbb{Z}/2\mathbb{Z}$-graded placed in degree $\dim \mathrm{H}^0(X, M)$ modulo 2. For each $i \geqslant 0$ we have a canonical

$\mathbb{Z}/2\mathbb{Z}$-graded isomorphism

$$_i\mathcal{B}^2 \widetilde{\to} \mathcal{A}|_{_i\widetilde{\mathrm{Bun}}_{G_k}}.$$

It yields a two-sheeted covering $_i\rho : {_i\mathrm{Bun}_{G_k}} \to {_i\widetilde{\mathrm{Bun}}_{G_k}}$ locally trivial in étale topology. Define a local system $_i\mathrm{Aut}$ on $_i\widetilde{\mathrm{Bun}}_{G_k}$ by

$$_i\mathrm{Aut} = \mathrm{Hom}_{S_2}(\mathrm{sign},\ {_i\rho_!}\bar{\mathbb{Q}}_\ell).$$

The perverse sheaf $\mathrm{Aut}_g \in \mathrm{P}(\widetilde{\mathrm{Bun}}_{G_k})$ (respectively, $\mathrm{Aut}_s \in \mathrm{P}(\widetilde{\mathrm{Bun}}_{G_k})$) is defined as the intermediate extension of $_0\mathrm{Aut}[\dim \mathrm{Bun}_G]$ (respectively, of $_1\mathrm{Aut}[\dim \mathrm{Bun}_G -1]$) under $_i\widetilde{\mathrm{Bun}}_{G_k} \hookrightarrow \widetilde{\mathrm{Bun}}_{G_k}$. The theta-sheaf Aut is defined by

$$\mathrm{Aut} = \mathrm{Aut}_g \oplus \mathrm{Aut}_s\,.$$

Let $P_k \subset G_k$ be the Siegel parabolic preserving the Lagrangian subsheaf $\mathcal{O}_X^k \subset \mathcal{O}_X^k \oplus \Omega^k$. Write Q_k for the Levi quotient of P_k, so $Q_k \widetilde{\to} \mathrm{GL}_k$ canonically.

Write $\nu_k : \mathrm{Bun}_{P_k} \to \mathrm{Bun}_{G_k}$ for the projection. As in [Lys06a, § 5.1], we extend it to a map $\tilde{\nu}_k : \mathrm{Bun}_{P_k} \to \widetilde{\mathrm{Bun}}_{G_k}$ defined as follows. The stack Bun_{P_k} classifies $L \in \mathrm{Bun}_k$ together with an exact sequence $0 \to \mathrm{Sym}^2 L \to ? \to \Omega \to 0$ of $\mathcal{O}_X$-modules. The induced exact sequence $0 \to L \to M \to L^* \otimes \Omega \to 0$ yields an isomorphism of $\mathbb{Z}/2\mathbb{Z}$-graded lines

$$\det \mathrm{R}\Gamma(X, M) \widetilde{\to} \det \mathrm{R}\Gamma(X, L) \otimes \det \mathrm{R}\Gamma(X, L^* \otimes \Omega) \widetilde{\to} \det \mathrm{R}\Gamma(X, L)^{\otimes 2}.$$

The map $\tilde{\nu}_k$ sends the above point to $(\mathcal{B}, M)$, where $\mathcal{B} = \det \mathrm{R}\Gamma(X, L)$ is equipped with the above isomorphism $\mathcal{B}^2 \widetilde{\to} \det \mathrm{R}\Gamma(X, M)$.

Recall the definition of the complex $S_{P,\psi}$ on Bun_{P_k} [Lys06a, Definition 3]. Denote by $\mathcal{V} \to \mathrm{Bun}_k$ the stack whose fibre over $L \in \mathrm{Bun}_k$ is $\mathrm{Hom}(L, \Omega)$. Write $\mathcal{V}_2 \to \mathrm{Bun}_k$ for the stack whose fibre over $L \in \mathrm{Bun}_k$ is $\mathrm{Hom}(\mathrm{Sym}^2 L, \Omega^2)$. We have a projection $\pi_2 : \mathcal{V} \to \mathcal{V}_2$ sending $s \in \mathrm{Hom}(L, \Omega)$ to $s \otimes s \in \mathrm{Hom}(\mathrm{Sym}^2 L, \Omega^2)$. We set

$$S_{P,\psi} \widetilde{\to} \mathrm{Four}_\psi(\pi_{2!}\bar{\mathbb{Q}}_\ell)[\dim.\mathrm{rel}],$$

where $\mathrm{Four}_\psi : \mathrm{D}(\mathcal{V}_2) \to \mathrm{D}(\mathrm{Bun}_P)$ denotes the Fourier transform functor, and dim. rel is the function of a connected component of Bun_k given by $\dim.\mathrm{rel}(L) = \dim \mathrm{Bun}_k - \chi(L)$, $L \in \mathrm{Bun}_k$.

The group S_2 acts on $\mathcal{V}$ sending $(L, s : L \to \Omega)$ to $(L, -s)$. This gives rise to a S_2-action on $S_{P,\psi}$. By [Lys06a, Remark 3], the S_2-invariants of $S_{P,\psi}$ are $S_{P,\psi,g}$ (respectively, $S_{P,\psi,s}$) over the connected component of Bun_{P_k} with $\chi(L)$ even (respectively, odd).

Let $^0\mathrm{Bun}_{P_k} \subset \mathrm{Bun}_{P_k}$ be the open substack given by $\mathrm{H}^0(X, \mathrm{Sym}^2 L) = 0$. By [Lys06a, Proposition 7] there is an isomorphism[1]

$$\mathfrak{r}_k : S_{P,\psi} \widetilde{\to} \tilde{\nu}_k^* \mathrm{Aut}[\dim.\mathrm{rel}(\nu_k)] \tag{6}$$

over $^0\mathrm{Bun}_{P_k}$; here $\dim.\mathrm{rel}(\nu_k) = \dim \mathrm{Bun}_{P_k} - \dim \mathrm{Bun}_{G_k}$ is a function of a connected component of Bun_{P_k}. From [Lys06a, § 2] it may be deduced that, in the case of a finite base field, the function 'trace of Frobenius' of $S_{P,\psi}$ descends with respect to $\tilde{\nu}_k : \mathrm{Bun}_{P_k} \to \widetilde{\mathrm{Bun}}_{G_k}$ over the whole of Bun_{P_k}. We claim that it is also true in the geometric setting.

Proposition 1. *The isomorphism* $\mathfrak{r}_k$ *extends naturally to an isomorphism over* Bun_{P_k}.

Proof. The proof is in several steps.

Step 1. For an effective divisor D on X denote by $_{D,P}\mathrm{Bun}_{G_k}$ the stack classifying $M \in \mathrm{Bun}_{G_k}$ together with a P_k-structure on $M|_D$. A point of $_{D,P}\mathrm{Bun}_{G_k}$ is given by $M \in \mathrm{Bun}_{G_k}$ together with

[1] The isomorphism $\mathfrak{r}_k$ is not canonical: once $\sqrt{-1} \in k$ is chosen, $\mathfrak{r}_k$ is well defined up to a sign.

a lagrangian $\mathcal{O}_D$-submodule $L_D \subset M|_D$. Denote by $p_D : \operatorname{Bun}_{P_k} \to {}_{D,P}\operatorname{Bun}_{G_k}$ the map sending $(L \subset M) \in \operatorname{Bun}_{P_k}$ to (M, L_D) with $L_D = L|_D$. Let $\nu_D : {}_{D,P}\operatorname{Bun}_{G_k} \to \operatorname{Bun}_{G_k}$ be the projection.

Pick a point $x \in X$ and a nonnegative integer i. Set $D = ix$. Let $a_D : \operatorname{Bun}_{P_k} \to \operatorname{Bun}_{P_k}$ be the map sending an exact sequence $0 \to \operatorname{Sym}^2 L \to ? \to \Omega \to 0$ to its push-forward with respect to the map $\operatorname{Sym}^2 L \hookrightarrow \operatorname{Sym}^2(L(D))$. Since $\operatorname{Hom}(\Omega, \operatorname{Sym}^2(L(D))/\operatorname{Sym}^2 L)$ acts freely and transitively on a fibre of a_D, the map a_D is an affine fibration of rank $k(k+1)i$.

We are going to establish a canonical isomorphism

$$(a_D)_! S_{P,\psi} \widetilde{\to} S_{P,\psi}[-k^2 i].$$

To do so, write $\pi_2^D : \mathcal{V}^D \to \mathcal{V}_2^D$ for the map obtained from $\pi_2 : \mathcal{V} \to \mathcal{V}_2$ by the base change $\operatorname{Bun}_k \to \operatorname{Bun}_k$ sending L to $L(D)$. So, a fibre of $\mathcal{V}_2^D \to \operatorname{Bun}_k$ is $\operatorname{Hom}(\operatorname{Sym}^2(L(D)), \Omega^2)$, and we have a natural map

$${}^{\mathrm{t}}a_D : \mathcal{V}_2^D \to \mathcal{V}_2,$$

the transpose of a_D. Since $({}^{\mathrm{t}}a_D)^* \pi_{2!}\bar{\mathbb{Q}}_\ell \widetilde{\to} (\pi_2^D)_!\bar{\mathbb{Q}}_\ell$ canonically, our assertion follows from the standard properties of the Fourier transform functor.

Step 2. Denote by ${}_x\mathcal{H}_{G_k}$ the Hecke stack classifying $M, M' \in \operatorname{Bun}_{G_k}$ together with an isomorphism of G_k-torsors $M \widetilde{\to} M'|_{X-x}$. Let $T_k \subset Q_k = \mathrm{GL}_k$ denote the maximal torus of diagonal matrices, its coweight lattice identifies with $\mathbb{Z}^k$. The preimage of the standard Borel subgroup of Q_k in P_k is a Borel subgroup of G_k, this also fixes the set of simple roots of G_k. Set $\omega_k = (1, \dots, 1)$, where 1 appears k times. This is a dominant coweight of G_k orthogonal to all of the roots of Q_k.

Denote by ${}_{D,P}\mathcal{H}_{G_k}$ the stack classifying $(M, M',\ M' \widetilde{\to} M|_{X-x}) \in {}_x\mathcal{H}_{G_k}$ such that M' is in the position $i\omega_k$ with respect to M at x, $L_D \subset M|_D$ with $(M, L_D) \in {}_{D,P}\operatorname{Bun}_{G_k}$ satisfying $L_D \cap (M'/M(-D)) = 0$. The latter intersection is taken inside $M(D)/M(-D)$, it makes sense because $L_D \subset M/M(-D)$ and

$$M(-D) \subset M' \subset M(D).$$

Denote by $a_{\mathcal{H},D} : {}_{D,P}\mathcal{H}_{G_k} \to {}_{D,P}\operatorname{Bun}_{G_k}$ the map sending the above point to (M, L_D). We have the following diagram, where the square is cartesian.

$$\begin{array}{ccccc}
\operatorname{Bun}_{P_k} & \xrightarrow{p_{\mathcal{H},D}} & {}_{D,P}\mathcal{H}_{G_k} & \xrightarrow{\nu_{\mathcal{H},D}} & \operatorname{Bun}_{G_k} \\
\downarrow{\scriptstyle a_D} & & \downarrow{\scriptstyle a_{\mathcal{H},D}} & & \\
\operatorname{Bun}_{P_k} & \xrightarrow{p_D} & {}_{D,P}\operatorname{Bun}_{G_k} & \xrightarrow{\nu_D} & \operatorname{Bun}_{G_k}
\end{array} \tag{7}$$

Here $p_{\mathcal{H},D}$ is the map sending $(L' \subset M')$ to (M', M, L_D), where $(L \subset M)$ is the image of $(L' \subset M')$ under a_D and $L_D = L|_D$. The map $\nu_{\mathcal{H},D} : {}_{D,P}\mathcal{H}_{G_k} \to \operatorname{Bun}_{G_k}$ sends the above point to M'.

Consider the diagram

$${}_{D,P}\tilde{\mathcal{H}}_{G_k} \xrightarrow{\tilde{a}_{\mathcal{H},D}} {}_{D,P}\widetilde{\operatorname{Bun}}_{G_k} \xrightarrow{\tilde{\nu}_D} \widetilde{\operatorname{Bun}}_{G_k}$$

obtained from ${}_{D,P}\mathcal{H}_{G_k} \stackrel{a_{\mathcal{H},D}}{\to} {}_{D,P}\operatorname{Bun}_{G_k} \stackrel{\nu_D}{\to} \operatorname{Bun}_{G_k}$ by the base change $\widetilde{\operatorname{Bun}}_{G_k} \to \operatorname{Bun}_{G_k}$. Now $\nu_{\mathcal{H},D}$ lifts to a map

$$\tilde{\nu}_{\mathcal{H},D} : {}_{D,P}\tilde{\mathcal{H}}_{G_k} \to \widetilde{\operatorname{Bun}}_{G_k}$$

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defined as follows. A point of ${}_{D,P}\tilde{\mathcal{H}}_{G_k}$ is given by $(\mathcal{B}, M, M', L_D \subset M|_D)$, where $\mathcal{B}$ is a one-dimensional ($\mathbb{Z}/2\mathbb{Z}$-graded purely of degree zero) vector space equipped with a $\mathbb{Z}/2\mathbb{Z}$-graded isomorphism $\mathcal{B}^2 \widetilde{\rightarrow} \det \mathrm{R}\Gamma(X, M)$. The map $\tilde{\nu}_{\mathcal{H},D}$ sends this point to $(\mathcal{B}', M')$, where $\mathcal{B}' = \mathcal{B} \otimes \det_k(L_D)^{-1}$ is equipped with an isomorphism

$$(\mathcal{B}')^2 \widetilde{\rightarrow} \det \mathrm{R}\Gamma(X, M') \tag{8}$$

that we are going to define.

For a vector bundle $\mathcal{N}$ on X write $\mathcal{N}_D = \mathcal{N}/\mathcal{N}(-D)$. We have an exact sequence of $\mathcal{O}_D$-modules

$$0 \to (M \cap M')/M(-D) \to M'/M(-D) \to (M + M')/M \to 0.$$

Note also that $(M + M')/M$ is the orthogonal complement of $(M \cap M')/M(-D)$ with respect to the perfect pairing of $\mathcal{O}_D$-modules $M_D \otimes M(D)_D \to \Omega(D)_D$ given by the symplectic form. By our assumptions, $(M \cap M')/M(-D) \subset M_D$ is a lagrangian $\mathcal{O}_D$-submodule such that

$$((M \cap M')/M(-D)) \oplus L_D \widetilde{\rightarrow} M_D \widetilde{\rightarrow} (((M + M')/M) \otimes \mathcal{O}(-D)) \oplus L_D.$$

The exact sequences of $\mathcal{O}_X$-modules $0 \to M(-D) \to M \to M_D \to 0$ and $0 \to M(-D) \to M' \to M'/M(-D) \to 0$ yield $\mathbb{Z}/2\mathbb{Z}$-graded isomorphisms

$$\det \mathrm{R}\Gamma(X, M') \widetilde{\rightarrow} \frac{\det \mathrm{R}\Gamma(X, M)}{\det_k(L_D)} \otimes \frac{\det_k(M(D)_D)}{\det_k(L_D \otimes \mathcal{O}(D))} \widetilde{\rightarrow} \frac{\det \mathrm{R}\Gamma(X, M)}{\det_k(L_D)^{\otimes 2}}$$

giving rise to (8).

To summarize, the diagram (7) is refined to the following commutative diagram

$$\begin{array}{ccccccc}
 & & \mathrm{Bun}_{P_k} & \xleftarrow{a_D} & \mathrm{Bun}_{P_k} & & \\
 & \swarrow{\scriptstyle \tilde{\nu}_k} & \downarrow{\scriptstyle \tilde{p}_D} & & \downarrow{\scriptstyle \tilde{p}_{\mathcal{H},D}} & \searrow{\scriptstyle \tilde{\nu}_k} & \\
\widetilde{\mathrm{Bun}}_{G_k} & \xleftarrow{\tilde{\nu}_D} & {}_{D,P}\widetilde{\mathrm{Bun}}_{G_k} & \xleftarrow{\tilde{a}_{\mathcal{H},D}} & {}_{D,P}\tilde{\mathcal{H}}_{G_k} & \xrightarrow{\tilde{\nu}_{\mathcal{H},D}} & \widetilde{\mathrm{Bun}}_{G_k}
\end{array} \tag{9}$$

where the middle square is cartesian. Here $\tilde{p}_D$ is the product map $(p_D \times \tilde{\nu}_k)$, and $\tilde{p}_{\mathcal{H},D}$ is the product map $(p_{\mathcal{H},D} \times \tilde{\nu}_k)$.

Step 3. Set ${}^{0,i}\mathrm{Bun}_P = a_D({}^0\mathrm{Bun}_P)$, this is an open substack of Bun_P. For $i \leqslant j$ we have ${}^{0,i}\mathrm{Bun}_P \subset {}^{0,j}\mathrm{Bun}_P$ and the union of all ${}^{0,i}\mathrm{Bun}_P$ equals Bun_P. We are going to extend $\mathfrak{r}_k$ to each ${}^{0,i}\mathrm{Bun}_P$ in a compatible way.

Now (6) and the diagram (9) yield an isomorphism over ${}^{0,i}\mathrm{Bun}_{P_k}$

$$S_{P,\psi}[-k^2 i] \widetilde{\rightarrow} (a_D)_! S_{P,\psi} \widetilde{\rightarrow} \tilde{p}_D^* (\tilde{a}_{\mathcal{H},D})_! (\tilde{\nu}_{\mathcal{H},D})^* \mathrm{Aut}[\dim.\mathrm{rel}], \tag{10}$$

where $\dim.\mathrm{rel} = \dim \mathrm{Bun}_{P_k} + \dim.\mathrm{rel}(a_D) - \dim \mathrm{Bun}_{G_k}$ and $\dim.\mathrm{rel}(a_D) = k(k+1)i$.

Restricting (10) to the open substack ${}^0\mathrm{Bun}_P \subset {}^{0,i}\mathrm{Bun}_P$ and applying (6) once again, we obtain an isomorphism of (shifted) perverse sheaves over ${}^0\mathrm{Bun}_P$

$$\tilde{p}_D^* \tilde{\nu}_D^* \mathrm{Aut} \widetilde{\rightarrow} \tilde{p}_D^* (\tilde{a}_{\mathcal{H},D})_! (\tilde{\nu}_{\mathcal{H},D})^* \mathrm{Aut}[k(k+1)i + k^2 i]. \tag{11}$$

Step 4. Denote by ${}^0_D\mathrm{Bun}_P \subset {}^0\mathrm{Bun}_P$ the open substack given by $\mathrm{H}^0(X, (\mathrm{Sym}^2 L)(D)) = 0$. Let us show that the map $p_D : {}^0_D\mathrm{Bun}_{P_k} \to {}_{D,P}\mathrm{Bun}_{G_k}$ is smooth.

Set $\mathfrak{p} = \mathrm{Lie}\, P_k$ and $\mathfrak{g} = \mathrm{Lie}\, G_k$. Let $\mathcal{F}_{P_k}$ be a k-point of Bun_{P_k} given by $(L \subset M)$. Let K denote the kernel of the composition

$$\mathfrak{g}_{\mathcal{F}_{P_k}} \to (\mathfrak{g}_{\mathcal{F}_{P_k}})|_D \to \mathrm{Hom}_{\mathcal{O}_D}(L_D, M_D/L_D).$$

Recall the following notion. For a 1-morphism $\mathrm{Spec}\, k \xrightarrow{x} \mathcal{X}$ to a stack $\mathcal{X}$ the tangent groupoid to x is the category, whose objects are pairs (x_1, α_1), where x_1 is a 1-morphism $\mathrm{Spec}\, k[\epsilon]/\epsilon^2 \to \mathcal{X}$ and

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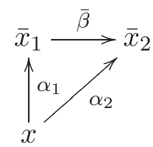
α is a 2-morphism $x \to \bar{x}_1$. Here $\bar{x}_1$ is the composition $\operatorname{Spec} k \hookrightarrow \operatorname{Spec} k[\epsilon]/\epsilon^2 \xrightarrow{x_1} \mathcal{X}$. A morphism from (x_1, α_1) to (x_2, α_2) is a 2-morphism $\beta : x_1 \to x_2$ such that the following diagram commutes.



The tangent groupoid to ${}_{D,P}\mathrm{Bun}_{G_k}$ at the k-point $p_D(\mathcal{F}_{P_k})$ is isomorphic to the stack quotient of $\mathrm{H}^1(X, K)$ by the trivial action of $\mathrm{H}^0(X, K)$. The natural map $\mathfrak{p}_{\mathcal{F}_{P_k}} \to \mathfrak{g}_{\mathcal{F}_{P_k}}$ factors through $K \subset \mathfrak{g}_{\mathcal{F}_{P_k}}$. We need to show that $\mathrm{H}^1(X, \mathfrak{p}_{\mathcal{F}_{P_k}}) \to \mathrm{H}^1(X, K)$ is surjective. We have an exact sequence $0 \to K/(\mathfrak{p}_{\mathcal{F}_{P_k}}) \to (\mathfrak{g}/\mathfrak{p})_{\mathcal{F}_{P_k}} \to (\Omega \otimes \mathrm{Sym}^2 L^*)_D \to 0$. So, $K/(\mathfrak{p}_{\mathcal{F}_{P_k}}) \widetilde{\to} (\Omega \otimes \mathrm{Sym}^2 L^*)(-D)$, the desired surjectivity follows.

It is easy to deduce that $\tilde{p}_D : {}^0_D\mathrm{Bun}_{P_k} \to {}_{D,P}\widetilde{\mathrm{Bun}}_{G_k}$ is smooth. One checks that it is also surjective and has connected fibres. So, (11) descends to an isomorphism of (shifted) perverse sheaves on ${}_{D,P}\widetilde{\mathrm{Bun}}_{G_k}$

$$\tilde{\nu}_D^* \operatorname{Aut} \widetilde{\to} (\tilde{a}_{\mathcal{H},D})_! (\tilde{\nu}_{\mathcal{H},D})^* \operatorname{Aut}[k(k+1)i + k^2 i].$$

Now from (10) we obtain an isomorphism over ${}^{0,i}\mathrm{Bun}_{P_k}$

$$S_{P,\psi} \widetilde{\to} \tilde{p}_D^* \tilde{\nu}_D^* \operatorname{Aut}[\dim.\operatorname{rel}(\nu_k)]. \qquad \square$$

For the rest of the paper we fix the isomorphism (6) over Bun_{P_k}, some of our results will depend on this choice.

3. Theta-lifting for the pair $\mathbb{GSp}_{2n}$, $\mathbb{GO}_{2m}$

3.1 Let $n, m \in \mathbb{N}$ and $\mathbb{G} = G = \mathbb{GSp}_{2n}$. Pick a maximal torus and a Borel subgroup $\mathbb{T}_{\mathbb{G}} \subset \mathbb{B}_{\mathbb{G}} \subset \mathbb{G}$. The stack Bun_G classifies $M \in \mathrm{Bun}_{2n}, \mathcal{A} \in \mathrm{Bun}_1$ with symplectic form $\bigwedge^2 M \to \mathcal{A}$. We have a ($\mathbb{Z}/2\mathbb{Z}$-graded) line bundle $\mathcal{A}_G$ on Bun_G with fibre $\det \mathrm{R}\Gamma(X, M)$ at $(M, \mathcal{A})$.

Let $\pi : \tilde{X} \to X$ be an étale degree two covering, σ the nontrivial automorphism of $\tilde{X}$ over X and $\Sigma = \{1, \sigma\}$. Let $\mathcal{E}$ be the σ-anti-invariants in $\pi_* \mathcal{O}$, it is equipped with a trivialization $\kappa : \mathcal{E}^2 \widetilde{\to} \mathcal{O}$. Let $\mathcal{E}_0$ denote the σ-anti-invariants in $\pi_* \bar{\mathbb{Q}}_\ell$, it is equipped with $\mathcal{E}_0^2 \widetilde{\to} \bar{\mathbb{Q}}_\ell$. Let g (respectively, $\tilde{g}$) denote the genus of X (respectively, of $\tilde{X}$).

Let $\mathbb{H} = \mathbb{GO}^0_{2m}$ be the connected component of unity of the split orthogonal similitude group $\mathbb{GO}_{2m}$ over $\operatorname{Spec} k$. Pick a maximal torus and a Borel subgroup $\mathbb{T}_{\mathbb{H}} \subset \mathbb{B}_{\mathbb{H}} \subset \mathbb{H}$. Pick $\tilde{\sigma} \in \mathbb{O}_{2m}(k)$ with $\tilde{\sigma}^2 = 1$ such that $\tilde{\sigma} \notin \mathbb{SO}_{2m}(k)$. We assume in addition that $\tilde{\sigma}$ preserves $\mathbb{T}_{\mathbb{H}}$ and $\mathbb{B}_{\mathbb{H}}$, so for $m \geqslant 2$ it induces the unique[2] nontrivial automorphism of the Dynkin diagram of $\mathbb{H}$. For $m = 1$ we identify $\mathbb{H} \widetilde{\to} \mathbb{G}_m \times \mathbb{G}_m$ in such a way that $\tilde{\sigma}$ permutes the two copies of $\mathbb{G}_m$.

Realize $\mathbb{H}$ as the subgroup of $\mathrm{GL}(k^{2m})$ preserving up to a multiple the symmetric form given by the matrix

$$\begin{pmatrix} 0 & E_m \\ E_m & 0 \end{pmatrix},$$

where $E_m \in \mathrm{GL}_m$ is the unity. Take $\mathbb{T}_H$ to be the maximal torus of diagonal matrices, $\mathbb{B}_H$ the Borel subgroup preserving for $i = 1, \ldots, m$ the isotropic subspace generated by the first i base vectors $\{e_1, \ldots, e_i\}$. Then one may take $\tilde{\sigma}$ interchanging e_m and e_{2m} and acting trivially on the orthogonal complement to $\{e_m, e_{2m}\}$.

[2]Except for $m = 4$. The group $\mathbb{GO}_8$ also has trilitarian outer forms, but we do not consider them.

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Consider the corresponding Σ-action on $\mathbb{H}$ by conjugation. Let $\tilde{H}$ be the group scheme on X, the twisting of $\mathbb{H}$ by the Σ-torsor $\pi : \tilde{X} \to X$.

The stack $\mathrm{Bun}_{\tilde{H}}$ classifies $V \in \mathrm{Bun}_{2m}$, $\mathcal{C} \in \mathrm{Bun}_1$, a nondegenerate symmetric form $\mathrm{Sym}^2 V \to \mathcal{C}$, and a compatible trivialization $\gamma : \mathcal{C}^{-m} \otimes \det V \widetilde{\to} \mathcal{E}$. This means that the composition

$$\mathcal{C}^{-2m} \otimes (\det V)^2 \stackrel{\gamma^2}{\to} \mathcal{E}^2 \widetilde{\to} \mathcal{O}$$

is the isomorphism induced by $V \widetilde{\to} V^* \otimes \mathcal{C}$. (Although det is involved, we view γ as ungraded.)

Let RCov^0 denote the stack classifying a line bundle $\mathcal{U}$ on X together with a trivialization $\mathcal{U}^{\otimes 2} \widetilde{\to} \mathcal{O}$. Its connected components are indexed by $\mathrm{H}^1_{et}(X, \mathbb{Z}/2\mathbb{Z})$, each connected component is isomorphic to the classifying stack $B(\mu_2)$.

Let Bun_H be the stack classifying $V \in \mathrm{Bun}_{2m}, \mathcal{C} \in \mathrm{Bun}_1$, and a symmetric form $\mathrm{Sym}^2 V \to \mathcal{C}$ such that the corresponding trivialization $(\mathcal{C}^{-m} \otimes \det V)^2 \widetilde{\to} \mathcal{O}$ lies in the component of RCov^0 given by $(\mathcal{E}, \kappa)$. Note that

$$\mathrm{Bun}_{\tilde{H}} \widetilde{\to} \mathrm{Spec}\, k \times_{\mathrm{RCov}^0} \mathrm{Bun}_H,$$

where the map $\mathrm{Spec}\, k \to \mathrm{RCov}^0$ is given by $(\mathcal{E}, \kappa)$. Write $\mathrm{Bun}^d_H \subset \mathrm{Bun}_H$ for the open substack given by $\deg \mathcal{C} = d$, and similarly for $\mathrm{Bun}^d_{\tilde{H}}$.

The projection $\rho_H : \mathrm{Bun}_{\tilde{H}} \to \mathrm{Bun}_H$ is a μ_2-torsor. By extension of scalars $\mu_2 \subset \bar{\mathbb{Q}}^*_\ell$ it yields a rank-one local system $\mathcal{N}$ on Bun_H, which we refer to as *the determinantal local system*.

Let $\mathcal{A}_H$ be the ($\mathbb{Z}/2\mathbb{Z}$-graded) line bundle on Bun_H with fibre $\det \mathrm{R}\Gamma(X, V)$ at $(V, \mathcal{C})$. Set

$$\mathrm{Bun}_{G,H} = \mathrm{Bun}_H \times_{\mathrm{Pic}\, X} \mathrm{Bun}_G,$$

where the map $\mathrm{Bun}_H \to \mathrm{Pic}\, X$ sends $(V, \mathcal{C}, \mathrm{Sym}^2 V \to \mathcal{C})$ to $\Omega \otimes \mathcal{C}^{-1}$. The map $\mathrm{Bun}_G \to \mathrm{Pic}\, X$ sends $(M, \bigwedge^2 M \to \mathcal{A})$ to $\mathcal{A}$. We have an isomorphism $\mathcal{C} \otimes \mathcal{A} \widetilde{\to} \Omega$ for a point of $\mathrm{Bun}_{G,H}$. Let

$$\tau : \mathrm{Bun}_{G,H} \to \mathrm{Bun}_{G_{2nm}}$$

be the map sending a point as above to $V \otimes M$ with symplectic form $\bigwedge^2 (V \otimes M) \to \Omega$.

Proposition 2. *For a point of* $\mathrm{Bun}_{G,H}$ *as above we have a canonical* $\mathbb{Z}/2\mathbb{Z}$*-graded isomorphism*

$$\det \mathrm{R}\Gamma(X, V \otimes M) \widetilde{\to} \frac{\det \mathrm{R}\Gamma(X, V)^{2n} \otimes \det \mathrm{R}\Gamma(X, M)^{2m}}{\det \mathrm{R}\Gamma(X, \mathcal{O})^{2nm} \otimes \det \mathrm{R}\Gamma(X, \mathcal{A})^{2nm}}. \tag{12}$$

More precisely, we have a canonical $\mathbb{Z}/2\mathbb{Z}$*-graded isomorphism of line bundles on* $\mathrm{Bun}_{G,H}$

$$\tau^* \mathcal{A}_{G_{2nm}} \widetilde{\to} \mathcal{A}_H^{2n} \otimes \mathcal{A}_G^{2m} \otimes \mathcal{A}_1^{-2nm} \otimes \det \mathrm{R}\Gamma(X, \mathcal{O})^{-2nm}.$$

Lemma 1. *We have the following.*

(i) *For any* $M \in \mathrm{Bun}_n$, $V \in \mathrm{Bun}_m$ *there is a canonical* $\mathbb{Z}/2\mathbb{Z}$*-graded isomorphism*

$$\det \mathrm{R}\Gamma(X, M \otimes V) \widetilde{\to} \frac{\det \mathrm{R}\Gamma(X, M)^{\otimes m} \otimes \det \mathrm{R}\Gamma(X, V)^{\otimes n}}{\det \mathrm{R}\Gamma(X, \mathcal{A}) \otimes \det \mathrm{R}\Gamma(X, \mathcal{B})} \otimes \frac{\det \mathrm{R}\Gamma(X, \mathcal{A} \otimes \mathcal{B})}{\det \mathrm{R}\Gamma(X, \mathcal{O})^{\otimes nm-1}},$$

where $\mathcal{A} = \det M$, $\mathcal{B} = \det V$.

(ii) *For any* $\mathcal{A}, \mathcal{B} \in \mathrm{Pic}\, X$ *there is a canonical isomorphism of* $\mathbb{Z}/2\mathbb{Z}$*-graded lines*

$$\frac{\det \mathrm{R}\Gamma(X, \mathcal{A} \otimes \mathcal{B}^m) \otimes \det \mathrm{R}\Gamma(X, \mathcal{A})^{m-1}}{\det \mathrm{R}\Gamma(X, \mathcal{A} \otimes \mathcal{B})^m} \widetilde{\to} \frac{\det \mathrm{R}\Gamma(X, \mathcal{B}^m) \otimes \det \mathrm{R}\Gamma(X, \mathcal{O})^{m-1}}{\det \mathrm{R}\Gamma(X, \mathcal{B})^m}.$$

Proof. (i) Denote by $A(M, V)$ the $\mathbb{Z}/2\mathbb{Z}$-graded vector space

$$\frac{\det \mathrm{R}\Gamma(X, M \otimes V)}{\det \mathrm{R}\Gamma(X, M)^{\otimes m} \otimes \det \mathrm{R}\Gamma(X, V)^{\otimes n}} \otimes \frac{\det \mathrm{R}\Gamma(X, \mathcal{A}) \otimes \det \mathrm{R}\Gamma(X, \mathcal{B})}{\det \mathrm{R}\Gamma(X, \mathcal{A} \otimes \mathcal{B})},$$

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where $\mathcal{A} = \det M$, $\mathcal{B} = \det V$. View A as a $\mathbb{Z}/2\mathbb{Z}$-graded line bundle on $\mathrm{Bun}_n \times \mathrm{Bun}_m$. Let us show that this line bundle is constant.

For an exact sequence of $\mathcal{O}_X$-modules $0 \to M \to M' \to M'/M \to 0$, where M'/M is a torsion sheaf of length one at $x \in X$, we have a canonical $\mathbb{Z}/2\mathbb{Z}$-graded isomorphism $A(M', V) \widetilde{\to} A(M, V)$. Similarly, for an exact sequence of $\mathcal{O}_X$-modules $0 \to V \to V' \to V'/V \to 0$, where V'/V is a torsion sheaf of length one at $x \in X$, we have a canonical $\mathbb{Z}/2\mathbb{Z}$-graded isomorphism $A(M, V') \widetilde{\to} A(M, V)$.

To conclude, note that $A(\mathcal{O}^n, \mathcal{O}^m) = \det \mathrm{R}\Gamma(X, \mathcal{O})^{\otimes 1-nm}$.

(ii) The proof is similar. □

Proof of Proposition 2. For a point $(M, \mathcal{A})$ of Bun_G the form $\bigwedge^2 M \to \mathcal{A}$ induces an isomorphism $\det M \widetilde{\to} \mathcal{A}^n$. So, by Lemma 1, for a point of $\mathrm{Bun}_{G,H}$ as above we get a $\mathbb{Z}/2\mathbb{Z}$-graded isomorphism

$$\det \mathrm{R}\Gamma(X, V \otimes M) \widetilde{\to} \frac{\det \mathrm{R}\Gamma(X, V)^{2n} \otimes \det \mathrm{R}\Gamma(X, M)^{2m}}{\det \mathrm{R}\Gamma(X, \mathcal{A}^n) \otimes \det \mathrm{R}\Gamma(X, \det V)} \otimes \frac{\det \mathrm{R}\Gamma(X, \mathcal{A}^n \otimes \det V)}{\det \mathrm{R}\Gamma(X, \mathcal{O})^{4nm-1}}. \tag{13}$$

Applying it to $M = \bigoplus_{i=1}^n (\mathcal{O} \oplus \mathcal{A})$ with the natural symplectic form $\bigwedge^2 M \to \mathcal{A}$, we obtain

$$\frac{\det \mathrm{R}\Gamma(X, \mathcal{O})^{2nm-1}}{\det \mathrm{R}\Gamma(X, \mathcal{A})^{2nm}} \widetilde{\to} \frac{\det \mathrm{R}\Gamma(X, \mathcal{A}^n \otimes \det V)}{\det \mathrm{R}\Gamma(X, \mathcal{A}^n) \otimes \det \mathrm{R}\Gamma(X, \det V)}.$$

Combining the latter formula with (13), one concludes the proof. □

3.2.1 By Proposition 2, we obtain a map $\tilde{\tau} : \mathrm{Bun}_{G,H} \to \widetilde{\mathrm{Bun}}_{G_{2nm}}$ sending $(\bigwedge^2 M \to \mathcal{A}, \mathrm{Sym}^2 V \to \mathcal{C}, \mathcal{A} \otimes \mathcal{C} \widetilde{\to} \Omega)$ to $(\bigwedge^2 (M \otimes V) \to \Omega, \mathcal{B})$. Here

$$\mathcal{B} = \frac{\det \mathrm{R}\Gamma(X, V)^n \otimes \det \mathrm{R}\Gamma(X, M)^m}{\det \mathrm{R}\Gamma(X, \mathcal{O})^{nm} \otimes \det \mathrm{R}\Gamma(X, \mathcal{A})^{nm}},$$

and $\mathcal{B}^2$ is identified with $\det \mathrm{R}\Gamma(X, M \otimes V)$ via (12).

Definition 1. For the diagram of projections

$$\mathrm{Bun}_H \stackrel{\mathfrak{q}}{\leftarrow} \mathrm{Bun}_{G,H} \stackrel{\mathfrak{p}}{\to} \mathrm{Bun}_G$$

define $F_G : \mathrm{D}(\mathrm{Bun}_H) \to \mathrm{D}(\mathrm{Bun}_G)$ by

$$F_G(K) = \mathfrak{p}_!(\tilde{\tau}^* \mathrm{Aut} \otimes \mathfrak{q}^* K)[\dim.\mathrm{rel}],$$

where $\dim.\mathrm{rel} = \dim \mathrm{Bun}_{G_n} - \dim \mathrm{Bun}_{G_{2nm}}$. Define $F_H : \mathrm{D}(\mathrm{Bun}_G) \to \mathrm{D}(\mathrm{Bun}_H)$ by

$$F_H(K) = \mathfrak{q}_!(\tilde{\tau}^* \mathrm{Aut} \otimes \mathfrak{p}^* K)[\dim.\mathrm{rel}],$$

where $\dim.\mathrm{rel} = \dim \mathrm{Bun}_{\mathbb{SO}_{2m}} - \dim \mathrm{Bun}_{G_{2nm}}$. Set also $F_{\tilde{H}} = \rho_H^* \circ F_H$. Replacing Aut by Aut_s (respectively, by Aut_g) in the above definitions, one defines the functors $F_{G,s}$, $F_{H,s}$, $F_{\tilde{H},s}$ (respectively, $F_{G,g}$, $F_{H,g}$, $F_{\tilde{H},g}$). We write $F_H^G = F_H$ when we need to express the dependence of F_H on G, and similarly for $F_G^H = F_G$.

Let $\mathrm{Bun}_{G,\tilde{H}}$ be obtained from $\mathrm{Bun}_{G,H}$ by the base change $\mathrm{Bun}_{\tilde{H}} \to \mathrm{Bun}_H$. By abuse of notation, the restriction of $\tilde{\tau} : \mathrm{Bun}_{G,H} \to \widetilde{\mathrm{Bun}}_{G_{2nm}}$ to $\mathrm{Bun}_{G,\tilde{H}}$ is also denoted by $\tilde{\tau}$.

3.2.2 Let $\Lambda_{\mathbb{H}}$ (respectively, $\check{\Lambda}_{\mathbb{H}}$) denote the coweight (respectively, weight) lattice for $\mathbb{H}$. Write $\Lambda^+_{\mathbb{H}}$ for the dominant coweights. The corresponding objects for $\mathbb{G}$ are denoted $\Lambda_{\mathbb{G}}, \check{\Lambda}_{\mathbb{G}}$ and so on.

For $m \geqslant 2$ let $\iota_m \in \mathbb{S}\mathrm{pin}_{2m}$ be the central element of order two such that $\mathbb{S}\mathrm{pin}_{2m} / \{\pm \iota_m\} \widetilde{\to} \mathbb{SO}_{2m}$. Here $\mathbb{S}\mathrm{pin}_{2m}$ and $\mathbb{SO}_{2m}$ denote the corresponding split groups over $\mathrm{Spec}\, k$. For $m \geqslant 2$ denote by $\mathrm{G}\mathbb{S}\mathrm{pin}_{2m}$ the quotient of $\mathbb{G}_m \times \mathbb{S}\mathrm{pin}_{2m}$ by the subgroup generated by $(-1, \iota)$. Let us convent that

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$\mathbb{GS}\mathrm{pin}_2 \widetilde{\to} \mathbb{G}_m \times \mathbb{G}_m$. The Langlands dual group is $\check{\mathbb{H}} \widetilde{\to} \mathbb{GS}\mathrm{pin}_{2m}$. We also have $\check{\mathbb{G}} \widetilde{\to} \mathbb{GS}\mathrm{pin}_{2n+1}$, where $\mathbb{GS}\mathrm{pin}_{2n+1}$ is the quotient of $\mathbb{G}_m \times \mathbb{S}\mathrm{pin}_{2n+1}$ by the diagonally embedded $\{\pm 1\}$.

Let $V_{\mathbb{H}}$ (respectively, $V_{\mathbb{G}}$) denote the standard representation of $\mathbb{SO}_{2m}$ (respectively, of $\mathbb{SO}_{2n+1}$).

Case $m \leqslant n$. Pick an inclusion $V_{\mathbb{H}} \hookrightarrow V_{\mathbb{G}}$ that is compatible with symmetric forms. It yields an inclusion $\check{\mathbb{H}} \hookrightarrow \check{\mathbb{G}}$, which we assume is compatible with the corresponding maximal tori. Pick an element $\sigma_{\mathbb{G}} \in \mathbb{SO}(V_{\mathbb{G}}) \widetilde{\to} \check{\mathbb{G}}_{ad}$ normalizing $\check{\mathbb{T}}_{\mathbb{G}}$ and preserving $V_{\mathbb{H}}$ and $\check{\mathbb{T}}_{\mathbb{H}} \subset \check{\mathbb{B}}_{\mathbb{H}}$. Let $\sigma_{\mathbb{H}} \in \mathbb{O}(V_{\mathbb{H}})$ be its restriction to $V_{\mathbb{H}}$. We assume that $\sigma_{\mathbb{H}}$ viewed as an automorphism of $(\check{\mathbb{H}}, \check{\mathbb{T}}_{\mathbb{H}})$ extends the action of Σ on the roots datum of $(\check{\mathbb{H}}, \check{\mathbb{T}}_{\mathbb{H}})$ defined in § 3.1.

In concrete terms, one may take $V_{\mathbb{G}} = k^{2n+1}$ with symmetric form given by the matrix

$$\begin{pmatrix} 0 & E_n & 0 \\ E_n & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where $E_n \in \mathrm{GL}_n$ is the unity. Take $\check{\mathbb{T}}_{\mathbb{G}}$ to be the maximal torus of diagonal matrices. Let $V_{\mathbb{H}} \subset V_{\mathbb{G}}$ be generated by $\{e_1, \ldots, e_m, e_{n+1}, \ldots, e_{n+m}\}$. Let $\check{\mathbb{T}}_{\mathbb{H}}$ be the torus of diagonal matrices and $\check{\mathbb{B}}_{\mathbb{H}}$ the Borel subgroup preserving for $i = 1, \ldots, m$ the isotropic subspace generated by $\{e_1, \ldots, e_i\}$. Then one may take $\sigma_{\mathbb{G}}$ permuting e_m and e_{n+m}, sending e_{2n+1} to $-e_{2n+1}$ and acting trivially on the other base vectors.

We let Σ act on $\check{\mathbb{H}}$ and $\check{\mathbb{G}}$ via the elements $\sigma_{\mathbb{H}}, \sigma_{\mathbb{G}}$. So, the inclusion $\check{\mathbb{H}} \hookrightarrow \check{\mathbb{G}}$ is Σ-equivariant and yields a morphism of the L-groups $\tilde{H}^L \to G^L$, where $\tilde{H}^L \widetilde{\to} \check{\mathbb{H}} \ltimes \Sigma$ and $G^L \widetilde{\to} \check{\mathbb{G}} \ltimes \Sigma$ (in the sense of Appendix B.2).

Case $m > n$. Pick an inclusion $V_{\mathbb{G}} \hookrightarrow V_{\mathbb{H}}$ compatible with symmetric forms. It yields an inclusion $\check{\mathbb{G}} \hookrightarrow \check{\mathbb{H}}$, which we assume compatible with the corresponding maximal tori. Let $\sigma_{\mathbb{G}}$ be the identical automorphism of $V_{\mathbb{G}}$. Extend it to an element $\sigma_{\mathbb{H}} \in \mathbb{O}(V_{\mathbb{H}})$ by requiring that $\sigma_{\mathbb{H}}$ preserves $\check{\mathbb{T}}_{\mathbb{H}} \subset \check{\mathbb{B}}_{\mathbb{H}}$ and $\sigma_{\mathbb{H}} \notin \mathbb{SO}(V_{\mathbb{H}})$, $\sigma_{\mathbb{H}}^2 = \mathrm{id}$.

Let Σ act on $\check{\mathbb{H}}$ and $\check{\mathbb{G}}$ via the elements $\sigma_{\mathbb{H}}, \sigma_{\mathbb{G}}$. The Σ-action on $(\check{\mathbb{H}}, \check{\mathbb{T}}_{\mathbb{H}})$ extends the Σ-action (defined in § 3.1) on the root datum of $(\check{\mathbb{H}}, \check{\mathbb{T}}_{\mathbb{H}})$. Again, we get a morphism of the L-groups $G^L \to \tilde{H}^L$. Note that $\check{\mathbb{G}} \times \Sigma \widetilde{\to} G^L$ is the direct product in this case.

As in Appendix B.2, in both cases the corresponding functoriality problem can be posed. As in Appendix B.1.1, for $\lambda \in \Lambda^+_{\mathbb{H}}$ (respectively, $\lambda \in \Lambda^+_G$) one defines the Hecke functors

$$\mathrm{H}^\lambda_{\tilde{H}} : \mathrm{D}(\mathrm{Bun}_{\tilde{H}}) \to \mathrm{D}(\tilde{X} \times \mathrm{Bun}_{\tilde{H}})$$

and

$$\mathrm{H}^\lambda_G : \mathrm{D}(\mathrm{Bun}_G) \to \mathrm{D}(X \times \mathrm{Bun}_G).$$

Write $V^\lambda_{\mathbb{H}}$ (respectively, $V^\lambda_{\mathbb{G}}$) for the irreducible representation of $\check{\mathbb{H}}$ (respectively, of $\check{\mathbb{G}}$) with highest weight λ.

Conjecture 1. We make the following conjectures.

(i) Case $m = n$. For $\lambda \in \Lambda^+_{\mathbb{G}}$ there is an isomorphism functorial in $K \in \mathrm{D}(\mathrm{Bun}_{\tilde{H}})$

$$(\pi \times \mathrm{id})^* \mathrm{H}^\lambda_G F_G(K) \widetilde{\to} \bigoplus_{\mu \in \Lambda^+_{\mathbb{H}}} (\mathrm{id} \boxtimes F_G) \mathrm{H}^\mu_{\tilde{H}}(K) \otimes \mathrm{Hom}_{\check{\mathbb{H}}}(V^\mu_{\mathbb{H}}, (V^\lambda_{\mathbb{G}})^*).$$

Here $\pi \times \mathrm{id} : \tilde{X} \times \mathrm{Bun}_G \to X \times \mathrm{Bun}_G$ and $\mathrm{id} \boxtimes F_G : \mathrm{D}(\tilde{X} \times \mathrm{Bun}_{\tilde{H}}) \to \mathrm{D}(\tilde{X} \times \mathrm{Bun}_G)$ is the corresponding theta-lifting functor.

(ii) Case $m = n+1$. For $\mu \in \Lambda^+_{\mathbb{H}}$ there is an isomorphism functorial in $K \in \mathrm{D}(\mathrm{Bun}_G)$

$$\mathrm{H}^{\mu}_{\tilde{H}} F_{\tilde{H}}(K) \widetilde{\to} \bigoplus_{\lambda \in \Lambda^+_{\mathbb{G}}} (\pi \times \mathrm{id})^*(\mathrm{id} \boxtimes F_{\tilde{H}}) \mathrm{H}^{\lambda}_G(K) \otimes \mathrm{Hom}_{\check{\mathbb{G}}}((V^{\lambda}_{\mathbb{G}})^*, V^{\mu}_{\mathbb{H}}).$$

Here $\pi \times \mathrm{id} : \tilde{X} \times \mathrm{Bun}_{\tilde{H}} \to X \times \mathrm{Bun}_{\tilde{H}}$ and $\mathrm{id} \boxtimes F_{\tilde{H}} : \mathrm{D}(X \times \mathrm{Bun}_G) \to \mathrm{D}(X \times \mathrm{Bun}_{\tilde{H}})$ is the corresponding theta-lifting functor.

In both cases these isomorphisms are compatible with the action of Σ on both sides (Σ acts on the Hecke operators for $\tilde{H}$ via (61)).

Remark 1. For other pairs (n, m) the relation between the theta-lifting functors and Hecke functors is expected to be essentially as in [Lys07] involving the SL_2 of Arthur.

3.2.3 Let $\mathrm{act} : \mathrm{Pic}\, X \times \mathrm{Bun}_G \to \mathrm{Bun}_G$ be the map sending $(\mathcal{L} \in \mathrm{Pic}\, X, M, \mathcal{A})$ to $(M \otimes \mathcal{L}, \mathcal{A} \otimes \mathcal{L}^2)$. Write also $\mathrm{act} : \mathrm{Pic}\, X \times \mathrm{Bun}_H \to \mathrm{Bun}_H$ for the map sending $(\mathcal{L} \in \mathrm{Pic}\, X, V, \mathcal{C})$ to $(V \otimes \mathcal{L}, \mathcal{C} \otimes \mathcal{L}^2)$.

Definition 2. For a rank-one local system $\mathcal{U}$ on X write $A\mathcal{U}$ for the automorphic local system on $\mathrm{Pic}\, X$ corresponding to $\mathcal{U}$. It is equipped with an isomorphism between the restriction of $A\mathcal{U}$ under $X^{(d)} \to \mathrm{Pic}\, X, D \mapsto \mathcal{O}(D)$ and $\mathcal{U}^{(d)}$, this defines $A\mathcal{U}$ up to a unique isomorphism.

Definition 3. For a rank-one local system $\mathcal{A}$ on X say that $K \in \mathrm{D}(\mathrm{Bun}_G)$ (respectively, $K \in \mathrm{D}(\mathrm{Bun}_H)$) has central character $\mathcal{A}$ if K is equipped with a $(\mathrm{Pic}\, X, A\mathcal{A})$-equivariant structure as in [Lys06a, Appendix A.1, Definition 7]. In particular, we have $\mathrm{act}^* K \widetilde{\to} A\mathcal{A} \boxtimes K$.

Remark 2. Using Lemma 1, one checks that for $K \in \mathrm{D}(\mathrm{Bun}_{\tilde{H}})$ (respectively, $K \in \mathrm{D}(\mathrm{Bun}_G)$) with central character χ the central character of $F_G(K)$ (respectively, of $F_{\tilde{H}}(K)$) is $\chi^{-1} \otimes \mathcal{E}_0^{\otimes n}$. The reason for that is as follows. Let $\mathcal{X}_{G,H}$ be the stack classifying $(M, \mathcal{A}) \in \mathrm{Bun}_G$, $(V, \mathcal{C}) \in \mathrm{Bun}_H$, $U \in \mathrm{Pic}\, X$ equipped with $\mathcal{A} \otimes \mathcal{C} \otimes U^2 \widetilde{\to} \Omega$. We have a commutative diagram

$$\begin{array}{ccc} \mathrm{Bun}_{G,H} & \xrightarrow{\tau} & \mathrm{Bun}_{G_{2nm}} \\ \uparrow{\scriptstyle m_G} & & \uparrow{\scriptstyle \tau} \\ \mathcal{X}_{G,H} & \xrightarrow{m_H} & \mathrm{Bun}_{G,H} \end{array}$$

where m_G (respectively, m_H) sends the above collection to $(M \otimes U, \mathcal{A} \otimes U^2) \in \mathrm{Bun}_G$, $(V, C) \in \mathrm{Bun}_H$ (respectively, to $(M, \mathcal{A}) \in \mathrm{Bun}_G$, $(V \otimes U, \mathcal{C} \otimes U^2) \in \mathrm{Bun}_H$). Then the diagram

$$\begin{array}{ccc} \mathrm{Bun}_{G,H} & \xrightarrow{\tilde{\tau}} & \mathrm{Bun}_{G_{2nm}} \\ \uparrow{\scriptstyle m_G} & & \uparrow{\scriptstyle \tilde{\tau}} \\ \mathcal{X}_{G,H} & \xrightarrow{m_H} & \mathrm{Bun}_{G,H} \end{array}$$

is not 2-commutative in general. The key observation is as follows. Consider a line bundle on $\mathrm{Pic}\, X$ whose fibre at $U \in \mathrm{Pic}\, X$ is

$$\vartheta(U) := \frac{\det \mathrm{R}\Gamma(X, U \otimes \mathcal{E}) \otimes \det \mathrm{R}\Gamma(X, \mathcal{O})}{\det \mathrm{R}\Gamma(X, U) \otimes \det \mathrm{R}\Gamma(X, \mathcal{E})}.$$

The tensor square of this line bundle is canonically trivialized, so defines a 2-sheeted covering of $\mathrm{Pic}\, X$. The corresponding local system of order two on $\mathrm{Pic}\, X$ is $A\mathcal{E}_0$.

3.3.1 Let $P \subset G$ be the Siegel parabolic, so Bun_P classifies $L \in \mathrm{Bun}_n$, $\mathcal{A} \in \mathrm{Bun}_1$, and an exact sequence of $\mathcal{O}_X$-modules $0 \to \mathrm{Sym}^2 L \to ? \to \mathcal{A} \to 0$. Write $\nu_P : \mathrm{Bun}_P \to \mathrm{Bun}_G$ for the projection. Let M_P be the Levi factor of P, so $\mathrm{Bun}_{M_P} \widetilde{\to} \mathrm{Bun}_n \times \mathrm{Pic}\, X$.

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Set $\mathrm{Bun}_{P,H} = \mathrm{Bun}_P \times_{\mathrm{Bun}_G} \mathrm{Bun}_{G,H}$ and $\mathrm{Bun}_{P,\tilde H} = \mathrm{Bun}_P \times_{\mathrm{Bun}_G} \mathrm{Bun}_{G,\tilde H}$. We have a commutative diagram

$$\begin{array}{ccc} \mathrm{Bun}_{G,H} & \xrightarrow{\tau} & \mathrm{Bun}_{G_{2nm}} \\ \uparrow & & \uparrow{\scriptstyle \nu_{2nm}} \\ \mathrm{Bun}_{P,H} & \xrightarrow{\tau_P} & \mathrm{Bun}_{P_{2nm}} \end{array} \qquad (14)$$

where τ_P sends $(V, \mathcal{C}, \mathrm{Sym}^2 V \to \mathcal{C})$ and

$$0 \to \mathrm{Sym}^2 L \to ? \to \mathcal{A} \to 0 \qquad (15)$$

to the extension

$$0 \to \mathrm{Sym}^2(V \otimes L) \to ? \to \Omega \to 0, \qquad (16)$$

which is the push-forward of (15) under the composition

$$\mathcal{A}^{-1} \otimes \mathrm{Sym}^2 L \,\tilde{\to}\, \mathcal{C} \otimes \Omega^{-1} \otimes \mathrm{Sym}^2 L \to \Omega^{-1} \otimes \mathrm{Sym}^2 V \otimes \mathrm{Sym}^2 L \to \Omega^{-1} \otimes \mathrm{Sym}^2(V \otimes L).$$

Here the second map is induced by the form $\mathcal{C} \to \mathrm{Sym}^2 V$. We have used the fact that for $(V, \mathcal{C}, \mathrm{Sym}^2 V \xrightarrow{h} \mathcal{C})$ of Bun_H the map $\mathrm{Sym}^2 V^* \,\tilde{\to}\, (\mathrm{Sym}^2 V) \otimes \mathcal{C}^{-2} \xrightarrow{h} \mathcal{C}^{-1}$ induces a section $\mathcal{C} \hookrightarrow \mathrm{Sym}^2 V$ of $\mathrm{Sym}^2 V \xrightarrow{h} \mathcal{C}$, so $\mathcal{C}$ is naturally a direct summand of $\mathrm{Sym}^2 V$.

Although (14) commutes, the following diagram is not 2-commutative

$$\begin{array}{ccc} \mathrm{Bun}_{G,\tilde H} & \xrightarrow{\tilde\tau} & \widetilde{\mathrm{Bun}}_{G_{2nm}} \\ \uparrow{\scriptstyle \nu_P \times \mathrm{id}} & & \uparrow{\scriptstyle \tilde\nu_{2nm}} \\ \mathrm{Bun}_{P,\tilde H} & \xrightarrow{\tau_P} & \mathrm{Bun}_{P_{2nm}} \end{array}$$

its non-commutativity is measured by the following lemma. Write $\alpha_P : \mathrm{Bun}_P \to \mathrm{Pic}\, X$ for the map sending (15) to $\det L$. Let $\alpha_{P,\tilde H}$ be the composition $\mathrm{Bun}_{P,\tilde H} \to \mathrm{Bun}_P \xrightarrow{\alpha_P} \mathrm{Pic}\, X$.

LEMMA 2. *There is a canonical isomorphism over* $\mathrm{Bun}_{P,\tilde H}$

$$(\nu_P \times \mathrm{id})^* \tilde\tau^* \mathrm{Aut} \,\tilde{\to}\, \tau_P^* \tilde\nu_{2nm}^* \mathrm{Aut} \otimes \alpha_{P,\tilde H}^* A\mathcal{E}_0.$$

Proof. Write $\widetilde{\mathrm{Bun}}_{P,\tilde H}$ for the restriction of the gerbe $\widetilde{\mathrm{Bun}}_{G_{2nm}} \to \mathrm{Bun}_{G_{2nm}}$ under $\tau \circ (\nu_P \times \mathrm{id}) : \mathrm{Bun}_{P,\tilde H} \to \mathrm{Bun}_{G_{2nm}}$. The map $\tilde\tau \circ (\nu_P \times \mathrm{id})$ yields a trivialization $\widetilde{\mathrm{Bun}}_{P,\tilde H} \,\tilde{\to}\, \mathrm{Bun}_{P,\tilde H} \times B(\mu_2)$ of this gerbe. So, $\tilde\nu_{2nm} \circ \tau_P$ gives rise to a map $\mathrm{Bun}_{P,\tilde H} \to B(\mu_2)$. The corresponding μ_2-torsor over $\mathrm{Bun}_{P,\tilde H}$ is calculated using Lemma 1. Namely, we have a line bundle on $\mathrm{Bun}_{P,\tilde H}$ whose fibre at $((15), V, \mathcal{C}, \mathrm{Sym}^2 V \to \mathcal{C}, \gamma)$ is

$$\frac{\vartheta(\mathcal{C}^m \otimes \det L)}{\vartheta(\mathcal{C}^m)}.$$

The tensor square of this line bundle is canonically trivialized and gives rise to the corresponding μ_2-torsor on $\mathrm{Bun}_{P,\tilde H}$. Our assertion follows by Remark 2. ∎

DEFINITION 4. Let $F_{P,\psi} : \mathrm{D}(\mathrm{Bun}_H) \to \mathrm{D}(\mathrm{Bun}_P)$ be the functor given by

$$F_{P,\psi}(K) = (\mathfrak{p}_P)_!(\mathfrak{q}_P^* K \otimes \tau_P^* S_{P,\psi})[\dim.\mathrm{rel}]$$

for the diagram of projections

$$\mathrm{Bun}_H \xleftarrow{\mathfrak{q}_P} \mathrm{Bun}_{P,H} \xrightarrow{\mathfrak{p}_P} \mathrm{Bun}_P,$$

where $\dim.\mathrm{rel} = \dim \mathrm{Bun}_{P,H} - \dim \mathrm{Bun}_H - \dim \mathrm{Bun}_{P_{2nm}}$ is a function of a connected component of $\mathrm{Bun}_{P,H}$. Replacing H by $\tilde H$ in the above diagram, one defines $F_{P,\psi} : \mathrm{D}(\mathrm{Bun}_{\tilde H}) \to \mathrm{D}(\mathrm{Bun}_P)$ by the same formula.

Corollary 1. *The isomorphism (6) yields an isomorphism* $F_{P,\psi}\otimes\alpha_P^*A\mathcal{E}_0\widetilde{\to}\nu_P^*F_G[\dim.\mathrm{rel}(\nu_P)]$ *of functors from* $\mathrm{D}(\mathrm{Bun}_{\tilde H})$ *to* $\mathrm{D}(\mathrm{Bun}_P)$.

3.3.2 Let $\mathcal{S}_P$ be the stack classifying $L\in\mathrm{Bun}_n, \mathcal{A}\in\mathrm{Pic}\,X$, and a section $\mathrm{Sym}^2 L\stackrel{s}{\to}\mathcal{A}\otimes\Omega$. Then $\mathcal{S}_P$ and Bun_P are dual (generalized) vector bundles over $\mathrm{Bun}_{M_P}\widetilde{\to}\mathrm{Bun}_n\times\mathrm{Pic}\,X$. Let $i_P:\mathrm{Bun}_{M_P}\hookrightarrow\mathcal{S}_P$ denote the zero section.

Let

$$\mathcal{V}_{H,P}\to\mathrm{Bun}_n\times\mathrm{Bun}_H$$

be the stack whose fibre over $(L\in\mathrm{Bun}_n, V,\mathcal{C},\mathrm{Sym}^2 V\to\mathcal{C})$ is $\mathrm{Hom}(V\otimes L,\Omega)$. We have a map $\mathfrak{p}_\mathcal{V}:\mathcal{V}_{H,P}\to\mathcal{S}_P$ sending $(V,\mathcal{C})\in\mathrm{Bun}_H$, $L\in\mathrm{Bun}_n$, $L\stackrel{t}{\to}V^*\otimes\Omega$ to $(L,\mathcal{A},s)$, where $\mathcal{A}=\Omega\otimes\mathcal{C}^{-1}$ and s is the composition

$$\mathrm{Sym}^2 L\stackrel{t\otimes t}{\to}\Omega^2\otimes\mathrm{Sym}^2 V^*\to\Omega^2\otimes\mathcal{C}^{-1}\widetilde{\to}\mathcal{A}\otimes\Omega.$$

Definition 5. Let $F_\mathcal{S}:\mathrm{D}(\mathrm{Bun}_H)\to\mathrm{D}(\mathcal{S}_P)$ be given by $F_\mathcal{S}(K)=(\mathfrak{p}_\mathcal{V})_!\mathfrak{q}_\mathcal{V}^*K[\dim.\mathrm{rel}(\mathfrak{q}_\mathcal{V})]$ for the diagram

$$\mathrm{Bun}_H\stackrel{\mathfrak{q}_\mathcal{V}}{\longleftarrow}\mathcal{V}_{H,P}\stackrel{\mathfrak{p}_\mathcal{V}}{\longrightarrow}\mathcal{S}_P, \qquad (17)$$

where $\mathfrak{q}_\mathcal{V}$ is the projection.

The following is immediate from the definitions.

Lemma 3. *There is a canonical isomorphism of functors* $F_{P,\psi}\widetilde{\to}\mathrm{Four}_\psi\circ F_\mathcal{S}$ *from* $\mathrm{D}(\mathrm{Bun}_H)$ *to* $\mathrm{D}(\mathrm{Bun}_P)$.

Let $\mathrm{CT}_P:\mathrm{D}(\mathrm{Bun}_G)\to\mathrm{D}(\mathrm{Bun}_{M_P})$ be the constant term functor given by $\mathrm{CT}_P(K)=\rho_P\nu_P^*K$ for the diagram $\mathrm{Bun}_G\stackrel{\nu_P}{\leftarrow}\mathrm{Bun}_P\stackrel{\rho_P}{\to}\mathrm{Bun}_{M_P}$, where ρ_P is the projection. Let $\alpha_{M_P}:\mathrm{Bun}_{M_P}\to\mathrm{Pic}\,X$ be the map sending $(L,\mathcal{A})$ to $\det L$.

Corollary 2. *The isomorphism (6) induces an isomorphism* $\mathrm{CT}_P\circ F_G\widetilde{\to}i_P^*F_\mathcal{S}\otimes\alpha_{M_P}^*A\mathcal{E}_0$ *of functors (up to a shift) from* $\mathrm{D}(\mathrm{Bun}_{\tilde H})$ *to* $\mathrm{D}(\mathrm{Bun}_{M_P})$.

Let $\mathcal{V}_{\tilde H,P}$ be obtained from $\mathcal{V}_{H,P}$ by the base change $\mathrm{Bun}_{\tilde H}\to\mathrm{Bun}_H$. Denote by

$$\xi_\mathcal{V}:\mathcal{V}_{H,P}\to\mathrm{Bun}_H\times_{\mathrm{Pic}\,X}\mathcal{S}_P \qquad (18)$$

the map $(\mathfrak{q}_\mathcal{V},\mathfrak{p}_\mathcal{V})$. The map $\mathcal{V}_{\tilde H,P}\to\mathrm{Bun}_{\tilde H}\times_{\mathrm{Pic}\,X}\mathcal{S}_P$ obtained from $\xi_\mathcal{V}$ by the base change $\mathrm{Bun}_{\tilde H}\to\mathrm{Bun}_H$ is again denoted by $\xi_\mathcal{V}$ by abuse of notation.

Define the complex ${}^0\mathcal{T}$ on $\mathrm{Bun}_{\tilde H}\times_{\mathrm{Pic}\,X}\mathcal{S}_P$ by

$${}^0\mathcal{T}=\xi_{\mathcal{V}!}\bar{\mathbb{Q}}_\ell[\dim\mathcal{V}_{\tilde H,P}].$$

The group S_2 acts on $\xi_\mathcal{V}$ changing the sign of $t:L\to V^*\otimes\Omega$, so S_2 acts also on ${}^0\mathcal{T}$. Let $\mathrm{Four}_\psi:\mathrm{D}(\mathrm{Bun}_{\tilde H}\times_{\mathrm{Pic}\,X}\mathcal{S}_P)\to\mathrm{D}(\mathrm{Bun}_{P,\tilde H})$ denote the Fourier transform. For the map $\tau_P:\mathrm{Bun}_{P,\tilde H}\to\mathrm{Bun}_{P_{2nm}}$ we have a S_2-equivariant isomorphism

$$\tau_P^*S_{P,\psi}[\dim.\mathrm{rel}(\tau_P)]\widetilde{\to}\mathrm{Four}_\psi({}^0\mathcal{T}).$$

Remark 3. If G_1 is a connected reductive group, which is not a torus, it is expected that for the projection $\mathrm{pr}_{G_1}:\mathrm{Bun}_{G_1}\to\mathrm{Bun}_{G_1/[G_1,G_1]}$ and a cuspidal complex $K\in\mathrm{D}(\mathrm{Bun}_{G_1})$ we have $(\mathrm{pr}_{G_1})_!K=0$. The reason to believe in this is that for K cuspidal and automorphic the eigenvalues of Hecke operators acting on K and on $\bar{\mathbb{Q}}_\ell$ are different, so that $\bar{\mathbb{Q}}_\ell$ and K are 'orthogonal'.

So, for $m>1$ it is expected that for the projection $\mathrm{pr}_{\tilde H}:\mathrm{Bun}_{\tilde H}\to\mathrm{Pic}\,X$ sending $(V,\mathcal{C})$ to $\mathcal{C}$ and a cuspidal $K\in\mathrm{D}(\mathrm{Bun}_{\tilde H})$ we have $(\mathrm{pr}_{\tilde H})_!K=0$. If this is true then for such cupidal K we have $\mathrm{CT}_P(F_G(K))=0$ (in particular, if $n=1$, then $F_G(K)$ is cuspidal).

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Indeed, by Corollary 2, a fibre of $\mathrm{CT}_P(F_G(K))$ over $(L, \mathcal{A} = \mathcal{C}^{-1} \otimes \Omega)$ is an integral over $(V, \mathcal{C}) \in \mathrm{Bun}_{\tilde{H}}$ equipped with a map $L \to V^* \otimes \Omega$, whose image is isotropic. However, we can first fix the isotropic subbundle of $V^* \otimes \Omega$ generated by the image of L and then integrate. The corresponding vanishing follows.

3.4 The case of split H

In this section we assume that the covering $\pi : \tilde{X} \to X$ split. Let $\tilde{Q} \subset \tilde{H} = \mathrm{G}\mathbb{O}^0_{2m}$ be the Siegel parabolic, then $\mathrm{Bun}_{\tilde{Q}}$ is the stack classifying $L \in \mathrm{Bun}_m, \mathcal{C} \in \mathrm{Bun}_1$, and an exact sequence $0 \to \bigwedge^2 L \to ? \to \mathcal{C} \to 0$ on X. The projection $\nu_{\tilde{Q}} : \mathrm{Bun}_{\tilde{Q}} \to \mathrm{Bun}_{\tilde{H}}$ sends the above point to $(V, \mathcal{C}, \mathrm{Sym}^2 V \to \mathcal{C}, \gamma)$, where V is included into an exact sequence $0 \to L \to V \to L^* \otimes \mathcal{C} \to 0$ and $\gamma : \det V \widetilde{\to} \mathcal{C}^m$.

Let $M_{\tilde{Q}}$ be the Levi factor of $\tilde{Q}$, so $\mathrm{Bun}_{M_{\tilde{Q}}} \widetilde{\to} \mathrm{Bun}_m \times \mathrm{Pic}\, X$. Let $\mathrm{Bun}_{G,\tilde{Q}}$ be the stack obtained from $\mathrm{Bun}_{G,H}$ by the base change $\mathrm{Bun}_{\tilde{Q}} \to \mathrm{Bun}_H$. Lemma 1 implies that the following diagram is 2-commutative

$$\begin{array}{ccc} \mathrm{Bun}_{G,H} & \xrightarrow{\tilde{\tau}} & \widetilde{\mathrm{Bun}}_{G_{2nm}} \\ \uparrow & & \uparrow \tilde{\nu}_{2nm} \\ \mathrm{Bun}_{G,\tilde{Q}} & \xrightarrow{\tau_{\tilde{Q}}} & \mathrm{Bun}_{P_{2nm}} \end{array}$$

where $\tau_{\tilde{Q}}$ sends $(M, \mathcal{A}, \bigwedge^2 M \to \mathcal{A},\, 0 \to \bigwedge^2 L \to ? \to \mathcal{C} \to 0)$ to the extension

$$0 \to \mathrm{Sym}^2(M \otimes L) \to ? \to \Omega \to 0,$$

which is the push-forward of $0 \to \mathcal{A} \otimes \bigwedge^2 L \to ? \to \Omega \to 0$ under the composition $\mathcal{A} \otimes \bigwedge^2 L \to \bigwedge^2 M \otimes \bigwedge^2 L \to \mathrm{Sym}^2(M \otimes L)$. Recall that we have a canonical direct sum decomposition $\mathrm{Sym}^2(M \otimes L) \widetilde{\to} (\mathrm{Sym}^2 M \otimes \mathrm{Sym}^2 L) \oplus (\bigwedge^2 M \otimes \bigwedge^2 L)$.

Definition 6. Let $F_{\tilde{Q},\psi} : \mathrm{D}(\mathrm{Bun}_G) \to \mathrm{D}(\mathrm{Bun}_{\tilde{Q}})$ be the functor given by

$$F_{\tilde{Q},\psi}(K) = \mathfrak{p}_{\tilde{Q}!}(\mathfrak{q}^*_{\tilde{Q}} K \otimes \tau^*_{\tilde{Q}} S_{P,\psi})[\dim.\mathrm{rel}]$$

for the diagram of projections

$$\mathrm{Bun}_G \xleftarrow{\mathfrak{q}_{\tilde{Q}}} \mathrm{Bun}_{G,\tilde{Q}} \xrightarrow{\mathfrak{p}_{\tilde{Q}}} \mathrm{Bun}_{\tilde{Q}},$$

where $\dim.\mathrm{rel} = \dim \mathrm{Bun}_{G,\tilde{Q}} - \dim \mathrm{Bun}_G - \dim \mathrm{Bun}_{P_{2nm}}$.

Corollary 3. *The isomorphism (6) induces an isomorphism* $F_{\tilde{Q},\psi} \widetilde{\to} \nu^*_{\tilde{Q}} F_{\tilde{H}}[\dim.\mathrm{rel}(\nu_{\tilde{Q}})]$ *of functors from* $\mathrm{D}(\mathrm{Bun}_G)$ *to* $\mathrm{D}(\mathrm{Bun}_{\tilde{Q}})$.

Let $\mathcal{S}_{\tilde{Q}}$ be the stack classifying $L \in \mathrm{Bun}_m, \mathcal{C} \in \mathrm{Pic}\, X$, and $\bigwedge^2 L \xrightarrow{s} \mathcal{C} \otimes \Omega$. Then $\mathcal{S}_{\tilde{Q}}$ and $\mathrm{Bun}_{\tilde{Q}}$ are dual (generalized) vector bundles over $\mathrm{Bun}_{M_{\tilde{Q}}}$.

Let $\mathcal{W}_{G,\tilde{Q}} \to \mathrm{Bun}_m \times \mathrm{Bun}_G$ be the stack whose fibre over $L \in \mathrm{Bun}_m, (M, \mathcal{A}) \in \mathrm{Bun}_G$ is $\mathrm{Hom}(M \otimes L, \Omega)$. We have a map $\mathfrak{p}_{\mathcal{W}} : \mathcal{W}_{G,\tilde{Q}} \to \mathcal{S}_{\tilde{Q}}$ sending $L \in \mathrm{Bun}_m, (M, \mathcal{A}) \in \mathrm{Bun}_G$ and $t : L \to M^* \otimes \Omega$ to $(L, \mathcal{C} = \Omega \otimes \mathcal{A}^{-1}, s)$, where s is the composition

$$\bigwedge^2 L \xrightarrow{\bigwedge^2 t} \Big(\bigwedge^2 M^*\Big) \otimes \Omega^2 \to \mathcal{A}^{-1} \otimes \Omega^2.$$

Define $F_{\mathcal{S}_{\tilde{Q}}} : \mathrm{D}(\mathrm{Bun}_G) \to \mathrm{D}(\mathcal{S}_{\tilde{Q}})$ by

$$F_{\mathcal{S}_{\tilde{Q}}}(K) = (\mathfrak{p}_{\mathcal{W}})_! \mathfrak{q}^*_{\mathcal{W}} K[\dim.\mathrm{rel}(\mathfrak{q}_{\mathcal{W}})]$$

for the diagram

$$\mathrm{Bun}_G \xleftarrow{\mathfrak{q}_{\mathcal{W}}} \mathcal{W}_{G,\tilde{Q}} \xrightarrow{\mathfrak{p}_{\mathcal{W}}} \mathcal{S}_{\tilde{Q}}.$$

Here $\mathfrak{q}_{\mathcal{W}}$ is the projection. From the definitions one obtains the following.

Lemma 4. *There is a canonical isomorphism of functors* $F_{\tilde{Q},\psi} \widetilde{\to} \mathrm{Four}_\psi \circ F_{\mathcal{S}_{\tilde{Q}}}$ *from* $\mathrm{D}(\mathrm{Bun}_G)$ *to* $\mathrm{D}(\mathrm{Bun}_{\tilde{Q}})$.

3.5 Weil representation and two-sheeted coverings

Write $\mathrm{Bun}_{G_n,\tilde{X}}$ for the stack classifying rank-$2n$ vector bundles W on $\tilde{X}$ with symplectic form $\bigwedge^2 W \to \Omega_{\tilde{X}}$. Let $\pi_n : \mathrm{Bun}_{G_n,\tilde{X}} \to \mathrm{Bun}_{G_{2n}}$ be the map sending the above point to $\pi_* W$ equipped with natural symplectic form $\bigwedge^2(\pi_* W) \to \Omega$. Denote by $\mathcal{A}_{G_n,\tilde{X}}$ the line bundle on $\mathrm{Bun}_{G_n,\tilde{X}}$ with fibre $\det \mathrm{R}\Gamma(\tilde{X}, W)$ at W. Since $\pi_n^* \mathcal{A}_{G_{2n}} \widetilde{\to} \mathcal{A}_{G_n,\tilde{X}}$ canonically, π_n lifts to a map

$$\tilde{\pi}_n : \widetilde{\mathrm{Bun}}_{G_n,\tilde{X}} \to \widetilde{\mathrm{Bun}}_{G_{2n}}.$$

Proposition 3. *There is a canonical isomorphism* ${}_{\tilde{X}}\mathrm{Aut} \widetilde{\to} \tilde{\pi}_n^* \mathrm{Aut}[\dim.\mathrm{rel}(\tilde{\pi}_n)]$ *preserving the generic and special parts.*

Proof. Let ${}_i\mathrm{Bun}_{G_k} \subset \mathrm{Bun}_{G_k}$ be the locally closed substack given by $\dim \mathrm{H}^0(X, M) = i$ for $M \in \mathrm{Bun}_{G_k}$. Let ${}_i\widetilde{\mathrm{Bun}}_{G_k}$ be the restriction of the μ_2-gerbe $\widetilde{\mathrm{Bun}}_{G_k} \to \mathrm{Bun}_{G_k}$ to ${}_i\mathrm{Bun}_{G_k}$. As in [Lys06a, Remark 1], we have a cartesian square, where the vertical arrows are canonical sections of the corresponding μ_2-gerbes.

$$\begin{array}{ccc}
{}_i\mathrm{Bun}_{G_n,\tilde{X}} & \xrightarrow{\pi_n} & {}_i\mathrm{Bun}_{G_{2n}} \\
\downarrow & & \downarrow \\
{}_i\widetilde{\mathrm{Bun}}_{G_n,\tilde{X}} & \xrightarrow{\tilde{\pi}_n} & {}_i\widetilde{\mathrm{Bun}}_{G_{2n}}
\end{array}$$

This gives a canonical normalization of the sought-for isomorphism over ${}_i\widetilde{\mathrm{Bun}}_{G_n,\tilde{X}}$ for $i = 0, 1$. It remains to show its existence.

To do so, consider the commutative diagram

$$\begin{array}{ccc}
\mathrm{Bun}_{P_n,\tilde{X}} & \xrightarrow{\tilde{\nu}_{n,\tilde{X}}} & \widetilde{\mathrm{Bun}}_{G_n,\tilde{X}} \\
\downarrow{\scriptstyle \pi_{n,P}} & & \downarrow{\scriptstyle \tilde{\pi}_n} \\
\mathrm{Bun}_{P_{2n}} & \xrightarrow{\tilde{\nu}_{2n}} & \widetilde{\mathrm{Bun}}_{G_{2n}}
\end{array}$$

where we denote by $\pi_{n,P}$ the following map. Given an exact sequence on $\tilde{X}$

$$0 \to \mathrm{Sym}^2 L \to ? \to \Omega_{\tilde{X}} \to 0, \qquad (19)$$

summate it with the sequence obtained by applying σ^*. The resulting exact sequence

$$0 \to \mathrm{Sym}^2(L \oplus \sigma^* L) \to ? \to \Omega_{\tilde{X}} \to 0$$

is equipped with descent data for $\tilde{X} \xrightarrow{\pi} X$, so yields an exact sequence $0 \to \mathrm{Sym}^2(\pi_* L) \to ? \to \Omega \to 0$, which is the image of (19) by $\pi_{n,P}$.

Let ${}_c\mathrm{Bun}_{n,\tilde{X}} \subset \mathrm{Bun}_{n,\tilde{X}}$ be the open substack classifying $W \in \mathrm{Bun}_{n,\tilde{X}}$ with $\mathrm{H}^0(\tilde{X}, W) = 0$. Let ${}_c\mathcal{V}_{n,\tilde{X}} \to {}_c\mathrm{Bun}_{n,\tilde{X}}$ be the vector bundle with fibre $\mathrm{Hom}(W, \Omega_{\tilde{X}})$ at W. Write ${}_c\mathrm{Bun}_{P_n,\tilde{X}}$ for the

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preimage of ${}_c\mathrm{Bun}_{n,\tilde X}$ under the projection $\mathrm{Bun}_{P_n,\tilde X} \to \mathrm{Bun}_{n,\tilde X}$. We obtain a commutative diagram

$$\begin{array}{ccc} \mathrm{Sym}^2\, {}_c\mathcal{V}^*_{n,\tilde X} & \longleftarrow & {}_c\mathrm{Bun}_{P_n,\tilde X} \\ \downarrow{\scriptstyle \pi_{\mathcal{V}}} & & \downarrow{\scriptstyle \pi_{n,P}} \\ \mathrm{Sym}^2\, {}_c\mathcal{V}^*_{2n} & \longleftarrow & {}_c\mathrm{Bun}_{P_{2n}} \end{array}$$

where the horizontal arrows are those of [Lys06a, § 5.2]. Here $\pi_{\mathcal{V}}$ is the map sending $L \in \mathrm{Bun}_{n,\tilde X}$, $b \in \mathrm{Sym}^2 H^1(\tilde X, L)$ to $\pi_* L \in \mathrm{Bun}_{2n}, b \in \mathrm{Sym}^2 \mathrm{H}^1(X, \pi_* L)$.

By [Lys06a, Proposition 7], it suffices to show that over ${}_c\mathrm{Bun}_{P_n,\tilde X}$ there is an isomorphism

$$\pi^*_{n,P} S_{P,\psi}[\dim.\mathrm{rel}(\pi_{n,P})] \widetilde{\to} S_{P,\psi}.$$

We have the sheaves S_ψ on $\mathrm{Sym}^2\, {}_c\mathcal{V}^*_{n,\tilde X}$ and on $\mathrm{Sym}^2\, {}_c\mathcal{V}^*_{2n}$ defined in [Lys06a, § 4.3]. Since $\pi^*_{\mathcal{V}} S_\psi \widetilde{\to} S_\psi$ up to a shift, our assertion follows from [Lys06a, § 5.2]. □

3.6 Whittaker-type functors

3.6.1 Write $\widetilde{\mathrm{Bun}}_P$ for the Drinfeld compactification of Bun_P introduced in [BG02, § 1.3.6]. So, $\widetilde{\mathrm{Bun}}_P$ classifies $(M, \mathcal{A}) \in \mathrm{Bun}_G$ together with a Lagrangian subsheaf $L \subset M$, $L \in \mathrm{Bun}_n$. Then $\mathrm{Bun}_P \subset \widetilde{\mathrm{Bun}}_P$ is the open substack given by the condition that L is a subbundle of M.

In the spirit of [Lys06b, § 7], let $\mathcal{Z}_1$ denote the stack obtained from $\mathrm{Bun}_{G,\tilde H}$ by the base change $\widetilde{\mathrm{Bun}}_P \to \mathrm{Bun}_G$. Let $\nu_{\mathcal{Z}} : \mathcal{Z}_1 \to \mathrm{Bun}_{G,\tilde H}$ be the projection.

Denote by $\pi_{2,1} : \mathcal{Z}_2 \to \mathcal{Z}_1$ the stack over $\mathcal{Z}_1$ with fibre consisting of all maps $s : \mathrm{Sym}^2 L \to \mathcal{A} \otimes \Omega$. A version of [Lys06b, Theorem 3] holds. Namely, one defines a Whittaker category $\mathrm{D}^W(\mathcal{Z}_2)$ as in [Lys06b, § 2.10]. We now give its description on strata.

For $d \geqslant 0$ let ${}^{\leqslant d}\mathcal{Z}_1 \subset \mathcal{Z}_1$ be the closed substack given by the condition that $\bigwedge^n L \hookrightarrow \bigwedge^n M$ has zeros of order at most d. Its open substack ${}^d\mathcal{Z}_1 \subset {}^{\leqslant d}\mathcal{Z}_1$ is given by the following: there is a subbundle $L' \subset M$ such that $L \subset L'$ is a subsheaf with $d = \deg(L'/L)$. Then ${}^{\leqslant d}\mathcal{Z}_1$ is stratified by ${}^i\mathcal{Z}_1$ for $0 \leqslant i \leqslant d$.

The stack ${}^d\mathcal{Z}_1$ classifies collections: a modification of rank-n vector bundles $L \subset L'$ on X with $\deg(L'/L) = d$, $\mathcal{A} \in \mathrm{Pic}\, X$, and an exact sequence $0 \to \mathrm{Sym}^2 L' \to ? \to \mathcal{A} \to 0$ on X, $(V, \mathcal{C}, \mathrm{Sym}^2 V \to \mathcal{C}, \gamma) \in \mathrm{Bun}_{\tilde H}$ with $\mathcal{C} \otimes \mathcal{A} \widetilde{\to} \Omega$.

Set

$${}^d\mathcal{Z}_2 = \mathcal{Z}_2 \times_{\mathcal{Z}_1} {}^d\mathcal{Z}_1 \quad \text{and} \quad {}^{\leqslant d}\mathcal{Z}_2 = \mathcal{Z}_2 \times_{\mathcal{Z}_1} {}^{\leqslant d}\mathcal{Z}_1.$$

Let ${}^d\mathcal{Z}'_2 \hookrightarrow {}^d\mathcal{Z}_2$ be the closed substack given by the following condition: s factors as

$$\mathrm{Sym}^2 L \hookrightarrow \mathrm{Sym}^2 L' \to \mathcal{A} \otimes \Omega.$$

Let ${}^d\chi : {}^d\mathcal{Z}'_2 \to \mathbb{A}^1$ be the pairing of s with the extension $0 \to \mathrm{Sym}^2 L' \to ? \to \mathcal{A} \to 0$.

Let ${}^d\mathcal{P}_2$ be the stack classifying $(V, \mathcal{C}, \mathrm{Sym}^2 V \to \mathcal{C}, \gamma) \in \mathrm{Bun}_{\tilde H}$, a modification of rank-n vector bundles $L \subset L'$ on X with $d = \deg(L'/L)$, and a section $s : \mathrm{Sym}^2 L' \to \mathcal{A} \otimes \Omega$ with $\mathcal{C} \otimes \mathcal{A} \widetilde{\to} \Omega$. The projection

$$\phi_2 : {}^d\mathcal{Z}'_2 \to {}^d\mathcal{P}_2$$

is smooth.

Lemma 5. *Any object of $\mathrm{D}^W({}^d\mathcal{Z}_2)$ is the extension by zero from ${}^d\mathcal{Z}'_2$. The functor*

$${}^dJ(K) = {}^d\chi^* \mathcal{L}_\psi \otimes \phi_2^* K[\dim.\mathrm{rel}]$$

provides an equivalence of categories ${}^dJ : \mathrm{D}({}^d\mathcal{P}_2) \to \mathrm{D}^W({}^d\mathcal{Z}_2)$ *and is exact for the perverse t-structures. Here* dim. rel *is the relative dimension of* ϕ_2.

Proposition 4. *There is an equivalence of categories* $W_{12} : \mathrm{D}(\mathcal{Z}_1) \widetilde{\to} \mathrm{D}^W(\mathcal{Z}_2)$, *which is exact for the perverse t-structures, and* $(\pi_{2,1})_!$ *is quasi-inverse to it. Moreover, for any* $K \in \mathrm{D}^W(\mathcal{Z}_2)$ *the natural map* $(\pi_{2,1})_! K \to (\pi_{2,1})_* K$ *is an isomorphism.*

3.6.2 We have naturally ${}^0\mathcal{P}_2 \widetilde{\to} \mathrm{Bun}_{\tilde{H}} \times_{\mathrm{Pic}\, X} \mathcal{S}_P$. Note that ${}^d\mathcal{Z}_1$ is of codimension d in $\mathcal{Z}_1$. Define the complex ${}^d\mathcal{T} \in \mathrm{D}({}^d\mathcal{P}_2)$ by

$$ {}^d\mathcal{T} = \mathrm{pr}^*({}^0\mathcal{T})[d + nd], $$

where $\mathrm{pr} : {}^d\mathcal{P}_2 \to {}^0\mathcal{P}_2$ is the projection forgetting L (the relative dimension of pr is nd). Set also ${}^dK_2 = {}^dJ({}^d\mathcal{T})$. Let S_2 act on ${}^dK_2$ via its action on ${}^0\mathcal{T}$. Using (6), for the $*$-restriction we obtain

$$ W_{12}(\nu_{\mathcal{Z}}^* \tilde{\tau}^* \mathrm{Aut})|_{{}^d\mathcal{Z}_2}[\dim.\mathrm{rel}(\tilde{\tau} \circ \nu_{\mathcal{Z}})] \widetilde{\to} {}^dK_2. $$

4. The pair $\mathbf{GL_2}$, $\mathbf{G\mathbb{O}_2}$

4.1 Keep the notation of § 3 assuming $n = 1, m = 1$. So, $G = \mathbb{G} = \mathrm{GL}_2$. In § 4 we assume that $\tilde{X}$ is connected.[3] Identify $\mathbb{H} = \mathrm{G}\mathbb{O}_2^0$ with $\mathbb{G}_m \times \mathbb{G}_m$ in such a way that the automorphism σ of $\mathbb{H}$ permutes the two copies of $\mathbb{G}_m$, so $\tilde{H} = \pi_* \mathbb{G}_m$. We have canonically $\Lambda_{\mathbb{H}} \widetilde{\to} \mathbb{Z}^2$, and σ sends a coweight $\mu = (\mu_1, \mu_2)$ to $\sigma\mu = (\mu_2, \mu_1)$.

We have a canonical isomorphism $\mathrm{Pic}\, \tilde{X} \widetilde{\to} \mathrm{Bun}_{\tilde{H}}$ sending $\mathcal{B} \in \mathrm{Pic}\, \tilde{X}$ to $V = \pi_* \mathcal{B}$, $\mathcal{C} = N(\mathcal{B})$ equipped with natural symmetric form $\mathrm{Sym}^2 V \to \mathcal{C}$ and isomorphism $\gamma : \mathcal{C} \widetilde{\to} \mathcal{E} \otimes \det V$. Here $N : \mathrm{Pic}\, \tilde{X} \to \mathrm{Pic}\, X$ is the norm map (cf. Appendix A.1). We also write σ for the map $\mathrm{Pic}\, \tilde{X} \to \mathrm{Pic}\, \tilde{X}$ sending $\mathcal{B}$ to $\sigma^* \mathcal{B}$. The following diagram is 2-commutative.

$$ \begin{array}{ccc} \mathrm{Pic}\, \tilde{X} & \xrightarrow{\sigma} & \mathrm{Pic}\, \tilde{X} \\ \downarrow{\scriptstyle \rho_H} & \swarrow{\scriptstyle \rho_H} & \\ \mathrm{Bun}_H & & \end{array} $$

Write $X^{(d)}$ for the dth symmetric power of X, we also view it as the scheme classifying effective divisors on X of degree d.

Let $\mathrm{Pic}^d \tilde{X}$ be the connected component of $\mathrm{Pic}\, \tilde{X}$ classifying $\mathcal{L} \in \mathrm{Pic}\, \tilde{X}$ with $\deg \mathcal{L} = d$. Write $\mathrm{Pic}'^d \tilde{X}$ for the stack classifying $\mathcal{L} \in \mathrm{Pic}^d \tilde{X}$ with a section $\mathcal{O} \to \mathcal{L}$. One defines $\mathrm{Pic}'^d X$ similarly. Let

$$ \pi'_{ex} : \mathrm{Pic}'^d \tilde{X} \to \mathrm{Pic}'^d X \times_{\mathrm{Pic}\, X} \mathrm{Pic}\, \tilde{X} $$

be the map sending $(\mathcal{L} \in \mathrm{Pic}^d \tilde{X}, \mathcal{O} \xrightarrow{t} \mathcal{L})$ to $(\mathcal{L}, \mathcal{O} \xrightarrow{Nt} N(\mathcal{L}))$. The group S_2 acts on π'_{ex} sending $(\mathcal{L}, t)$ to $(\mathcal{L}, -t)$. We have an open immersion $X^{(d)} \hookrightarrow \mathrm{Pic}'^d X$ corresponding to nonzero sections. Restricting π'_{ex} to the corresponding open substacks, one obtains a map

$$ \pi' : \tilde{X}^{(d)} \to X^{(d)} \times_{\mathrm{Pic}\, X} \mathrm{Pic}\, \tilde{X} $$

By abuse of notation, denote also by π the direct image map $\pi : \tilde{X}^{(d)} \to X^{(d)}$.

Lemma 6. *We have the following.*

(i) *For any local system* $\tilde{E}$ *on* $\tilde{X}$ *we have* $\pi_* \tilde{E}^{(d)} \widetilde{\to} (\pi_* \tilde{E})^{(d)}$. *For a rank-one local system* χ *on* X *we have* $\pi^*(\chi^{(d)}) \widetilde{\to} (\pi^* \chi)^{(d)}$.

[3] Some of our results extend to the case of non-connected $\tilde{X}$, but this case reduces to the study of renormalized geometric Eisenstein series from [BG02].

(ii) *For each* $d \geqslant 0$ *both the* S_2*-invariants and anti-invariants in* $\pi'_!\bar{\mathbb{Q}}_\ell[d]$ *are irreducible perverse sheaves. If* $d > 2\tilde{g} - 2$*, then the same holds for* $(\pi'_{ex})_!\bar{\mathbb{Q}}_\ell[d]$.

Proof. (ii) The map π'_{ex} is finite. Write $\underline{\mathrm{Pic}}\,\tilde{X}$ for the Picard scheme of $\tilde{X}$ and similarly for X. We have a μ_2-gerbe

$$\underline{\mathfrak{r}} : X^{(d)} \times_{\mathrm{Pic}\, X} \mathrm{Pic}\,\tilde{X} \to X^{(d)} \times_{\underline{\mathrm{Pic}}\, X} \underline{\mathrm{Pic}}\,\tilde{X}.$$

Step 1. Let us show that the S_2-invariants in $\pi'_!\bar{\mathbb{Q}}_\ell[d]$ is an irreducible perverse sheaf. It suffices to show that $\underline{\mathfrak{r}}_!\pi'_!\bar{\mathbb{Q}}_\ell[d]$ is an irreducible perverse sheaf.

Let Z denote the image of the (finite) map $\underline{\mathfrak{r}}\circ\pi'$ (with reduced scheme structure). The projection $p_Z : Z \to X^{(d)}$ is a finite map. Take a rank-one local system $\tilde{E}$ on $\tilde{X}$ that does not descend to X. Let $A\tilde{E}$ denote the corresponding automorphic local system on $\mathrm{Pic}\,\tilde{X}$ (cf. Definition 2). Since $(p_Z)_!$ sends a nonzero perverse sheaf to a nonzero perverse sheaf, it suffices to show that

$$(p_Z)_!((\underline{\mathfrak{r}}_!\pi'_!\bar{\mathbb{Q}}_\ell[d]) \otimes (\bar{\mathbb{Q}}_\ell \boxtimes A\tilde{E}))$$

is an irreducible perverse sheaf. However, the latter identifies with $(\pi_!\tilde{E})^{(d)}[d]$, so is irreducible.

Step 2. Let η_Z (respectively, η) denote the generic point of Z (respectively, of $\tilde{X}^{(d)}$). From Step 1 it follows that the map $\underline{\mathfrak{r}}\circ\pi'$ yields an isomorphism $\eta \widetilde{\to} \eta_Z$. So, the restriction of the μ_2-gerbe $\underline{\mathfrak{r}}$ to η_Z is trivial. For any map $\xi = (\mathrm{id}, \xi') : \eta \to \eta \times B(\mu_2)$, the S_2-anti-invariants in $\xi_!\bar{\mathbb{Q}}_\ell$ is an irreducible local system. It follows that the S_2-anti-invariants in $\pi'_!\bar{\mathbb{Q}}_\ell[d]$ is an irreducible perverse sheaf.

Step 3. For $d > 2\tilde{g} - 2$ the stack $\mathrm{Pic}'^d\,\tilde{X}$ is smooth. Since $(\pi'_{ex})_!\bar{\mathbb{Q}}_\ell[d]$ is the Goresky–MacPherson extension from $X^{(d)} \times_{\mathrm{Pic}\, X} \mathrm{Pic}\,\tilde{X}$, from Steps 1 and 2 we learn that both the S_2-invariants and anti-invariants in $(\pi'_{ex})_!\bar{\mathbb{Q}}_\ell[d]$ are irreducible perverse sheaves. □

For the map $\tilde{\tau} : \mathrm{Bun}_{G,\tilde{H}} \to \widetilde{\mathrm{Bun}}_{G_2}$ set

$$\mathrm{Aut}_{G,\tilde{H},g} = \tilde{\tau}^*\,\mathrm{Aut}_g[\dim.\mathrm{rel}(\tilde{\tau})] \quad \text{and} \quad \mathrm{Aut}_{G,\tilde{H},s} = \tilde{\tau}^*\,\mathrm{Aut}_s[\dim.\mathrm{rel}(\tilde{\tau})]$$

and $\mathrm{Aut}_{G,\tilde{H}} = \mathrm{Aut}_{G,\tilde{H},g} \oplus \mathrm{Aut}_{G,\tilde{H},s}$.

PROPOSITION 5. *We have the following.*

(i) *Both* $\mathrm{Aut}_{G,\tilde{H},g}$ *and* $\mathrm{Aut}_{G,\tilde{H},s}$ *are irreducible perverse sheaves, and we have* $\mathbb{D}(\mathrm{Aut}_{G,\tilde{H}}) \widetilde{\to} \mathrm{Aut}_{G,\tilde{H}}$ *canonically.*

(ii) *The sheaf* $\mathrm{Aut}_{G,\tilde{H}}$ *is ULA with respect to* $\mathrm{Bun}_{G,\tilde{H}} \to \mathrm{Bun}_{\tilde{H}}$.

Proof. (i) Consider the map $\mathrm{Bun}_{P,\tilde{H}} \to \mathrm{Bun}_{G,\tilde{H}}$ obtained from $\mathrm{Bun}_P \to \mathrm{Bun}_G$ by base change $\mathrm{Bun}_{G,\tilde{H}} \to \mathrm{Bun}_G$. Let ${}^0\mathrm{Bun}_{P,\tilde{H}}$ be the open substack of $\mathrm{Bun}_{P,\tilde{H}}$ given by $2\deg L + \deg \mathcal{C} < 0$. We have a commutative diagram

$$\begin{array}{ccc} \mathrm{Bun}_{G,\tilde{H}} & \xrightarrow{\tau} & \mathrm{Bun}_{G_2} \\ \uparrow & & \uparrow \scriptstyle{\nu_2} \\ {}^0\mathrm{Bun}_{P,\tilde{H}} & \xrightarrow{\tau_P} & {}^0\mathrm{Bun}_{P_2} \end{array}$$

where the vertical arrows are smooth and surjective. By [Lys06a, Proposition 7] and Lemma 2, it suffices to show that both $\tau_P^* S_{P,\psi,g}[\dim.\mathrm{rel}]$ and $\tau_P^* S_{P,\psi,s}[\dim.\mathrm{rel}]$ are irreducible perverse sheaves over each connected component of ${}^0\mathrm{Bun}_{P,\tilde{H}}$.

Recall the following notation introduced in § 3.3.2. The stack $\mathcal{S}_P$ classifies $L \in \mathrm{Pic}\,X$, $\mathcal{A} \in \mathrm{Pic}\,X$ and a map $L^{\otimes 2} \to \mathcal{A} \otimes \Omega$. Let ${}^0\mathcal{S}_P \subset \mathcal{S}_P$ be the open substack given by $2\deg L - \deg \mathcal{A} + \deg \Omega < 0$.

The stack $\mathcal{V}_{\tilde{H},P}$ classifies $L \in \operatorname{Pic} X$, $\mathcal{B} \in \operatorname{Pic} \tilde{X}$ and $\pi^* L \xrightarrow{t} \mathcal{B}^\star$. (We may view t as a map $L \otimes V \to \Omega$ for $V = \pi_* \mathcal{B}$.) The map $\xi_{\mathcal{V}} : \mathcal{V}_{\tilde{H},P} \to \operatorname{Bun}_{\tilde{H}} \times_{\operatorname{Pic} X} \mathcal{S}_P$ sends $(L, \mathcal{B}, t)$ to $(L, \mathcal{B}, s)$, where $s : L^2 \to N(\mathcal{B}^\star)$ is the norm of t. Set

$$^0\mathcal{V}_{\tilde{H},P} = \xi_{\mathcal{V}}^{-1}(\operatorname{Bun}_{\tilde{H}} \times_{\operatorname{Pic} X} {}^0\mathcal{S}_P).$$

Note that $^0\mathcal{V}_{\tilde{H},P}$ is smooth (here the connectedness of $\tilde{X}$ is essentially used).

By definition, $\tau_P^* S_{P,\psi}[\dim.\mathrm{rel}]$ is the Fourier transform of $(\xi_{\mathcal{V}})_! \bar{\mathbb{Q}}_\ell[\dim \mathcal{V}_{\tilde{H},P}]$. From Lemma 6(ii) it follows that $(\xi_{\mathcal{V}})_! \bar{\mathbb{Q}}_\ell[\dim \mathcal{V}_{\tilde{H},P}]$ is a direct sum of two irreducible perverse sheaves over $\operatorname{Bun}_{\tilde{H}} \times_{\operatorname{Pic} X} {}^0\mathcal{S}_P$. We are done.

(ii) We need the following general observation. If $f : Y \to S$ is a vector bundle over a smooth base S and $K \in \mathrm{D}(Y)$ is ULA with respect to f, then $\operatorname{Four}_\psi(K)$ is ULA with respect to the projection $\check{Y} \to S$.

Apply this to the vector bundle $v : \operatorname{Bun}_{\tilde{H}} \times_{\operatorname{Pic} X} {}^0\mathcal{S}_P \to {}^0(\operatorname{Bun}_{\tilde{H}} \times_{\operatorname{Pic} X} \operatorname{Bun}_{M_P})$, where the base classifies pairs $\mathcal{B} \in \operatorname{Pic} \tilde{X}, L \in \operatorname{Pic} X$ with $2 \deg L + \deg \mathcal{B} < 0$.

Since the projection $^0\mathcal{V}_{\tilde{H},P} \to {}^0(\operatorname{Bun}_{\tilde{H}} \times_{\operatorname{Pic} X} \operatorname{Bun}_{M_P})$ is smooth, $(\xi_{\mathcal{V}})_! \bar{\mathbb{Q}}_\ell$ is ULA with respect to v, so its Fourier transform is also ULA over $^0(\operatorname{Bun}_{\tilde{H}} \times_{\operatorname{Pic} X} \operatorname{Bun}_{M_P})$.

Since $^0\operatorname{Bun}_{P,\tilde{H}} \to \operatorname{Bun}_{G,\tilde{H}}$ is smooth and surjective, our assertion follows (the ULA property is local in the smooth topology on the source). □

For our particular pair $(\tilde{H}, G)$ the projection $\operatorname{Bun}_{G,\tilde{H}} \to \operatorname{Bun}_G$ is proper (this phenomenon does not happen for $m > 1$). So, Proposition 5 implies that F_G commutes with the Verdier duality.

4.2 Hecke property

4.2.1.1 For a dominant coweight λ of G write $\overline{\mathcal{H}}_G^\lambda$ for the Hecke stack classifying $x \in X, M, M' \in \operatorname{Bun}_G$ with an isomorphism $\beta : M \widetilde{\to} M'|_{X-x}$ such that M' is in a position $\leqslant \lambda$ with respect to M in the sense of [BG02]. We have a diagram

$$X \times \operatorname{Bun}_G \xleftarrow{\operatorname{supp} \times p_G} \overline{\mathcal{H}}_G^\lambda \xrightarrow{p'_G} \operatorname{Bun}_G,$$

where p_G (respectively, p'_G) sends (x, M, M', β) to M (respectively, M').

We fix an inclusion $\check{\mathbb{H}} \hookrightarrow \check{\mathbb{G}} = \mathrm{GL}_2$ as the maximal torus of diagonal matrices. This yields isomorphisms $\Lambda_{\mathbb{H}} \widetilde{\to} \Lambda_{\mathbb{G}} \widetilde{\to} \mathbb{Z}$. Given a coweight $\mu = (\mu_1, \mu_2) \in \Lambda_{\mathbb{H}}$ such that $\lambda - \mu$ vanishes in $\pi_1(G)$ (that is, $\lambda_1 + \lambda_2 = \mu_1 + \mu_2$), consider the diagram

$$\begin{array}{ccccc}
\tilde{X} \times \operatorname{Bun}_{\tilde{H},G} & \xleftarrow{\operatorname{supp} \times p_{\tilde{H},G}} & \overline{\mathcal{H}}^{\mu,\lambda}_{\tilde{H},G} & \xrightarrow{p'_{\tilde{H},G}} & \operatorname{Bun}_{\tilde{H},G} \\
\downarrow & & \downarrow & & \\
X \times \operatorname{Bun}_G & \xleftarrow{\operatorname{supp} \times p_G} & \overline{\mathcal{H}}^\lambda_G & &
\end{array}$$

where $\overline{\mathcal{H}}^{\mu,\lambda}_{\tilde{H},G}$ classifies collections: $(\mathcal{B}, M) \in \operatorname{Bun}_{\tilde{H},G}$, $\tilde{x} \in \tilde{X}$ for which we set $x = \pi(\tilde{x})$ and $(x, M, M', \beta) \in \overline{\mathcal{H}}^\lambda_G$. The map $p_{\tilde{H},G}$ forgets (β, M'), so the left square is cartesian. The map $p'_{\tilde{H},G}$ sends the above collection to $(\mathcal{B}', M') \in \operatorname{Bun}_{\tilde{H},G}$, where $\mathcal{B}' = \mathcal{B}(\mu_1 \tilde{x} + \mu_2 \sigma(\tilde{x}))$.

The Hecke functor $\mathrm{H}^{\mu,\lambda} : \mathrm{D}(\operatorname{Bun}_{\tilde{H},G}) \to \mathrm{D}(\tilde{X} \times \operatorname{Bun}_{\tilde{H},G})$ is given by

$$\mathrm{H}^{\mu,\lambda}(K) = (\operatorname{supp} \times p_{\tilde{H},G})_! (\mathrm{IC}_{\overline{\mathcal{H}}^{\mu,\lambda}_{\tilde{H},G}} \otimes (p'_{\tilde{H},G})^* K)[-\dim \operatorname{Bun}_{\tilde{H},G}]. \tag{20}$$

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We have

$$\mathrm{pr}^* \, \mathrm{IC}_{\overline{\mathcal{H}}^{\lambda}_{G}}[\dim.\mathrm{rel}] \widetilde{\to} \mathrm{IC}_{\overline{\mathcal{H}}^{\mu,\lambda}_{\tilde{H},G}},$$

where $\dim.\mathrm{rel} = \dim \mathrm{Bun}_{G,\tilde{H}} - \dim \mathrm{Bun}_G$. By Appendix B.1.3, $\mathrm{H}^{\mu,\lambda}$ commutes with the Verdier duality. As in Appendix B.1.2, we have a canonical isomorphism

$$\delta_\sigma : (\sigma \times \mathrm{id})^* \circ \mathrm{H}^{\mu,\lambda} \widetilde{\to} \mathrm{H}^{\sigma\mu,\lambda} \tag{21}$$

(we used that σ acts trivially on the dominant coweights of G).

Write V^λ for the irreducible representation of $\check{\mathbb{G}}$ with highest weight λ. For a $\check{\mathbb{G}}$-representation V and $\mu \in \Lambda_{\mathbb{H}}$ denote by $V(\mu)$ the μ-weight space of $\check{\mathbb{H}}$ in V.

Theorem 1. *For $\lambda \in \Lambda^+_{\mathbb{G}}, \mu \in \Lambda_{\mathbb{H}}$ such that $\lambda - \mu$ vanishes in $\pi_1(G)$ the sheaf $\mathrm{H}^{\mu,\lambda}(\mathrm{Aut}_{G,\tilde{H}})$ is perverse, and we have*

$$\mathrm{H}^{\mu,\lambda}(\mathrm{Aut}_{G,\tilde{H}}) \widetilde{\to} \bigoplus_{\nu} \mathrm{H}^{\nu,0}(\mathrm{Aut}_{G,\tilde{H}}) \otimes (V^\lambda)^*(\nu - \mu). \tag{22}$$

In other words, the sum (without multiplicities) is over the coweights $\nu = (-a, a)$ such that $\lambda_1 - \mu_1 \geqslant a \geqslant \mu_2 - \lambda_1$. If $\lambda_1 - \lambda_2$ is even, then this isomorphism preserves generic and special parts, otherwise it interchanges them.

Remark 4. The isomorphism (22) is compatible with the action of $\Sigma = \{1, \sigma\}$ on both sides, that is, the following diagram commutes.

$$\begin{array}{ccc}
(\sigma \times \mathrm{id})^* \mathrm{H}^{\mu,\lambda}(\mathrm{Aut}_{G,\tilde{H}}) & \xrightarrow{\sim} & (\sigma \times \mathrm{id})^* (\bigoplus_\nu \mathrm{H}^{\nu,0}(\mathrm{Aut}_{G,\tilde{H}}) \otimes (V^\lambda)^*(\nu - \mu)) \\
\downarrow \delta_\sigma & & \downarrow \delta_\sigma \\
\mathrm{H}^{\sigma\mu,\lambda}(\mathrm{Aut}_{G,\tilde{H}}) & \xrightarrow{\sim} & \bigoplus_\nu \mathrm{H}^{\sigma\nu,0}(\mathrm{Aut}_{G,\tilde{H}}) \otimes (V^\lambda)^*(\sigma\nu - \sigma\mu)
\end{array}$$

4.2.1.2 Recall that $\mathrm{Bun}_{\tilde{H},P}$ denotes the stack obtained from $\mathrm{Bun}_{\tilde{H},G}$ by the base change $\mathrm{Bun}_P \stackrel{\nu_P}{\to} \mathrm{Bun}_G$. Recall the stack ${}^{\leqslant d}\mathcal{Z}_1$ (cf. § 3.6.1). We have a commutative diagram,

$$\begin{array}{ccccc}
\tilde{X} \times \mathrm{Bun}_{\tilde{H},P} & \xleftarrow{\mathrm{supp} \times p_{\tilde{H},P}} & \overline{\mathcal{H}}^{\mu,\lambda}_{\tilde{H},P} & \xrightarrow{p'_{\tilde{H},P}} & {}^{\leqslant \lambda_1 - \lambda_2}\mathcal{Z}_1 \\
\downarrow \mathrm{id} \times \nu_P & & \downarrow & & \downarrow \nu_{\mathcal{Z}} \\
\tilde{X} \times \mathrm{Bun}_{\tilde{H},G} & \xleftarrow{\mathrm{supp} \times p_{\tilde{H},G}} & \overline{\mathcal{H}}^{\mu,\lambda}_{\tilde{H},G} & \xrightarrow{p'_{\tilde{H},G}} & \mathrm{Bun}_{\tilde{H},G}
\end{array}$$

where the left square is cartesian, thus defining $\overline{\mathcal{H}}^{\mu,\lambda}_{\tilde{H},P}$, and the map $p'_{\tilde{H},P}$ sends $(\tilde{x}, \mathcal{B}, L \hookrightarrow M, \beta : M \widetilde{\to} M' \mid_{X-x})$ to $(\mathcal{B}', L(-\lambda_1 x) \hookrightarrow M')$. Here $\mathcal{B}' = \mathcal{B}(\mu_1 \tilde{x} + \mu_2 \sigma(\tilde{x}))$. Write also $\mathcal{A}' = \det M' \widetilde{\to} \mathcal{A}(-(\lambda_1 + \lambda_2)x)$ and $L' = L(-\lambda_1 x)$.

Define the Hecke functor

$$\mathrm{H}^{\mu,\lambda} : \mathrm{D}({}^{\leqslant \lambda_1 - \lambda_2}\mathcal{Z}_1) \to \mathrm{D}(\tilde{X} \times \mathrm{Bun}_{\tilde{H},P})$$

by

$$\mathrm{H}^{\mu,\lambda}(K) = (\mathrm{supp} \times p_{\tilde{H},P})_!(\mathrm{pr}^* \, \mathrm{IC}_{\overline{\mathcal{H}}^{\lambda}_{G}} \otimes (p'_{\tilde{H},P})^* K)[(\lambda_2 - \lambda_1) - \dim \mathrm{Bun}_G].$$

We normalize it so that (in view of Theorem 1) it should preserve perversity. The term $(\lambda_2 - \lambda_1)$ appears, because the dimension of $\mathrm{Bun}_{\tilde{H},P}$ depends on a connected component. So,

$$(\mathrm{id} \times \nu_P)^* \mathrm{H}^{\mu,\lambda}(K)[\dim.\mathrm{rel}(\nu_P)] \widetilde{\to} \mathrm{H}^{\mu,\lambda}(\nu_{\mathcal{Z}}^* K[\dim.\mathrm{rel}(\nu_{\mathcal{Z}})]).$$

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We have a commutative diagram

$$\begin{array}{ccccc}
\mathcal{Z}_2^{\leqslant \lambda_1-\lambda_2} & \xleftarrow{p_{\mathcal{Z}}} & \overline{\mathcal{H}}^{\mu,\lambda}_{\tilde{H},P} \times_{\mathcal{Z}_1} \mathcal{Z}_2 & \xrightarrow{p'_{\mathcal{Z}}} & {}^{\leqslant \lambda_1-\lambda_2}\mathcal{Z}_2 \\
\downarrow {\scriptstyle \pi_{2,1}} & & \downarrow & & \downarrow {\scriptstyle \pi_{2,1}} \\
\tilde{X} \times \mathrm{Bun}_{\tilde{H},P} & \xleftarrow{\mathrm{supp} \times p_{\tilde{H},P}} & \overline{\mathcal{H}}^{\mu,\lambda}_{\tilde{H},P} & \xrightarrow{p'_{\tilde{H},P}} & {}^{\leqslant \lambda_1-\lambda_2}\mathcal{Z}_1
\end{array}$$

where the right square is cartesian. Here $\mathcal{Z}_2^{\leqslant i}$ is the stack over $\tilde{X} \times \mathrm{Bun}_{\tilde{H},P}$ whose fibre over $(\tilde{x}, \mathcal{B}, 0 \to L \to M \to L^* \otimes \mathcal{A} \to 0)$ is $\mathrm{Hom}(L^2, \mathcal{A} \otimes \Omega(ix))$, where $x = \pi(\tilde{x})$. By definition, $p_{\mathcal{Z}}$ is the map that forgets (β, M').

Define the Hecke functor $\mathrm{H}^{\mu,\lambda} : \mathrm{D}({}^{\leqslant \lambda_1-\lambda_2}\mathcal{Z}_2) \to \mathrm{D}(\mathcal{Z}_2^{\leqslant \lambda_1-\lambda_2})$ by

$$\mathrm{H}^{\mu,\lambda}(K) = p_{\mathcal{Z}!}(\mathrm{pr}^* \mathrm{IC}_{\overline{\mathcal{H}}^{\lambda}_G} \otimes (p'_{\mathcal{Z}})^* K)[(\lambda_2 - \lambda_1) - \dim \mathrm{Bun}_G], \tag{23}$$

it commutes with the functor $(\pi_{2,1})_!$.

4.2.2 *Grothendieck group calculation.* Set $K_1 = \nu_{\mathcal{Z}}^* \mathrm{Aut}_{G,\tilde{H}}[\dim.\mathrm{rel}(\nu_{\mathcal{Z}})]$ and $K_2 = W_{12}(K_1)$. Recall that $\mathrm{Bun}_{\tilde{H},P}$ classifies $\mathcal{B} \in \mathrm{Pic}\,\tilde{X}$, $L \in \mathrm{Pic}\,X$, and an exact sequence $0 \to L^2 \to ? \to \mathcal{A} \to 0$ on X with $\mathcal{C} = N(\mathcal{B})$ and $\mathcal{A} = \Omega \otimes \mathcal{C}^{-1}$. Let ${}_c\mathrm{Bun}_{\tilde{H},P} \subset \mathrm{Bun}_{\tilde{H},P}$ be the open substack given by

$$2 \deg L + \deg \mathcal{C} + 2(\lambda_1 - \lambda_2) < 0. \tag{24}$$

The projection ${}_c\mathrm{Bun}_{\tilde{H},P} \to \mathrm{Bun}_{\tilde{H},G}$ is smooth and surjective. We will derive Theorem 1 from a description of the complex

$$(\mathrm{id} \times \nu_P)^* \mathrm{H}^{\mu,\lambda}(\mathrm{Aut}_{G,\tilde{H}})[\dim.\mathrm{rel}(\nu_P)] \,\tilde{\to}\, \mathrm{H}^{\mu,\lambda}(K_1)$$

over $\tilde{X} \times {}_c\mathrm{Bun}_{\tilde{H},P}$. The latter follows from the Hecke property of K_2 for (23).

By equivariance, $\mathrm{H}^{\mu,\lambda}(K_2)$ is the extension by zero from the closed substack $\mathcal{Z}_2^{\leqslant 0} \hookrightarrow \mathcal{Z}_2^{\leqslant \lambda_1-\lambda_2}$. Write $\mathcal{Z}_2^i \subset \mathcal{Z}_2^{\leqslant i}$ for the open substack given by the condition that $L^2 \xrightarrow{s} \mathcal{A} \otimes \Omega(ix)$ does not have a zero at x. Let ${}_c\mathcal{Z}_2^{\leqslant i}$ be the preimage of $\tilde{X} \times {}_c\mathrm{Bun}_{\tilde{H},P}$ under $\pi_{2,1} : \mathcal{Z}_2^{\leqslant i} \to \tilde{X} \times \mathrm{Bun}_{\tilde{H},P}$. Set ${}_c\mathcal{Z}_2^i = {}_c\mathcal{Z}_2^{\leqslant i} \cap \mathcal{Z}_2^i$. Set also

$${}^dY^i = p_{\mathcal{Z}}^{-1}({}_c\mathcal{Z}_2^i) \cap (p'_{\mathcal{Z}})^{-1}({}^d\mathcal{Z}'_2).$$

Let $\mathcal{K}^i$ denote the $*$-restriction of $\mathrm{H}^{\mu,\lambda}(K_2)$ to ${}_c\mathcal{Z}_2^i$. We can similarly define the category $\mathrm{D}^W({}_c\mathcal{Z}_2^i)$, then $\mathcal{K}^i \in \mathrm{D}^W({}_c\mathcal{Z}_2^i)$.

Lemma 7. *The complex $\mathcal{K}^0$ (respectively, $\mathcal{K}^i$ for $i < 0$) is placed in non-positive (respectively, strictly negative) perverse degrees. The zeroth perverse cohomology of $\mathcal{K}^0$ identifies with*

$$\bigoplus_{\nu} \mathrm{H}^{\nu,0}({}^0K_2) \otimes (V^{\lambda})^*(\nu - \mu)|_{{}_c\mathcal{Z}_2^0}. \tag{25}$$

Proof. Denote by ${}^d\mathcal{K}^i$ the $*$-restriction of

$$(\mathrm{pr}^* \mathrm{IC}_{\overline{\mathcal{H}}^{\lambda}_G} \otimes (p'_{\mathcal{Z}})^*({}^dK_2))[(\lambda_2 - \lambda_1) - \dim \mathrm{Bun}_G] \tag{26}$$

to ${}^dY^i$ followed by the direct image $p_{\mathcal{Z}!}$. Let S_2 act on ${}^d\mathcal{K}^i$ via its action on ${}^dK_2$ (cf. § 3.6.2). We are reduced to the following lemma. $\square$

Lemma 8. *The complex ${}^d\mathcal{K}^i$ is placed in perverse degrees at most zero. The inequality is strict for all terms except ${}^0\mathcal{K}^0$. The zeroth perverse cohomology of ${}^0\mathcal{K}^0$ is S_2-equivariantly isomorphic to (25).*

Proof. Recall the following diagram.

$$\begin{array}{ccccc} {}^dY^i & \xrightarrow{p'_{\mathcal{Z}}} & {}^d\mathcal{Z}'_2 & \xrightarrow{{}^d\chi} & \mathbb{A}^1 \\ \downarrow{\scriptstyle p_{\mathcal{Z}}} & & \downarrow{\scriptstyle \phi_2} & & \\ {}_c\mathcal{Z}^i_2 & & {}^d\mathcal{P}_2 & & \end{array}$$

The scheme ${}^dY^i$ is empty unless $i \leqslant (\lambda_1 - \lambda_2) - 2d$.

Assume $i \leqslant (\lambda_1 - \lambda_2) - 2d$, then the map $p_{\mathcal{Z}} : {}^dY^i \to \mathcal{Z}^i_2$ can be seen as a (twisted) projection

$${}_c\mathcal{Z}^i_2 \,\tilde{\times}\, (\overline{\mathrm{Gr}}^{\lambda}_G \cap S^{\lambda'}) \to {}_c\mathcal{Z}^i_2$$

for $\lambda' = (\lambda_1 - d, \lambda_2 + d)$.

Recall that $\overline{\mathcal{H}}^{\lambda}_G$ is a twisted product $(X \times \mathrm{Bun}_G) \,\tilde{\times}\, \overline{\mathrm{Gr}}^{\lambda}_G$, where the projection to Bun_G corresponds to p_G. We have $\mathrm{IC}_{\overline{\mathcal{H}}^{\lambda}_G} \,\tilde{\to}\, \mathrm{IC}_{X \times \mathrm{Bun}_G} \boxtimes \mathcal{A}_{\lambda}$, where $\mathcal{A}_{\lambda} \in \mathrm{Sph}(\mathrm{Gr}_G)$ is the spherical sheaf on Gr_G corresponding to λ.

View ${}_c\mathcal{Z}^i_2$ as the stack classifying $\tilde{x} \in \tilde{X}$, $\mathcal{B} \in \mathrm{Pic}\,\tilde{X}$, an exact sequence $0 \to L \to M \to L^* \otimes \mathcal{A} \to 0$ on X, where $\mathcal{C} \otimes \mathcal{A} \,\tilde{\to}\, \Omega$ and (24) holds, and a section $s : L^2 \to \mathcal{A} \otimes \Omega(ix)$ that has no zero at x. Here $x = \pi(\tilde{x})$ and $\mathcal{C} = N(\mathcal{B})$.

View ${}^dY^i$ as the stack over ${}_c\mathcal{Z}^i_2$, whose fibre over the above point is the scheme of pairs (M', β) with $M' \in \mathrm{Bun}_G$ and $\beta : M \,\tilde{\to}\, M'|_{X-x}$ such that M' is in a position at most λ with respect to M, and $\bar{L} \subset M'$ is a subbundle. Here $\bar{L} = L((d - \lambda_1)x)$.

View ${}^d\mathcal{Z}'_2$ as the stack classifying $\mathcal{B}' \in \mathrm{Pic}\,\tilde{X}$, a modification $L' \subset \bar{L}$ of line bundles on X with $\deg(\bar{L}/L') = d$, an exact sequence $0 \to \bar{L}^2 \to ? \to \mathcal{A}' \to 0$, and a section $s : \bar{L}^2 \to \mathcal{A}' \otimes \Omega$. Here $N(\mathcal{B}') \otimes \mathcal{A}' \,\tilde{\to}\, \Omega$.

The map $p'_{\mathcal{Z}}$ sends the above collection to $\mathcal{B}' = \mathcal{B}(\mu_1\tilde{x} + \mu_2\sigma(\tilde{x}))$, $L(-\lambda_1 x) = L' \subset \bar{L} = L((d - \lambda_1)x)$, $s : \bar{L}^2 \to \mathcal{A}' \otimes \Omega$, and $0 \to \bar{L}^2 \to ? \to \mathcal{A}' \to 0$. Note that $\mathcal{A}' = \mathcal{A}(-(\lambda_1 + \lambda_2)x)$.

Now it is convenient to think of ${}^d\mathcal{P}_2$ as the stack classifying a modification $L' \subset \bar{L}$ of line bundles on X with $\deg(\bar{L}/L') = d$, $\mathcal{B}' \in \mathrm{Pic}\,\tilde{X}$, and a section $s : \bar{L}^2 \to \mathcal{A}' \otimes \Omega$, where $N(\mathcal{B}') \otimes \mathcal{A}' \,\tilde{\to}\, \Omega$.

Denote by $p'_{\mathcal{P}} : {}_c\mathcal{Z}^i_2 \to {}^d\mathcal{P}_2$ the map sending the above collection to $\mathcal{B}' = \mathcal{B}(\mu_1\tilde{x} + \mu_2\sigma(\tilde{x}))$, $L(-\lambda_1 x) = L' \subset \bar{L} = L((d - \lambda_1)x)$, and $s : \bar{L}^2 \to \mathcal{A}' \otimes \Omega$. It fits into the following commutative diagram.

$$\begin{array}{ccccc} {}^dY^i & \xrightarrow{p'_{\mathcal{Z}}} & {}^d\mathcal{Z}'_2 & \xrightarrow{{}^d\chi} & \mathbb{A}^1 \\ \downarrow{\scriptstyle p_{\mathcal{Z}}} & & \downarrow{\scriptstyle \phi_2} & & \\ {}_c\mathcal{Z}^i_2 & \xrightarrow{p'_{\mathcal{P}}} & {}^d\mathcal{P}_2 & & \end{array}$$

If $i \leqslant 0$ then by [FGV02, 7.2.7(2)] the map

$${}^d\chi \circ p'_{\mathcal{Z}} : \mathcal{Z}^i_2 \,\tilde{\times}\, (\overline{\mathrm{Gr}}^{\lambda}_G \cap S^{\lambda'}) \to \mathbb{A}^1$$

identifies with

$$\mathcal{Z}^i_2 \,\tilde{\times}\, (\overline{\mathrm{Gr}}^{\lambda}_G \cap S^{\lambda'}) \xrightarrow{{}^0\chi \times \chi^{\lambda'}_{\nu}} \mathbb{A}^1 \times \mathbb{A}^1 \xrightarrow{\mathrm{sum}} \mathbb{A}^1,$$

where $\nu = (0, i)$, and $\chi^{\lambda'}_{\nu}$ is the notation from [FGV02]. By [FGV02, Theorem 1], the complex

$$\mathrm{R}\Gamma_c(\overline{\mathrm{Gr}}^{\lambda}_G \cap S^{\lambda'}, \mathcal{A}_{\lambda} \otimes (\chi^{\lambda'}_{\nu})^* \mathcal{L}_{\psi})$$

is placed in degree $\langle 2\lambda', \check{\rho} \rangle = \lambda_1 - \lambda_2 - 2d$ and equals $\mathrm{Hom}_{\check{G}}(V^{\lambda} \otimes V^{\nu}, V^{\nu + \lambda'})$.

A connected component of ${}^dY^i$ maps to a pair of connected components of ${}^d\mathcal{Z}'_2$ and ${}_c\mathcal{Z}^i_2$. For such pair of components we have

$$\dim {}_c\mathcal{Z}^i_2 - \dim {}^d\mathcal{Z}'_2 = 1 + 3d + i + 2(\lambda_2 - \lambda_1). \tag{27}$$

The $*$-restriction of (26) to ${}^dY^i$ identifies with

$$((p'_{\mathcal{P}})^*({}^d\mathcal{T}) \otimes {}^0\chi^*\mathcal{L}_\psi)\tilde{\boxtimes}(\mathcal{A}_\lambda \otimes (\chi^{\lambda'}_\nu)^*\mathcal{L}_\psi|_{\overline{\mathrm{Gr}}^\lambda_G \cap S^{\lambda'}})[1 + (\lambda_2 - \lambda_1) + \dim.\mathrm{rel}(\phi_2)].$$

The condition (24) guarantees that, over $p'_{\mathcal{Z}}({}^dY^i)$, the complex ${}^dK_2$ is placed in the usual cohomological degree $-d - \dim({}^d\mathcal{Z}'_2)$. Using (27), we learn that ${}^d\mathcal{K}^i$ is placed in usual cohomological degree $-\dim {}_c\mathcal{Z}^i_2 + i$. Since $i \leqslant 0$, it is placed in perverse degree at most zero, and the inequality is strict unless $i = 0$.

For $i = 0$ we have $\mathrm{Hom}_{\check{G}}(V^\lambda, V^{\lambda'}) = 0$ unless $\lambda = \lambda'$. So, only ${}^0\mathcal{K}^0$ contributes to the zeroth perverse cohomology of $\mathcal{K}^0$.

It remains to analyze the zeroth perverse cohomology of the $*$-restriction of ${}^0\mathcal{T}$ under $p'_{\mathcal{P}} : \mathcal{Z}^0_2 \to {}^0\mathcal{P}_2$. We have to consider the space of sections $t : \pi^*L' \to (\mathcal{B}')^\star$, that is,

$$t : \pi^*L \to \mathcal{B}^* \otimes \Omega_{\tilde{X}}((\lambda_1 - \mu_1)\tilde{x} + (\lambda_1 - \mu_2)\sigma(\tilde{x}))$$

such that $Nt : L^2 \to \mathcal{A} \otimes \Omega$ has no zero at x. This means that $t : \pi^*L \to \mathcal{B}^* \otimes \Omega_{\tilde{X}}(a\tilde{x} - a\sigma(\tilde{x}))$ for some $a \in \mathbb{Z}$ such that

$$a\tilde{x} - a\sigma(\tilde{x}) \leqslant (\lambda_1 - \mu_1)\tilde{x} + (\lambda_1 - \mu_2)\sigma(\tilde{x})$$

as divisors on $\tilde{X}$. Our assertion follows. □

Remark 5. In the above proof we have that $i - \dim {}_c\mathcal{Z}^i_2 = -\dim {}_c\mathcal{Z}^0_2$ does not depend on i, so that $\mathrm{H}^{\mu,\lambda}(K_2)|_{{}_c\mathcal{Z}^{\leqslant 0}_2}$ is placed in the usual cohomological degree $-\dim {}_c\mathcal{Z}^0_2$.

Lemma 9. *The complex $\mathrm{H}^{\mu,\lambda}(K_1)$ over $\tilde{X} \times {}_c\mathrm{Bun}_{\tilde{H},P}$ is placed in perverse degrees at most zero, and its zeroth perverse cohomology identifies with*

$$\bigoplus_\nu \mathrm{H}^{\nu,0}(K_1) \otimes (V^\lambda)^*(\nu - \mu)|_{\tilde{X} \times {}_c\mathrm{Bun}_{\tilde{H},P}}.$$

Proof. The intersection of a fibre of ${}_c\mathrm{Bun}_{P,\tilde{H}} \to \mathrm{Bun}_{G,\tilde{H}}$ with each connected component of ${}_c\mathrm{Bun}_{P,\tilde{H}}$ is either connected or empty. So, by Proposition 5, K_1 is an irreducible perverse sheaf over each connected component of ${}_c\mathrm{Bun}_{P,\tilde{H}}$.

Let ${}^0_c\mathcal{Z}_2$ be the preimage of ${}_c\mathrm{Bun}_{P,\tilde{H}}$ under $\pi_{2,1} : {}^0\mathcal{Z}_2 \to \mathrm{Bun}_{P,\tilde{H}}$. By Proposition 4, K_2 is an irreducible perverse sheaf over each connected component of ${}^0_c\mathcal{Z}_2$. So, if ν is a coweight of $\mathrm{G}\mathbb{O}^0_2$ that vanishes in $\pi_1(G)$, then $\mathrm{H}^{\nu,0}(K_2)$ is an irreducible perverse sheaf over each connected component of ${}_c\mathcal{Z}^{\leqslant 0}_2$.

The functor $(\pi_{2,1})_! : \mathrm{D}^W(\mathcal{Z}_2) \to \mathrm{D}(\mathcal{Z}_1)$ is exact for the perverse t-structures (and commutes with Hecke functors). Our assertion now follows from Lemma 7. □

Proof of Theorem 1. Recall that the Hecke functor (20) commutes with Verdier duality. Since $\mathrm{Aut}_{G,\tilde{H}}$ is self-dual, our assertion follows from Lemma 9.

The S_2-equivariance statement from Lemma 8 combined with [Lys06a, Remark 3] imply the last assertion about generic and special parts. Indeed, for $L \in \mathrm{Pic}\, X, \mathcal{B} \in \mathrm{Pic}\, \tilde{X}$ and $L' = L(-\lambda_1 x)$, $\mathcal{B}' = \mathcal{B}(\mu_1\tilde{x} + \mu_2\tilde{x})$ we have $\chi(L' \otimes \pi_*\mathcal{B}') - \chi(L \otimes \pi_*\mathcal{B}) = \lambda_2 - \lambda_1$. □

4.2.3 Let us derive from Theorem 1 that F_G commutes with Hecke operators. As in Appendix B.1.3, for $\mu = (\mu_1, \mu_2) \in \Lambda_{\mathbb{H}}$ we have a Hecke functor $\mathrm{H}^\mu_{\tilde{H}} : \mathrm{D}(\mathrm{Bun}_{\tilde{H}}) \to \mathrm{D}(\tilde{X} \times \mathrm{Bun}_{\tilde{H}})$.

It is given by

$$\mathrm{H}^{\mu}_{\tilde H}(K) = (p'_{\tilde H})^* K[1]$$

for the map $p'_{\tilde H} : \tilde X \times \mathrm{Bun}_{\tilde H} \to \mathrm{Bun}_{\tilde H}$ sending $(\tilde x, \mathcal{B})$ to $\mathcal{B}(-\mu_1 \tilde x - \mu_2 \sigma(\tilde x))$, here $\mathcal{B} \in \mathrm{Pic}\, \tilde X$.

Corollary 4. *We have the following.*

(i) *For the map* $\pi \times \mathrm{id} : \tilde X \times \mathrm{Bun}_G \to X \times \mathrm{Bun}_G$ *and a dominant coweight* λ *of* G *we have an isomorphism of functors*

$$(\pi \times \mathrm{id})^* \circ \mathrm{H}^{\lambda}_G \circ F_G \,\tilde\to\, \bigoplus_{\mu} (\mathrm{id} \boxtimes F_G) \circ \mathrm{H}^{\mu}_{\tilde H} \otimes (V^{\lambda})^*(\mu) \tag{28}$$

from $\mathrm{D}(\mathrm{Bun}_{\tilde H})$ *to* $\mathrm{D}(\tilde X \times \mathrm{Bun}_G)$. *This isomorphism is compatible with the action of* $\Sigma = \{1, \sigma\}$ *on both sides. It is understood that* Σ *acts on* $\bigoplus_{\mu} \mathrm{H}^{\mu}_{\tilde H} \otimes (V^{\lambda})^*(\mu)$ *via the isomorphisms (61). So,*

$$\mathrm{H}^{\lambda}_G \circ F_G \,\tilde\to\, \mathrm{Hom}_{\Sigma}\Big(\mathrm{triv}, \bigoplus_{\mu} (\pi \times \mathrm{id})_! \circ (\mathrm{id} \boxtimes F_G) \circ \mathrm{H}^{\mu}_{\tilde H} \otimes (V^{\lambda})^*(\mu)\Big).$$

Here $\mathrm{id} \boxtimes F_G$ *is the corresponding functor* $\mathrm{D}(\tilde X \times \mathrm{Bun}_{\tilde H}) \to \mathrm{D}(\tilde X \times \mathrm{Bun}_G)$. *If* $\lambda_1 - \lambda_2$ *is even, then (28) preserves the generic and special parts of* F_G*; otherwise it interchanges them.*

(ii) *If* K *is an automorphic sheaf on* $\mathrm{Bun}_{\tilde H}$ *corresponding to a rank-one local system* $\tilde E$ *on* $\tilde X$, *then* $F_G(K) \in \mathrm{D}(\mathrm{Bun}_G)$ *is an automorphic sheaf corresponding to the local system* $(\pi_* \tilde E)^*$.

Proof. (i) Take $\tilde\mu = (\tilde\mu_1, 0)$ with $\tilde\mu_1 = \lambda_1 + \lambda_2$. Consider the diagram

$$\begin{array}{ccccccc}
\tilde X \times \mathrm{Bun}_{\tilde H, G} & \xleftarrow{\mathrm{supp} \times p_{\tilde H, G}} & \overline{\mathcal{H}}^{\tilde\mu, \lambda}_{\tilde H, G} & \xrightarrow{p'_{\tilde H, G}} & \mathrm{Bun}_{\tilde H, G} & \xrightarrow{\mathfrak{q}} & \mathrm{Bun}_{\tilde H} \\
\downarrow{\scriptstyle \mathrm{id} \times \mathfrak{p}} & & \downarrow & & \downarrow{\scriptstyle \mathfrak{p}} & & \\
\tilde X \times \mathrm{Bun}_G & \longleftarrow & \tilde X \times_X \overline{\mathcal{H}}^{\lambda}_G & \longrightarrow & \mathrm{Bun}_G & &
\end{array}$$

where both squares are cartesian. By Theorem 1, for $K \in \mathrm{D}(\mathrm{Bun}_{\tilde H})$ we obtain an isomorphism

$$\begin{aligned}
&(\pi \times \mathrm{id})^* \mathrm{H}^{\lambda}_G F_G(K) \\
&\quad \tilde\to \bigoplus_{\nu} (\mathrm{id} \times \mathfrak{p})_! (\mathrm{H}^{\nu, 0}(\mathrm{Aut}_{G, \tilde H}) \otimes (V^{\lambda})^*(\nu - \tilde\mu) \otimes \mathrm{H}^{-\tilde\mu}_{\tilde H}(K))[-1 - \dim \mathrm{Bun}_{\tilde H}] \\
&\quad \tilde\to \bigoplus_{\nu} (\mathrm{id} \boxtimes F_G) \mathrm{H}^{\nu - \tilde\mu}_{\tilde H}(K) \otimes (V^{\lambda})^*(\nu - \tilde\mu).
\end{aligned}$$

The assertion about generic and special parts also follows from Theorem 1. ∎

Corollary 5. *For the map* $\pi \times \mathrm{id} : \tilde X \times \mathrm{Bun}_{\tilde H} \to X \times \mathrm{Bun}_{\tilde H}$ *and a dominant coweight* λ *of* G *we have*

$$(\pi \times \mathrm{id})^* \circ (\mathrm{id} \boxtimes F_{\tilde H}) \circ \mathrm{H}^{\lambda}_G \,\tilde\to\, \bigoplus_{\mu} \mathrm{H}^{\mu}_{\tilde H} \circ F_{\tilde H} \otimes (V^{\lambda})(-\mu). \tag{29}$$

This isomorphism is compatible with the action of $\Sigma = \{1, \sigma\}$ *on both sides. So,*

$$(\mathrm{id} \boxtimes F_{\tilde H}) \circ \mathrm{H}^{\lambda}_G \,\tilde\to\, \mathrm{Hom}_{\Sigma}\Big(\mathrm{triv}, \bigoplus_{\mu} (\pi \times \mathrm{id})_! \circ \mathrm{H}^{\mu}_{\tilde H} \circ F_{\tilde H} \otimes (V^{\lambda})(-\mu)\Big). \tag{30}$$

Here $\mathrm{id} \boxtimes F_{\tilde H} : \mathrm{D}(X \times \mathrm{Bun}_G) \to \mathrm{D}(X \times \mathrm{Bun}_{\tilde H})$ *is the corresponding functor. If* $\lambda_1 - \lambda_2$ *is even then (29) preserves the generic and special parts of* $F_{\tilde H}$, *otherwise it interchanges them.*

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Proof. For a coweight $\tilde{\mu}$ such that $\lambda - \tilde{\mu}$ vanishes in $\pi_1(G)$ we have a commutative diagram

$$\begin{array}{ccccccc}
\mathrm{Bun}_G & \xleftarrow{p'_G} & \overline{\mathcal{H}}^{\lambda}_G & \longleftarrow & \overline{\mathcal{H}}^{\tilde{\mu},\lambda}_{\tilde{H},G} & \xrightarrow{\mathrm{supp}\times p'_{\tilde{H},G}} & \tilde{X}\times \mathrm{Bun}_{G,\tilde{H}} \xrightarrow{\mathrm{id}\times\mathfrak{q}} \tilde{X}\times\mathrm{Bun}_{\tilde{H}} \\
 & & \downarrow{\scriptstyle \mathrm{supp}\times p_G} & & \downarrow{\scriptstyle \mathrm{supp}\times p_{\tilde{H},G}} & & \swarrow{\scriptstyle \mathrm{id}\times p'_{\tilde{H}}} \\
 & & X\times \mathrm{Bun}_G & \xleftarrow{\pi\times\mathfrak{p}} & \tilde{X}\times\mathrm{Bun}_{G,\tilde{H}} & \xrightarrow{\mathrm{id}\times\mathfrak{q}} & \tilde{X}\times\mathrm{Bun}_{\tilde{H}}
\end{array}$$

where the left square is cartesian. Here $\mathrm{supp}\times p'_{\tilde{H},G}$ sends $(\tilde{x},\mathcal{B},M\,\tilde{\to}\,M'|_{X-x})$ to $(\tilde{x},M',\mathcal{B}')$ with $\mathcal{B}'=\mathcal{B}(\tilde{\mu}_1\tilde{x}+\tilde{\mu}_2\sigma(\tilde{x}))$. The map $p'_{\tilde{H}}$ sends $(\tilde{x},\mathcal{B}')$ to $\mathcal{B}'(-\tilde{\mu}_1\tilde{x}-\tilde{\mu}_2\sigma(\tilde{x}))$.

In this notation we have

$$\mathrm{H}^{-\tilde{\mu},-w_0(\lambda)}(K)\,\tilde{\to}\,(\mathrm{supp}\times p'_{\tilde{H},G})_!(\mathrm{IC}_{\overline{\mathcal{H}}^{\tilde{\mu},\lambda}_{\tilde{H},G}}\otimes p^*_{\tilde{H},G}K)[-\dim\mathrm{Bun}_{\tilde{H},G}].$$

So, for $K\in \mathrm{D}(\mathrm{Bun}_G)$ the above diagram yields an isomorphism

$$(\pi\times\mathrm{id})^*(\mathrm{id}\boxtimes F_{\tilde{H}})\mathrm{H}^{\lambda}_G(K)\,\tilde{\to}\,(\mathrm{id}\times p'_{\tilde{H}})_!(\mathrm{id}\times\mathfrak{q})_!(\mathrm{H}^{-\tilde{\mu},-w_0(\lambda)}(\mathrm{Aut}_{G,\tilde{H}})\otimes\mathfrak{p}^*K)[-\dim\mathrm{Bun}_G]. \qquad (31)$$

By Theorem 1,

$$\mathrm{H}^{-\tilde{\mu},-w_0(\lambda)}(\mathrm{Aut}_{G,\tilde{H}})\,\tilde{\to}\,\bigoplus_{\nu}\mathrm{H}^{\nu,0}(\mathrm{Aut}_{G,\tilde{H}})\otimes(V^{-w_0(\lambda)})^*(\nu+\tilde{\mu}).$$

So, the right-hand side of (31) identifies with $\bigoplus_\nu \mathrm{H}^{-\tilde{\mu}-\nu}F_{\tilde{H}}(K)\otimes V^{\lambda}(\nu+\tilde{\mu})$. ∎

4.3 Recall that $\mathcal{S}_P$ classifies: $L\in\mathrm{Bun}_1, \mathcal{A}\in\mathrm{Bun}_1$ and $L^2\xrightarrow{s}\mathcal{A}\otimes\Omega$. We have open immersion $j_d:\mathrm{Pic}\,X\times X^{(d)}\hookrightarrow\mathcal{S}_P$ sending (L,D) to $L,\mathcal{A}=\Omega^{-1}\otimes L^2(D)$ with the canonical inclusion $L^2\hookrightarrow\mathcal{A}\otimes\Omega$.

Definition 7. Let $\tilde{E}$ be a rank-one local system on $\tilde{X}$. Recall that $A\tilde{E}$ denotes the corresponding automorphic local system on $\mathrm{Pic}\,\tilde{X}$ (cf. Definition 2). Set $E=\pi_*\tilde{E}$. Define the perverse sheaf $\tilde{E}_H\in P(\mathrm{Bun}_H)$ by

$$\tilde{E}_H=\rho_{H\,!}A\tilde{E}[\dim\mathrm{Bun}_H].$$

Lemma 10. *For $d\geqslant 0$ we have canonically*

$$j_d^*F_{\mathcal{S}}(\tilde{E}_H)\,\tilde{\to}\,AN(\tilde{E}^*)\boxtimes(E^*)^{(d)}\otimes AN(\tilde{E})_\Omega[\dim\mathcal{S}_P].$$

Proof. The stack $\mathcal{V}_{\tilde{H},P}$ classifies $\mathcal{B}\in\mathrm{Pic}\,\tilde{X}$, $L\in\mathrm{Pic}\,X$, and a map $t:L\otimes\pi_*\mathcal{B}\to\Omega$. The datum of t is equivalent to a datum of $t:\pi^*L\to\mathcal{B}^\star$. We have $\mathcal{C}=N(\mathcal{B})$.

Let $\tilde{\mathfrak{p}}_{\mathcal{V}}$ be the composition $\mathcal{V}_{\tilde{H},P}\to\mathcal{V}_{H,P}\xrightarrow{\mathfrak{p}_{\mathcal{V}}}\mathcal{S}_P$, it sends the above point to $(L,\mathcal{A}=\Omega\otimes\mathcal{C}^{-1},s)$, where $s:L^2\to\mathcal{A}\otimes\Omega$ equals the norm of t. We have a cartesian square

$$\begin{array}{ccc}
\mathrm{Pic}\,X\times\tilde{X}^{(d)} & \xrightarrow{j_{\mathcal{V},d}} & \mathcal{V}_{\tilde{H},P} \\
\downarrow{\scriptstyle \mathrm{id}\times\pi} & & \downarrow{\scriptstyle \tilde{\mathfrak{p}}_{\mathcal{V}}} \\
\mathrm{Pic}\,X\times X^{(d)} & \xrightarrow{j_d} & \mathcal{S}_P
\end{array}$$

where $j_{\mathcal{V},d}$ sends $(L,\tilde{D})$ to $L,\mathcal{B}=(\pi^*L(\tilde{D}))^\star$ with the canonical inclusion $t:\pi^*L\hookrightarrow\mathcal{B}^\star$. We have canonically

$$j^*_{\mathcal{V},d}\,\mathfrak{q}^*_{\mathcal{V}}A\tilde{E}\,\tilde{\to}\,AN(\tilde{E}^*)\boxtimes(\tilde{E}^*)^{(d)}\otimes AN(\tilde{E})_\Omega.$$

Our assertion follows by Lemma 6(i). ∎

Since $\mathrm{Pic}^d\,\tilde{X}$ is connected, the covering $\rho_H:\mathrm{Pic}^d\,\tilde{X}\to\mathrm{Bun}^d_H$ is nontrivial, and $\mathcal{N}$ is a nontrivial local system on each Bun^d_H.

Lemma 11. *The following conditions are equivalent:*

(i) *E is irreducible;*

(ii) *$\tilde{E}$ does not descend to a rank-one local system on X;*

(iii) *$A\tilde{E}$ does not descend with respect to $\operatorname{Pic}\tilde{X} \xrightarrow{N} \operatorname{Pic} X$;*

(iv) *the local system $\rho_{H\,!}A\tilde{E}$ on Bun_H is irreducible.*

Definition 8. For an irreducible rank-k local system W on X denote by Aut_W the corresponding automorphic sheaf on Bun_k normalized as in [FGV02]. By [FGV02], if λ is the dominant weight of the standard representation of $\check{\mathrm{GL}}_k$, then $\mathrm{H}^{\lambda}_{\mathrm{GL}_k}(\operatorname{Aut}_W) \widetilde{\to} W^* \boxtimes \operatorname{Aut}_W[1]$.

Recall the normalization of Aut_W for $k = 2$. The map $\nu_P : \operatorname{Bun}_P \to \operatorname{Bun}_G$ sends $0 \to L^2 \to ? \to \mathcal{A} \to 0$ to M (included into $0 \to L \to M \to L^* \otimes \mathcal{A} \to 0$). First, one considers the complex, say $\mathcal{K}_W$, on $\operatorname{Pic} X \times X^{(d)}$ whose fibre at L, D is

$$(A \det W)_{L\otimes\Omega^{-1}} \otimes W_D^{(d)}[\dim \mathcal{S}_P].$$

Then $\operatorname{Four}_\psi((j_d)_!\mathcal{K}_W)$ identifies with $\nu_P^* \operatorname{Aut}_W[\dim.\operatorname{rel}(\nu_P)]$ over the components of Bun_P for which $\deg(\mathcal{A}\otimes\Omega) \geqslant 2\deg L$. The sheaf Aut_W is perverse and irreducible on each connected component of Bun_G.

To fix notation for Eisenstein series, denote by $\overline{\operatorname{Bun}}_P$ the stack classifying $M \in \operatorname{Bun}_2, L \in \operatorname{Pic} X$ and an inclusion of coherent sheaves $L \hookrightarrow M$. Write $\overline{\operatorname{Bun}}_P^{d,d_1}$ for the connected component of $\overline{\operatorname{Bun}}_P$ given by $\deg L = d_1$ and $\deg M + \deg\Omega = 2d_1 + d$. We have a diagram

$$\operatorname{Pic} X \times \operatorname{Pic} X \xleftarrow{\bar{\mathfrak{q}}_P} \overline{\operatorname{Bun}}_P \xrightarrow{\bar{\mathfrak{p}}_P} \operatorname{Bun}_G,$$

where $\bar{\mathfrak{q}}_P$ sends $(L \subset M)$ to $(L, L^{-1}\otimes\det M)$, and $\bar{\mathfrak{p}}_P$ is the projection. For rank-one local systems E_1, E_2 on X set

$$\operatorname{Aut}_{E_1\oplus E_2} = (AE_2)_\Omega^{-1} \otimes (\bar{\mathfrak{p}}_P)_!\bar{\mathfrak{q}}_P^*(AE_1 \boxtimes AE_2)[\dim \overline{\operatorname{Bun}}_P].$$

This normalization is compatible with the above in the following sense. If E_1 and E_2 are not isomorphic, then $\operatorname{Four}_\psi((j_d)_{!*}\mathcal{K}_{E_1\oplus E_2})$ descends (over some open substack of Bun_P) to $\operatorname{Aut}_{E_1\oplus E_2}$, and we have the functional equation $\operatorname{Aut}_{E_1\oplus E_2} \widetilde{\to} \operatorname{Aut}_{E_2\oplus E_1}$ (cf. [BG02]). Write

$$\operatorname{Aut}_{E_1\oplus E_2} \widetilde{\to} \bigoplus_{(d,d_1)\in\mathbb{Z}^2} \operatorname{Aut}^{d,d_1}_{E_1\oplus E_2}, \tag{32}$$

where $\operatorname{Aut}_E^{d,d_1}$ is the contribution of $\overline{\operatorname{Bun}}_P^{d,d_1}$.

Recall the map $\alpha_P : \operatorname{Bun}_P \to \operatorname{Pic} X$ sending $0 \to \operatorname{Sym}^2 L \to ? \to \mathcal{A} \to 0$ to L. Let ${}^0\mathcal{S}_P \subset \mathcal{S}_P$ be the open substack classifying inclusions $L^2 \hookrightarrow \mathcal{A}\otimes\Omega$ with $L, \mathcal{A} \in \operatorname{Pic} X$.

Proposition 6. *We have the following.*

(1) *If E is irreducible then, over the connected components of Bun_P given by $\deg L < 0$, there exists an isomorphism*

$$F_{P,\psi}(\tilde{E}_H) \widetilde{\to} \nu_P^* \operatorname{Aut}_{E^*} \otimes (A\mathcal{E}_0)_\Omega \otimes \alpha_P^* A\mathcal{E}_0[\dim.\operatorname{rel}(\nu_P)]. \tag{33}$$

So, (6) gives rise to an isomorphism of perverse sheaves on Bun_G

$$F_G(\tilde{E}_H) \widetilde{\to} \operatorname{Aut}_{E^*} \otimes (A\mathcal{E}_0)_\Omega \tag{34}$$

(2) *Assume $\tilde{E} = \bar{\mathbb{Q}}_\ell$. Then over the components of $\mathcal{S}_P$ given by $\deg(\mathcal{A}\otimes\Omega) - 2\deg L > 3g-3$ the sheaf $F_{\mathcal{S}}(\tilde{E}_H)$ is perverse, the Goresky–MacPherson extension from ${}^0\mathcal{S}_P$. Both (33) (for $\deg L$ small enough) and (34) remain valid, where now $E \widetilde{\to} \bar{\mathbb{Q}}_\ell \oplus \mathcal{E}_0$.*

Proof. (1) Since $N_!(A\tilde{E}) = 0$, our assertion follows from Lemma 10 combined with Corollary 1.

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(2) The components of $\mathcal{V}_{\tilde{H},P}$ given by $\deg \mathcal{B}+2\deg L<0$ are smooth (this is where the connectedness of $\tilde{X}$ is essential). The fibres of $N:\operatorname{Pic}\tilde{X}\to\operatorname{Pic}X$ are of dimension $g-1$, so over the corresponding components of $\mathcal{S}_P$ the map $\tilde{\mathfrak{p}}_{\mathcal{V}}:\mathcal{V}_{\tilde{H},P}\to\mathcal{S}_P$ is small. The first assertion follows. The second assertion is obtained from Lemma 10. ☐

Remark 6. (i) Proposition 6 implies that the constant term $\mathrm{CT}_P(\mathrm{Aut}_{\bar{\mathbb{Q}}_\ell\oplus\mathcal{E}_0})$ is essentially the cohomology of the Prym variety (cf. Appendix A.1).

(ii) The formula (34) for $\tilde{E}=\bar{\mathbb{Q}}_\ell$ is a version of the classical theorem of Siegel (its proof given by Weil can be found in [Wei65]; a version proved by Waldspurger is found in [Wal85, § I.5, Proposition 2].

(iii) If $\tilde{E}=\bar{\mathbb{Q}}_\ell$, then $\Sigma=\{1,\sigma\}$ acts naturally on $\tilde{E}_H$ and, hence, on $F_G(\tilde{E}_H)$. Let Σ acts on $\mathrm{Aut}_{\mathcal{E}_0\oplus\bar{\mathbb{Q}}_\ell}$ via (34). The σ-invariants of $\mathrm{Aut}_{\mathcal{E}_0\oplus\bar{\mathbb{Q}}_\ell}$ are $\bigoplus_{d,d_1}\mathrm{Aut}^{d,d_1}_{\mathcal{E}_0\oplus\bar{\mathbb{Q}}_\ell}$, the sum over $(d,d_1)\in\mathbb{Z}^2$ with d_1 even.

(iv) The stack $\mathrm{Bun}_{G,\tilde{H}}\times_{\mathrm{Bun}_G}\mathrm{Bun}_{G,\tilde{H}}$ splits as a disjoint union of the open substacks $\mathcal{U}^0\sqcup\mathcal{U}^1$, where $\mathcal{U}^a$ is given by the condition that $\mathcal{B}_1\otimes\mathcal{B}_2^{-1}\in\mathrm{Bun}^a_{U_\pi}$ for a point $(\mathcal{B}_1,\mathcal{B}_2\in\operatorname{Pic}\tilde{X}, M\in\mathrm{Bun}_2, N(\mathcal{B}_1)\otimes\det M\widetilde{\to}N(\mathcal{B}_2)\otimes\det M\widetilde{\to}\Omega)$ of $\mathrm{Bun}_{G,\tilde{H}}\times_{\mathrm{Bun}_G}\mathrm{Bun}_{G,\tilde{H}}$ (cf. Appendix A.1). So, the restriction of $F_G(\tilde{E}_H)$ under $\mathrm{Bun}_{G,\tilde{H}}\to\mathrm{Bun}_G$ is naturally a direct sum $\mathcal{K}^0\oplus\mathcal{K}^1$, where $\mathcal{K}^a$ is the contribution of $\mathcal{U}^a$. If $\tilde{E}=\bar{\mathbb{Q}}_\ell$, then (32) is not a refinement of the decomposition $\mathcal{K}^0\oplus\mathcal{K}^1$.

To see this, consider the line bundle $\mathcal{E}^{(d)}$ on $X^{(d)}$, the dth symmetric power of $\mathcal{E}$. Its tensor square is canonically trivialized, so it defines a μ_2-torsor ${}_\pi X^{(d)}\to X^{(d)}$. A fibre of the latter map over $D\in X^{(d)}$ can also be seen as the set of connected components of the stack of pairs $(\mathcal{B},\kappa)$, where $\mathcal{B}\in\operatorname{Pic}\tilde{X}$, $\kappa:N(\mathcal{B})\widetilde{\to}\mathcal{O}(D)$. The restriction of the covering ${}_\pi X^{(d)}\to X^{(d)}$ under $\pi:\tilde{X}^{(d)}\to X^{(d)}$ has a distinguished section, and $\pi_*(\bar{\mathbb{Q}}_\ell^{(d)})\big|_{{}_\pi X^{(d)}}\widetilde{\to}K^0\oplus K^1$, where K^0 is the contribution of the distinguished section.

Recall that $(\bar{\mathbb{Q}}_\ell\oplus\mathcal{E}_0)^{(d)}\widetilde{\to}\bigoplus_{k=0}^d(\mathrm{sym}_{k,d-k})_!(\bar{\mathbb{Q}}_\ell\boxtimes\mathcal{E}_0^{(d-k)})$, where $\mathrm{sym}_{k,d-k}:X^{(k)}\times X^{(d-k)}\to X^{(d)}$ is the sum of divisors. So,

$$K^0\oplus K^1\widetilde{\to}\left(\bigoplus_{k=0}^d(\mathrm{sym}_{k,d-k})_!\bar{\mathbb{Q}}_\ell\boxtimes\mathcal{E}_0^{(d-k)}\right)\bigg|_{{}_\pi X^{(d)}},$$

but the right-hand side is *not* a refinement of the decomposition of the left-hand side.

4.4 Local Rankin–Selberg-type convolutions

4.4.1 Recall the following Laumon's construction for GL_2. Let Bun_2' be the stack classifying $M\in\mathrm{Bun}_2$ with nonzero section $\Omega\hookrightarrow M$. To a local system E on X one associates a complex Laum_E on Bun_2' defined as follows.

Let $\mathcal{Q}$ be the stack classifying collections $(L_1\subset L_2\subset M)$ with $L_1\widetilde{\to}\Omega$, $L_2/L_1\widetilde{\to}\mathcal{O}_X$, where $L_2\subset M$ is a modification of locally free $\mathcal{O}_X$-modules of rank two. Let $ev_{\mathcal{Q}}:\mathcal{Q}\to\mathbb{A}^1$ be the map sending the above point to the class of $0\to L_1\to L_2\to L_2/L_1\to 0$. Let $\mathfrak{q}_{\mathcal{Q}}:\mathcal{Q}\to\mathrm{Sh}_0$ be the map sending the above point to M/L_2, here Sh_0 is the stack of torsion sheaves on X. Write $\mathcal{L}_E$ for Laumon's sheaf corresponding to E (see [FGV02]). Let $\mathfrak{p}_{\mathcal{Q}}:\mathcal{Q}\to\mathrm{Bun}_2'$ be the map forgetting L_2. Set

$$\mathrm{Laum}_E=\mathfrak{p}_{\mathcal{Q}!}(\mathfrak{q}_{\mathcal{Q}}^*\mathcal{L}_E\otimes ev_{\mathcal{Q}}^*\mathcal{L}_\psi)[\dim\mathcal{Q}].$$

Consider the map $\mathfrak{q}':\overline{\mathrm{Bun}}_P\to\operatorname{Pic}X\times\mathrm{Bun}_2'$ sending $(L\subset M)$ to $(L\otimes\Omega^{-1},\Omega\subset M\otimes L^{-1}\otimes\Omega)$. For local systems E,E_1 on X, where E_1 is of rank one, set

$$\mathrm{Laum}_{E,E_1}=(AE_1)_\Omega\otimes(\mathfrak{q}')^*(A(E_1\otimes\det E)\boxtimes\mathrm{Laum}_E)[\dim\operatorname{Pic}X].$$

Let $\mathrm{Laum}^{d,d_1}_{E,E_1}$ denote the restriction of Laum_{E,E_1} to the connected component $\overline{\mathrm{Bun}}^{d,d_1}_P$ of $\overline{\mathrm{Bun}}_P$ given by $\deg M + \deg \Omega = 2d_1 + d$ and $\deg L = d_1$. Recall the projection $\bar{\mathfrak{p}}_P : \overline{\mathrm{Bun}}_P \to \mathrm{Bun}_G$. By [FGV02, § 7.9], for E irreducible and $d \geqslant 0$ we have

$$\mathrm{Laum}^{d,d_1}_{E,\bar{\mathbb{Q}}_\ell} \widetilde{\to} \bar{\mathfrak{p}}_P^* \,\mathrm{Aut}_E[\dim.\mathrm{rel}(\bar{\mathfrak{p}}_P)]$$

over $\overline{\mathrm{Bun}}^{d,d_1}_P$, and

$$\bar{\mathfrak{p}}_{P\,!}\mathrm{Laum}_{E,E_1} \widetilde{\to} \mathrm{Aut}_{E_1\oplus\bar{\mathbb{Q}}_\ell} \otimes \mathrm{Aut}_E[-\dim \mathrm{Bun}_G].$$

Denote by

$$\mathrm{mult}^{d,d_1} : \mathrm{Pic}^{d_1} X \times \tilde{X}^{(d)} \to \mathrm{Pic}^{2d_1+d}\,\tilde{X}$$

the map sending $(L, \tilde{D})$ to $(\pi^* L)(\tilde{D})$. Note that mult^{0,d_1} is a (representable) Galois S_2-covering over its image, the corresponding automorphism of $\mathrm{Pic}^{d_1} X$ sends L to $L \otimes \mathcal{E}$. Let $\epsilon : \mathrm{Pic}\,\tilde{X} \widetilde{\to} \mathrm{Pic}\,\tilde{X}$ be the involution sending $\mathcal{B}$ to $\mathcal{B}^\star$. The following is closely related to the main result of [Lys02].

Theorem 2. *For any local systems E, E_1 on X with $\mathrm{rk}(E_1) = 1$, $\mathrm{rk}(E) = 2$ there is an isomorphism*

$$F_{\tilde{H}}(\bar{\mathfrak{p}}_{P\,!}\mathrm{Laum}^{d,d_1}_{E,E_1}) \widetilde{\to} (A \det E)_{\Omega^{-1}} \otimes \epsilon_!\, \mathrm{mult}^{d,d_1}_!(A(\det E \otimes E_1 \otimes \mathcal{E}_0) \boxtimes (\pi^* E)^{(d)})[d+g-1] \quad (35)$$

depending only on a choice of (6). If d is even (respectively, odd), then $F_{\tilde{H},s}$ (respectively, $F_{\tilde{H},g}$) sends $\bar{\mathfrak{p}}_{P\,!}\mathrm{Laum}^{d,d_1}_{E,E_1}$ to zero. In particular, for E irreducible we have

$$\begin{aligned} &F_{\tilde{H}}(\mathrm{Aut}_{E_1\oplus\bar{\mathbb{Q}}_\ell} \otimes \mathrm{Aut}_E) \\ &\quad \widetilde{\to} \bigoplus_{d\geqslant 0} (A \det E)_{\Omega^{-1}} \otimes \epsilon_!\, \mathrm{mult}^{d,d_1}_!(A(\det E \otimes E_1 \otimes \mathcal{E}_0) \boxtimes (\pi^* E)^{(d)})[d+g-1+\dim \mathrm{Bun}_G], \end{aligned}$$

where d_1 is a function of a connected component of $\mathrm{Bun}_{\tilde{H}}$ given by $2d_1 + d = \deg(\mathcal{B}^\star)$ for $\mathcal{B} \in \mathrm{Bun}_{\tilde{H}}$. The sum is over $d \geqslant 0$ such that $d_1 \in \mathbb{Z}$.

Remark 7. Write $s^a : X^{(a)} \times \tilde{X}^{(d-2a)} \to \tilde{X}^{(d)}$ for the map sending $(D, \tilde{D})$ to $\pi^* D + \tilde{D}$. For any rank-two local system E on X the sheaf $(\pi^* E)^{(d)}$ admits a filtration with successive quotients being

$$s^a_!((\det E)^{(a)} \boxtimes \pi^*(E^{(d-2a)}))$$

for $0 \leqslant a \leqslant d/2$. This follows from the fact that for a two-dimensional $\bar{\mathbb{Q}}_\ell$-vector space E we have

$$\mathrm{Sym}^{a+b} E \otimes \mathrm{Sym}^a E \widetilde{\to} \bigoplus_{i=0}^{a} (\det E)^{\otimes i} \otimes \mathrm{Sym}^{2a+b-2i} E.$$

On the other hand, the complex $\bar{\mathfrak{p}}_{P\,!}\mathrm{Laum}^{d,d_1}_{E,E_1}$ has a filtration indexed by $a \geqslant 0$ coming from a stratification of $\overline{\mathrm{Bun}}_P$. The corresponding stratum of $\overline{\mathrm{Bun}}_P$ is given by the condition that there is a divisor $D \in X^{(a)}$ such that $L(D) \subset M$ is a subbundle. One may check that the corresponding graded complexes of the left- and the right-hand side of (35) coincide.

4.4.2 Following [FGV02], for a local system E on X denote by $\mathrm{Av}^d_E : \mathrm{D}(\mathrm{Bun}_G) \to \mathrm{D}(\mathrm{Bun}_G)$ the following averaging functor. Let Mod^d_2 be the stack classifying a modification $M \subset M'$ of rank-two vector bundles on X with $\deg(M'/M) = d$. Let $\mathfrak{s} : \mathrm{Mod}^d_2 \to \mathrm{Sh}_0$ be the map sending this point to M'/M. We have a diagram

$$\mathrm{Bun}_G \xleftarrow{p} \mathrm{Mod}^d_2 \xrightarrow{p'} \mathrm{Bun}_G,$$

where p (respectively, p') sends $(M \subset M')$ to M (respectively, to M'). Set $\mathrm{Av}^d_E(K) = p'_!(p^* K \otimes \mathcal{L}^d_E)[2d](d)$.

By [FGV02, Proposition 9.5], for an irreducible rank-two local system W on X we have

$$\mathrm{Av}^d_E(\mathrm{Aut}_W) \widetilde{\to} \mathrm{Aut}_W \otimes \mathrm{R}\Gamma(X^{(d)}, (E \otimes W^*)^{(d)})[d].$$

Write S_d for the symmetric group on d elements, set $\Sigma^d = \mathrm{Aut}_{X^d}(\tilde{X}^d)$. We have a semi-direct product $\Sigma^d \rtimes S_d$ acting on $\tilde{X}^d$, it fits into an exact sequence $1 \to \Sigma^d \to \Sigma^d \rtimes S_d \to S_d \to 1$. For a dominant coweight λ of G the functor

$$\bigoplus_{\mu_1,\dots,\mu_d} \mathrm{H}^{\mu_1}_{\tilde{H}} \circ \cdots \circ \mathrm{H}^{\mu_d}_{\tilde{H}} \otimes (V^\lambda)(-\mu_1) \otimes \cdots \otimes (V^\lambda)(-\mu_d)$$

is naturally a functor from $\mathrm{D}(\mathrm{Bun}_{\tilde{H}})$ to the equivariant derived category $\mathrm{D}^{\Sigma^d \rtimes S_d}(\tilde{X}^d \times \mathrm{Bun}_{\tilde{H}})$. So, we can introduce the functor

$$(\mathrm{H}^\lambda_{\tilde{H},G})^{\boxtimes d} : \mathrm{D}(\mathrm{Bun}_{\tilde{H}}) \to \mathrm{D}^{S_d}(X^d \times \mathrm{Bun}_{\tilde{H}})$$

given by

$$(\mathrm{H}^\lambda_{\tilde{H},G})^{\boxtimes d} = \mathrm{Hom}_{\Sigma^d}\Bigl(\mathrm{triv}, (\pi \times \mathrm{id})_! \bigoplus_{\mu_1,\dots,\mu_d} \mathrm{H}^{\mu_1}_{\tilde{H}} \circ \cdots \circ \mathrm{H}^{\mu_d}_{\tilde{H}} \otimes (V^\lambda)(-\mu_1) \otimes \cdots \otimes (V^\lambda)(-\mu_d)\Bigr),$$

where, by abuse of notation, we have written $\pi \times \mathrm{id} : \tilde{X}^d \times \mathrm{Bun}_{\tilde{H}} \to X^d \times \mathrm{Bun}_{\tilde{H}}$ for the projection.

For a local system E on X let $\mathrm{Av}^d_E : \mathrm{D}(\mathrm{Bun}_{\tilde{H}}) \to \mathrm{D}(\mathrm{Bun}_{\tilde{H}})$ denote the averaging functor given by

$$\mathrm{Av}^d_E(K) = \mathrm{Hom}_{S_d}(\mathrm{triv}, (\mathrm{pr}_2)_!(\mathrm{pr}_1^* E^{\boxtimes d} \otimes (\mathrm{H}^\lambda_{\tilde{H},G})^{\boxtimes d}(K)),$$

where $\lambda = (1, 0)$, and pr_i are the two projections from $X^d \times \mathrm{Bun}_{\tilde{H}}$ to X^d and $\mathrm{Bun}_{\tilde{H}}$, respectively.

Proposition 7. *For any local system E on X we have a canonical isomorphism of functors*

$$F_{\tilde{H}} \circ \mathrm{Av}^d_E \widetilde{\to} \mathrm{Av}^d_E \circ F_{\tilde{H}}$$

from $\mathrm{D}(\mathrm{Bun}_G)$ to $\mathrm{D}(\mathrm{Bun}_{\tilde{H}})$. If d is even, then this isomorphism preserves the generic and special parts of $F_{\tilde{H}}$, otherwise it interchanges them.

Proof. Take $\lambda = (1, 0)$. By [Gai04, 1.8], the functor $(\mathrm{H}^\lambda_G)^{\boxtimes d}$ maps $\mathrm{D}(\mathrm{Bun}_G)$ to the equivariant derived category $\mathrm{D}^{S_d}(X^d \times \mathrm{Bun}_G)$. We have a canonical isomorphism of functors from $\mathrm{D}(\mathrm{Bun}_G)$ to itself

$$\mathrm{Av}^d_E \widetilde{\to} \mathrm{Hom}_{S_d}(\mathrm{triv},\ (\mathrm{pr}_2)_!(\mathrm{pr}_1^* E^{\boxtimes d} \otimes (\mathrm{H}^\lambda_G)^{\boxtimes d}))$$

where pr_i are the two projections from $X^d \times \mathrm{Bun}_G$ to X^d and Bun_G, respectively. Applying (30) d times we get a S_d-equivariant isomorphism

$$(\mathrm{id} \boxtimes F_{\tilde{H}}) \circ (\mathrm{H}^\lambda_G)^{\boxtimes d} \widetilde{\to} (\mathrm{H}^\lambda_{\tilde{H},G})^{\boxtimes d} \circ F_{\tilde{H}},$$

where $\mathrm{id} \boxtimes F_{\tilde{H}} : \mathrm{D}(X^d \times \mathrm{Bun}_G) \to \mathrm{D}(X^d \times \mathrm{Bun}_{\tilde{H}})$ is the corresponding functor. If d is even, then this isomorphism preserves the generic and special parts of $F_{\tilde{H}}$, otherwise it interchanges them. Our assertion follows. □

Proof of Theorem 2. The proof proceeds in two steps.

Step 1. Case $d = 0$. Let $\mathcal{X}_1$ be the stack classifying $L \in \mathrm{Pic}^{d_1} X$, $\mathcal{B} \in \mathrm{Pic}\, \tilde{X}$, and an isomorphism $\xi : L^2 \otimes \mathcal{C} \widetilde{\to} \Omega^2$, where $\mathcal{C} = N(\mathcal{B})$. We have a diagram of projections

$$\mathrm{Pic}^{d_1} X \stackrel{q_1}{\leftarrow} \mathcal{X}_1 \stackrel{p_1}{\to} \mathrm{Pic}\, \tilde{X},$$

where q_1 (respectively, p_1) sends the above point to L (respectively, to $\mathcal{B}$).

Let $p_{\mathcal{X}} : \mathcal{X} \to \mathcal{X}_1$ denote the stack whose fibre over a point of $\mathcal{X}_1$ is the stack of exact sequences of $\mathcal{O}_X$-modules

$$0 \to L \to M \to L \otimes \Omega^{-1} \to 0. \tag{36}$$

Consider the diagram

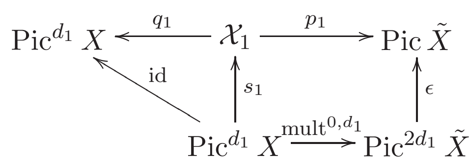
$$\mathrm{Bun}_{P_2} \xleftarrow{\tau_P} \mathrm{Bun}_{P,\tilde{H}} \xleftarrow{q_{\mathcal{X}}} \mathcal{X} \xrightarrow{ev_{\mathcal{X}}} \mathbb{A}^1,$$

where $ev_{\mathcal{X}}$ is the map sending a point of $\mathcal{X}$ to the class of (36), and $q_{\mathcal{X}}$ sends a point of $\mathcal{X}$ to the collection $(L \subset M, \mathcal{B}, \xi : \mathcal{C} \otimes \det M \widetilde{\to} \Omega)$ with $\mathcal{C} = N(\mathcal{B})$. Using (6) and Corollary 1 we obtain

$$F_{\tilde{H}}(\bar{\mathfrak{p}}_{P\,!}\mathrm{Laum}^{0,d_1}_{E,E_1}) \widetilde{\to} (A \det E)_{\Omega^{-1}} \otimes p_{1\,!}(p_{\mathcal{X}\,!}(q^*_{\mathcal{X}}\tau^*_P S_{P,\psi} \otimes ev^*_{\mathcal{X}}\mathcal{L}_\psi) \otimes q_1^* A(E_1 \otimes \det E \otimes \mathcal{E}_0))[b],$$

where $b = \dim \mathrm{Bun}_{\mathbb{SO}_2} - \dim \mathrm{Bun}_{P_2}$, here $\dim \mathrm{Bun}_{P_2}$ is the dimension of (the unique) connected component of Bun_{P_2} containing $\tau_P(q_{\mathcal{X}}(\mathcal{X}))$.

Consider the map $s_1 : \mathrm{Pic}^{d_1} X \to \mathcal{X}_1$ sending L to the collection $(L, \mathcal{B}, \xi)$, where $\mathcal{B} = (\pi^* L)^\star$ and $\xi : L^2 \otimes \mathcal{C} \widetilde{\to} \Omega^2$ is the natural isomorphism with $\mathcal{C} = N(\mathcal{B})$. The following diagram commutes.



Recall that S_2 acts on $S_{P,\psi}$ (cf. § 2.1). By definition of $S_{P,\psi}$, we get a S_2-equivariant isomorphism

$$s_{1\,!}\bar{\mathbb{Q}}_\ell[g-1] \widetilde{\to} p_{\mathcal{X}\,!}(q^*_{\mathcal{X}}\tau^*_P S_{P,\psi} \otimes ev^*_{\mathcal{X}}\mathcal{L}_\psi)[b].$$

Note that s_1 is a (respresentable) S_2-covering over its image. We have a 2-automorphism η of the identity functor $\mathrm{id}_{\mathcal{X}_1}$ acting on $(L, \mathcal{B}, \xi)$ as -1 on L and trivially on $\mathcal{B}$. Since η acts as -1 on $\mathrm{Hom}_{S_2}(\mathrm{sign}, s_{1\,!}\bar{\mathbb{Q}}_\ell)$ and trivially on $q_1^* A(E_1 \otimes \det E \otimes \mathcal{E}_0)$, it follows that

$$p_{1\,!}(q_1^* A(E_1 \otimes \det E \otimes \mathcal{E}_0) \otimes \mathrm{Hom}_{S_2}(\mathrm{sign}, s_{1\,!}\bar{\mathbb{Q}}_\ell)) = 0. \tag{37}$$

We have used that $\mathrm{R}\Gamma_c(B(\mu_2), W) = 0$, where W is the nontrivial rank-one local system on $B(\mu_2)$ corresponding to the S_2-covering $\mathrm{Spec}\, k \to B(\mu_2)$.

Note that the genus of $\tilde{X}$ is odd. For a point of $\mathcal{X}$ as above, we have $\chi(L \otimes \pi_*\mathcal{B}) = 0 \bmod 2$. By [Lys06a, Remark 3], we obtain $(S_{P,\psi})^{S_2} \widetilde{\to} S_{P,\psi,g}$ over the connected component of Bun_{P_2} containing $\tau_P(q_{\mathcal{X}}(\mathcal{X}))$. From (37) it follows that $F_{\tilde{H},s}(\bar{\mathfrak{p}}_{P\,!}\mathrm{Laum}^{0,d_1}_{E,E_1}) = 0$. So,

$$F_{\tilde{H},g}(\bar{\mathfrak{p}}_{P\,!}\mathrm{Laum}^{0,d_1}_{E,E_1}) \widetilde{\to} (A \det E)_{\Omega^{-1}} \otimes \epsilon_!\, \mathrm{mult}^{0,d_1}_!\, A(E_1 \otimes \det E \otimes \mathcal{E}_0)[g-1].$$

Step 2. For $d \geqslant 0$ we have $\mathrm{Av}^d_E(\bar{\mathfrak{p}}_!\mathrm{Laum}^{0,d_1}_{E,E_1}) \widetilde{\to} \mathrm{Laum}^{d,d_1}_{E,E_1}$. By Step 1 and Proposition 7, we obtain

$$F_{\tilde{H}}(\bar{\mathfrak{p}}_{P\,!}\mathrm{Laum}^{d,d_1}_{E,E_1}) \widetilde{\to} (A \det E)_{\Omega^{-1}} \otimes \mathrm{Av}^d_E(\epsilon_!\, \mathrm{mult}^{0,d_1}_!\, A(E_1 \otimes \det E \otimes \mathcal{E}_0))[g-1].$$

It is easy to check that for any $K \in \mathrm{D}(\mathrm{Pic}^{d_1} X)$ we have

$$\mathrm{Av}^d_E(\epsilon_!\, \mathrm{mult}^{0,d_1}_!\, K) \widetilde{\to} \epsilon_!\, \mathrm{mult}^{d,d_1}_!(K \boxtimes (\pi^* E)^{(d)})[d].$$

Our assertion follows. ◻

5. The case $H = \mathrm{G}\mathbb{O}_4$

5.1 Keep the notation of § 3 assuming $m = 2$.

Remark 8. Given k-vector spaces V_1, V_2 of dimension two we have a canonical symmetric form $\mathrm{Sym}^2(V_1 \otimes V_2) \to \det V_1 \otimes \det V_2$. One may obtain a compatible isomorphism

$$\gamma_{V_1,V_2} : \det(V_1 \otimes V_2) \widetilde{\to} (\det V_1 \otimes \det V_2)^2$$

as follows. Denote by St (respectively, det) the standard (respectively, the determinantal) representation of GL_2. Fix an isomorphism $\gamma_{\mathrm{St}} : \det(\mathrm{St} \boxtimes \mathrm{St}) \widetilde{\to} (\det \boxtimes \det)^2$ of $\mathrm{GL}_2 \times \mathrm{GL}_2$-representations compatible with the above symmetric form. It yields the desired isomorphism as follows. Given V_i pick an isomorphism of vector spaces $b_i : V_i \widetilde{\to} \mathrm{St}$ and define γ_{V_1,V_2} by the following commutative diagram.

$$\begin{array}{ccc} \det(V_1 \otimes V_2) & \xrightarrow{\gamma_{V_1,V_2}} & (\det V_1 \otimes \det V_2)^2 \\ \downarrow{\scriptstyle b_1 \otimes b_2} & & \downarrow{\scriptstyle b_1 \otimes b_2} \\ \det(\mathrm{St} \boxtimes \mathrm{St}) & \xrightarrow{\gamma_{\mathrm{St}}} & (\det \boxtimes \det)^2 \end{array}$$

Then γ_{V_1,V_2} does not depend on b_i. We have $\gamma_{V_2,V_1} = -\gamma_{V_1,V_2}$.

Denote by $\mathrm{Bun}_{k,\tilde{X}}$ the stack of rank k vector bundles on $\tilde{X}$. Denote by $\rho : \mathrm{Bun}_{2,\tilde{X}} \to \mathrm{Bun}_{\tilde{H}}$ the map sending W to $(V, \mathcal{C}, \mathrm{Sym}^2 V \xrightarrow{h} \mathcal{C}, \gamma)$, where $V \in \mathrm{Bun}_4$ is the descent of $W \otimes \sigma^* W$ equipped with natural descent data, $\mathcal{C} = N(\det W)$, and h is the descent of the canonical symmetric form $\mathrm{Sym}^2(W \otimes \sigma^* W) \to \det W \otimes \sigma^* \det W$. The compatible trivialization

$$\det(W \otimes \sigma^* W) \widetilde{\to} (\det W \otimes \sigma^* \det W)^2$$

descends to $\gamma : \mathcal{C}^{-2} \otimes \det V \widetilde{\to} \mathcal{E}$. The map ρ is smooth and surjective.

Another way to spell the same construction is as follows. We have an exact sequence $1 \to \mathbb{G}_m \to \mathrm{GL}_2 \times \mathrm{GL}_2 \to \mathrm{G}\mathbb{O}_4^0 \to 1$, where the first map sends $x \in \mathbb{G}_m$ to (x, x^{-1}). Then we can think of the automorphism $\tilde{\sigma}$ of $\mathrm{G}\mathbb{O}_4^0$ chosen in § 3.1 as an automorphism of this exact sequence permuting the two factors of $\mathrm{GL}_2 \times \mathrm{GL}_2$. The corresponding twisting of this exact sequence by the Σ-torsor $\pi : \tilde{X} \to X$ gives an exact sequence $1 \to U_\pi \to \pi_* \mathrm{GL}_2 \to \tilde{H} \to 1$.

We have $\mathrm{Bun}_{\pi_* \mathrm{GL}_2} \widetilde{\to} \mathrm{Bun}_{2,\tilde{X}}$. The stack Bun_{U_π} classifies $\mathcal{B} \in \mathrm{Pic}\, \tilde{X}$ equipped with an isomorphism $N(\mathcal{B}) \widetilde{\to} \mathcal{O}$. The above map ρ is the extension of scalars under $\pi_* \mathrm{GL}_2 \to \tilde{H}$. Write also σ for the automorphism of $\mathrm{Bun}_{\tilde{H}}$ sending $(V, \mathcal{C}, \mathrm{Sym}^2 V \to \mathcal{C}, \gamma)$ to $(V, \mathcal{C}, \mathrm{Sym}^2 V \to \mathcal{C}, -\gamma)$. Then the following diagram is 2-commutative.

$$\begin{array}{ccccc} \mathrm{Bun}_{2,\tilde{X}} & \xrightarrow{\rho} & \mathrm{Bun}_{\tilde{H}} & \xrightarrow{\rho_H} & \mathrm{Bun}_H \\ \downarrow{\scriptstyle \sigma^*} & & \downarrow{\scriptstyle \sigma} & \nearrow{\scriptstyle \rho_H} & \\ \mathrm{Bun}_{2,\tilde{X}} & \xrightarrow{\rho} & \mathrm{Bun}_{\tilde{H}} & & \end{array}$$

Let $\mathrm{Bun}^d_{2,\tilde{X}} \subset \mathrm{Bun}_{2,\tilde{X}}$ be the substack given by $\deg W = d$. Recall that Bun^d_H is given by $\deg \mathcal{C} = d$. For $\tilde{X}$ connected, the irreducibility of $\mathrm{Bun}^d_{2,\tilde{X}}$ and surjectivity of ρ imply that the stack $\mathrm{Bun}^d_{\tilde{H}}$ is irreducible, so $\mathcal{N}$ is a nontrivial local system on each Bun^d_H in this case.

Let $\tilde{E}$ be an irreducible rank-two local system on $\tilde{X}$. Let $\mathrm{Aut}_{\tilde{E}}$ be the corresponding automorphic sheaf on $\mathrm{Bun}_{2,\tilde{X}}$ normalized as in [FGV02] (cf. also Definition 8). We fix a rank-one local system χ on X and an isomorphism $\pi^* \chi \widetilde{\to} \det \tilde{E}$. This provides descent data for $\mathrm{Aut}_{\tilde{E}}$ for the map $\mathrm{Bun}_{2,\tilde{X}} \to \mathrm{Bun}_{\tilde{H}}$, so we obtain a perverse sheaf, say $K_{\tilde{E},\chi,\tilde{H}}$ on $\mathrm{Bun}_{\tilde{H}}$.

For $\tilde{X}$ connected, the group stack Bun_{U_π} has two connected components (cf. Appendix A.1), write $\mathrm{Bun}^0_{U_\pi}$ for its connected component of unity.

Definition 9. The quotient of $\mathrm{Bun}_{2,\tilde{X}}$ by the action of $\mathrm{Bun}^0_{U_\pi}$ is a μ_2-torsor over $\mathrm{Bun}_{\tilde{H}}$, we denote by $\mathcal{S}N$ the corresponding local system (of order two) on $\mathrm{Bun}_{\tilde{H}}$. We refer to it as *the spinorial norm.*

We have $K_{\tilde{E},\chi,\tilde{H}} \otimes \mathcal{S}N \widetilde{\to} K_{\tilde{E},\chi\otimes\mathcal{E}_0,\tilde{H}}$. The central character of $K_{\tilde{E},\chi,\tilde{H}}$ is χ.

The local system $\pi_*\tilde{E}^*$ is equipped with a natural symplectic form $\bigwedge^2(\pi_*\tilde{E}^*) \to \chi^{-1}$, so gives rise to a $\check{G}$-local system $E_{\check{G}}$ on X, where $G = \mathrm{GSp}_4$ for $n = 2$.

If $\tilde{X}$ splits, then we fix a numbering of connected components of $\tilde{X}$. Then $\tilde{E}$ becomes a pair of irreducible rank-two local systems E_1, E_2 on X. We obtain $\mathrm{Bun}_{2,\tilde{X}} \widetilde{\to} \mathrm{Bun}_2 \times \mathrm{Bun}_2$ and $\mathrm{Aut}_{\tilde{E}} = \mathrm{Aut}_{E_1} \boxtimes \mathrm{Aut}_{E_2}$. The descent datum for $\det \tilde{E}$ becomes $\det E_1 \widetilde{\to} \det E_2 \widetilde{\to} \chi$. For $\tilde{X}$ split we have an exact sequence $0 \to \mathbb{Z}/2\mathbb{Z} \to \pi_1(\tilde{H}) \xrightarrow{\lambda} \mathbb{Z} \to 0$, and $\mathrm{Bun}^d_{\tilde{H}}$ has two connected components $\mathrm{Bun}^\theta_{\tilde{H}}$ for $\theta \in \lambda^{-1}(d)$. For d odd the stack Bun^d_H is connected and the covering $\rho_H : \mathrm{Bun}^d_{\tilde{H}} \to \mathrm{Bun}^d_H$ splits. For d even Bun^d_H has two connected components Bun^θ_H, $\theta \in \lambda^{-1}(d)$, and the covering $\rho_H : \mathrm{Bun}^\theta_{\tilde{H}} \to \mathrm{Bun}^\theta_H$ is nontrivial.

If E is an irreducible rank-two local system on X such that π^*E is irreducible, then the perverse sheaf $K_{\pi^*E,\det E,\tilde{H}}$ has natural descent data with respect to $\rho_H : \mathrm{Bun}_{\tilde{H}} \to \mathrm{Bun}_H$, thus defining a perverse sheaf $K_{E,H}$ on Bun_H. Recall the local system $\mathcal{E}_0$ on X (cf. § 3.1).

Conjecture 2. If $n = 2$, then we have the following.

(1) If $\tilde{E}$ does not descend with respect to $\tilde{X} \xrightarrow{\pi} X$, then $F_G(\rho_{H!}K_{\tilde{E},\chi,\tilde{H}}) \in \mathrm{D}(\mathrm{Bun}_G)$ is a cuspidal automorphic sheaf on Bun_G for $E_{\check{G}}$. (For non-connected $\tilde{X}$ our assumption says that E_1, E_2 are non-isomorphic irreducible rank-two local systems on X equipped with $\det E_1 \widetilde{\to} \det E_2$.)

(2) If E is an irreducible rank-two local system on X with π^*E irreducible, then we have two cases. If there is an isomorphism $E \widetilde{\to} E \otimes \mathcal{E}_0$ on X, then $F_G(K_{E,H}) \in \mathrm{D}(\mathrm{Bun}_G)$ is isomorphic to the geometric Eisenstein series (for the Siegel parabolic of $\check{G}$), otherwise it is a cuspidal automorphic sheaf on Bun_G for $E_{\check{G}}$. (In particular, for $\tilde{X}$ non-connected we obtain an Eisenstein series this way.)

Question. In case (2) of Conjecture 2 for connected $\tilde{X}$, what about $F_G(K_{E,H} \otimes \mathcal{N})$?

5.2 In the rest of § 5 we assume, in addition, that $n = 1$. Let E be an irreducible rank-two local system on X. Assume that its restriction $\tilde{E} = \pi^*E$ is still irreducible.

The following is a geometric version of a theorem of Shimizu (see [Wal85, Theorem 1], it is also an argument supporting Conjecture 1 in the case $n = 1, m = 2$.

Proposition 8. *For $\tilde{X}$ split we have $F_{\tilde{H}}(\mathrm{Aut}_{E^*}) \widetilde{\to} A(\det E)_\Omega \otimes K_{\tilde{E},\mathcal{E}_0\otimes\det E,\tilde{H}}$. This isomorphism depends on a choice of (6).*

Proof. The proof proceeds in two steps.

Step 1. Let $j_d : \mathrm{Bun}_2 \times X^{(d)} \hookrightarrow \mathcal{S}_{\tilde{Q}}$ be the open immersion sending (L, D) to $L, \mathcal{C} = \bigwedge^2 L \otimes \Omega^{-1}(D)$ with canonical inclusion $\bigwedge^2 L \hookrightarrow \mathcal{C} \otimes \Omega$. Let $\mathcal{S}^d_{\tilde{Q}}$ be the open substack of $\mathcal{S}_{\tilde{Q}}$ given by $\deg(\mathcal{C} \otimes \Omega) - \deg L = d$. We claim that

$$F_{\mathcal{S}_{\tilde{Q}}}(\mathrm{Aut}_{E^*})|_{\mathcal{S}^d_{\tilde{Q}}} \widetilde{\to} j_{d!}(\mathrm{Aut}_E \boxtimes E^{(d)})[d].$$

Recall the stack $\mathcal{W}_{G,\tilde{Q}}$ classifying $M \in \mathrm{Bun}_2, L \in \mathrm{Bun}_2$, and $t : L \to M^* \otimes \Omega$. Let ${}^0\mathcal{W}_{G,\tilde{Q}} \subset \mathcal{W}_{G,\tilde{Q}}$ be the open substack given by the condition $t : L \hookrightarrow M^* \otimes \Omega$ is an inclusion. From cuspidality of Aut_{E^*} it follows that only ${}^0\mathcal{W}_{G,\tilde{Q}}$ contributes to $F_{\mathcal{S}_{\tilde{Q}}}(\mathrm{Aut}_{E^*})|_{\mathcal{S}^d_{\tilde{Q}}}$, so the latter is extension by zero under j_d.

Let $\epsilon : \mathrm{Bun}_2 \to \mathrm{Bun}_2$ be the involution sending M to $M^* \otimes \Omega$. Then $\epsilon^* \mathrm{Aut}_E \widetilde{\to} \mathrm{Aut}_{E^*}$ canonically. Our assertion follows from Hecke property of Aut_E.

Step 2. The map $\nu_{\tilde Q}:\mathrm{Bun}_{\tilde Q}\to\mathrm{Bun}_{\tilde H}$ sends $(L\in\mathrm{Bun}_2, 0\to\bigwedge^2 L\to V_1\to\mathcal{C}\to 0)$ to $V=L^*\otimes V_1$ with symmetric form $\mathrm{Sym}^2 V\to\mathcal{C}$. From Step 1 we obtain

$$\nu_{\tilde Q}^* F_{\tilde H}(\mathrm{Aut}_{E^*})\,\tilde{\to}\, A(\det E)_\Omega\otimes\nu_{\tilde Q}^* K_{\tilde E,\det E,\tilde H}.$$

There is an open substack ${}^0\mathrm{Bun}_{\tilde Q}\subset\mathrm{Bun}_{\tilde Q}$ with the following properties. The projection ${}^0\mathrm{Bun}_{\tilde Q}\to\mathrm{Bun}_{\tilde H}$ is smooth and surjective with connected fibres, and $\nu_{\tilde Q}^* K_{\tilde E,\tilde H}[\dim.\mathrm{rel}(\nu_{\tilde Q})]$ is a perverse sheaf over ${}^0\mathrm{Bun}_{\tilde Q}$. Our assertion follows. $\square$

Note that Proposition 8 (at least the corresponding non-canonical isomorphism) would also follow from Conjecture 1. We conjecture that Proposition 8 remains valid for $\tilde X$ nonsplit.

6. Bessel periods for $\mathbf{G}\mathbb{S}\mathbf{p_4}$

6.1.1 Keep the notation of § 5.1. In §§ 6.1.1 through 6.2 we assume $n=2$. Recall the stack $\mathcal{S}_P$ classifying $L\in\mathrm{Bun}_2,\mathcal{A}\in\mathrm{Bun}_1$, and $\mathrm{Sym}^2 L\stackrel{s}{\to}\mathcal{A}\otimes\Omega$. Denote by $\mathcal{S}_P^r\subset\mathcal{S}_P$ the open substack given by

$$2\deg(\mathcal{A}\otimes\Omega)-2\deg L=r.$$

Let ${}^{rss}X^{(r)}\subset X^{(r)}$ be the open subscheme classifying divisors $x_1+\cdots+x_r$ on X with x_i pairwise distinct. Let ${}^{rss}\mathcal{S}_P^r\subset\mathcal{S}_P^r$ be the open substack given by the condition that $L\stackrel{s}{\hookrightarrow}L^*\otimes\mathcal{A}\otimes\Omega$ and $\mathrm{div}(L^*\otimes\mathcal{A}\otimes\Omega/L)\in{}^{rss}X^{(r)}$. Set

$$\mathrm{RCov}^r=\mathrm{Pic}\,X\times_{\mathrm{Pic}\,X}{}^{rss}X^{(r)},$$

where the map ${}^{rss}X^{(r)}\to\mathrm{Pic}\,X$ sends D to $\mathcal{O}(-D)$, and $\mathrm{Pic}\,X\to\mathrm{Pic}\,X$ takes a line bundles to its tensor square. It is understood that ${}^{rss}X^{(0)}=\mathrm{Spec}\,k$.

We have a map $\mathfrak{p}_1:{}^{rss}\mathcal{S}_P^r\to\mathrm{RCov}^r$ sending the above point to $\mathcal{E}_\phi=(\mathcal{A}\otimes\Omega)^{-1}\otimes\det L$ with the induced inclusion $\mathcal{E}_\phi^2\hookrightarrow\mathcal{O}_X$.

Lemma 12 [Lys06b, 7.7.2]. *For $r\geqslant 0$ the stack RCov^r classifies two-sheeted coverings $\phi:Y\to X$ ramified exactly at $D_X\in{}^{rss}X^{(r)}$ with Y smooth. The stack ${}^{rss}\mathcal{S}_P^r$ identifies with that classifying collections $D_X\in{}^{rss}X^{(r)}$, a two-sheeted covering $\phi:Y\to X$ ramified exactly at D_X, and $\mathcal{B}\in\mathrm{Pic}\,Y$.*

The identification in Lemma 12 sends $\mathcal{B}\in\mathrm{Pic}\,Y$ to $L=\phi_*\mathcal{B}$ with symmetric form $\mathrm{Sym}^2 L\stackrel{s}{\to}N(\mathcal{B})\,\tilde{\to}\,\mathcal{A}\otimes\Omega$. Note that s admits a canonical section $N(\mathcal{B})(-D_X)\hookrightarrow\mathrm{Sym}^2 L$, which is a vector subbundle of $\mathrm{Sym}^2 L$. Let $\mathfrak{q}_1:{}^{rss}\mathcal{S}_P^r\to\mathrm{Bun}_2$ be the map sending $(L,\mathcal{A},s)$ as above to L.

Lemma 13. *For $r>4(g-1)$ the map $\mathfrak{q}_1:{}^{rss}\mathcal{S}_P^r\to\mathrm{Bun}_2$ is smooth.*

Proof. Since the projection ${}^{rss}\mathcal{S}_P^r\to\mathrm{RCov}^r$ is smooth, the stack ${}^{rss}\mathcal{S}_P^r$ is smooth. Since Bun_2 is also smooth, it suffices to show that the fibre of $\mathfrak{q}_1$ over a field-valued point $L\in\mathrm{Bun}_2$ is smooth.

Let us calculate the tangent space to the fibre of $\mathfrak{q}_1$ at a point $(L,\mathcal{A},s)$. For brevity, write $\mathcal{C}=\mathcal{A}\otimes\Omega$. Let K denote the cokernel of $\mathcal{O}\stackrel{s}{\hookrightarrow}\mathcal{C}\otimes\mathrm{Sym}^2 L^*$. The sheaf K is locally free. The tangent space in question identifies with $\mathrm{H}^0(X,K)$. We claim that $\mathrm{H}^1(X,K)=0$.

Indeed, suppose that $(L,\mathcal{A},s)$ is given by a collection: a two-sheeted covering $\phi:Y\to X$ ramified at $D_X\in{}^{rss}X^{(r)}$ and a line bundle $\mathcal{B}$ on Y. So, $L\,\tilde{\to}\,\phi_*\mathcal{B}$ and $s:\mathrm{Sym}^2 L\to\mathcal{C}\,\tilde{\to}\,N(\mathcal{B})$ is the natural symmetric form. Let $D_Y\in{}^{rss}Y^{(r)}$ be the ramification divisor of ϕ, so $D_X=\phi_*D_Y$. Then $\mathrm{Sym}^2 L$

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is included into the following cartesian square.

$$\begin{array}{ccc} N(\mathcal{B}) \oplus \phi_*(\mathcal{B}^2) & \longrightarrow & \mathcal{B}^2|_{D_Y} \oplus \mathcal{B}^2|_{D_Y} \\ \uparrow & & \uparrow \scriptstyle{\mathrm{diag}} \\ \mathrm{Sym}^2 L & \longrightarrow & \mathcal{B}^2_{D_Y} \end{array}$$

Let σ_ϕ be the nontrivial automorphism of Y over X. We have $K^* \widetilde{\to} \phi_*(\mathcal{B} \otimes \sigma_\phi^* \mathcal{B}^{-1}(-D_Y))$. So, $\mathrm{H}^0(X, K^* \otimes \Omega) \widetilde{\to} \mathrm{H}^0(Y, \phi^*\Omega \otimes \mathcal{B} \otimes \sigma_\phi^* \mathcal{B}^{-1}(-D_Y)) = 0$, because the degree of the corresponding line bundle on Y is $4(g-1) - r < 0$.

As $\mathfrak{q}_1$ is separable, and the dimensions of the tangent spaces to the fibres are constant, $\mathfrak{q}_1$ is smooth. □

Fix a two-sheeted covering $\phi : Y \to X$ ramified at $D_X \in {}^{rss}X^{(r)}$. Write σ_ϕ for the nontrivial automorphism of Y over X, and $\mathcal{E}_\phi$ for the σ_ϕ-anti-invariants in $\phi_*\mathcal{O}_Y$, it is equipped with $\mathcal{E}_\phi^2 \widetilde{\to} \mathcal{O}(-D_X)$. Write $D_Y \in Y^{(r)}$ for the ramification divisor of ϕ, so $D_X = \phi_* D_Y$. Recall that ϕ is recovered from $(\mathcal{E}_\phi, D_X)$ as $\mathrm{Spec}(\mathcal{O}_X \oplus \mathcal{E}_\phi)$, where the structure of a $\mathcal{O}_X$-algebra on $\mathcal{O}_X \oplus \mathcal{E}_\phi$ is given by $\mathcal{E}_\phi^2 \hookrightarrow \mathcal{O}_X$.

Recall the stack Bun_{U_ϕ}, its connected components $\mathrm{Bun}^a_{U_\phi}$ are indexed by $a \in \mathbb{Z}/2\mathbb{Z}$ (cf. Appendix A.1). Let ${}^0\mathrm{Bun}_{U_\phi} \subset \mathrm{Bun}_{U_\phi}$ be the open substack given by $\mathrm{H}^0(Y, \mathcal{V} \otimes \phi^*\Omega) = 0$ for $\mathcal{V} \in \mathrm{Bun}_{U_\phi}$ equipped with $N(\mathcal{V}) \widetilde{\to} \mathcal{O}_X$.

Let ${}^0\mathrm{Pic}\, Y$ be the preimage of ${}^0\mathrm{Bun}_{U_\phi}$ under $e_\phi : \mathrm{Pic}\, Y \to \mathrm{Bun}_{U_\phi}$ (cf. Appendix A.1). Denote by $\phi_1 : \mathrm{Pic}\, Y \to \mathrm{Bun}_2$ the map sending $\mathcal{B}$ to $\phi_*\mathcal{B}$.

For $g = 0$ we have ${}^0\mathrm{Bun}_{U_\phi} = \mathrm{Bun}_{U_\phi}$. If $g = 1$, then ${}^0\mathrm{Bun}_{U_\phi} \subset \mathrm{Bun}_{U_\phi}$ is given by the condition that $\mathcal{V}$ is not isomorphic to $\mathcal{O}_Y$.

Lemma 14. *We have the following.*

(i) *If $r \geqslant 4g - 4$, then ${}^0\mathrm{Bun}^a_{U_\phi}$ is nonempty for each $a \in \mathbb{Z}/2\mathbb{Z}$. So, the intersection of ${}^0\mathrm{Pic}\, Y$ with each connected component of $\mathrm{Pic}\, Y$ is nonempty.*

(ii) *The restriction of $\phi_1 : \mathrm{Pic}\, Y \to \mathrm{Bun}_2$ to the open substack ${}^0\mathrm{Pic}\, Y \subset \mathrm{Pic}\, Y$ is smooth.*

Proof. (i) Write $\mathrm{Ker}\, \underline{N}$ for the kernel of the norm map $\underline{N} : \underline{\mathrm{Pic}}\, Y \to \underline{\mathrm{Pic}}\, X$ (cf. Appendix A.1). Let ${}^0\mathrm{Ker}\, \underline{N}$ be the open subscheme given by $\mathrm{H}^0(Y, \mathcal{V} \otimes \phi^*\Omega) = 0$ for $\mathcal{V} \in \mathrm{Ker}\, \underline{N}$. Then ${}^0\mathrm{Bun}_{U_\phi}$ is the preimage of ${}^0\mathrm{Ker}\, \underline{N}$ under the projection $\mathrm{Bun}_{U_\phi} \to \mathrm{Ker}\, \underline{N}$.

Let Z denote the preimage of Ω^2 under the map $X^{(4g-4)} \to \underline{\mathrm{Pic}}\, X$ sending D to $\mathcal{O}(D)$. Here $\underline{\mathrm{Pic}}\, X$ is the Picard scheme of X. Let Z' be the preimage of Z under $\phi : Y^{(4g-4)} \to X^{(4g-4)}$. We have $Z' = \emptyset$ for $g = 0$, $Z' = \mathrm{Spec}\, k$ for $g = 1$, and $\dim Z' = 3g - 4$ for $g > 1$. Then ${}^0\mathrm{Ker}\, \underline{N}$ is the complement to the image of the map $Z' \to \mathrm{Ker}\, \underline{N}$ sending D to $(\phi^*\Omega^{-1})(D)$. Since each connected component of $\mathrm{Ker}\, \underline{N}$ is of dimension $g - 1 + r/2$, our assertion follows.

(ii) Since both ${}^0\mathrm{Pic}\, Y$ and Bun_2 are smooth, it suffices to check that for $\mathcal{B} \in {}^0\mathrm{Pic}\, Y$ the natural map $\mathrm{H}^1(Y, \mathcal{O}) \to \mathrm{H}^1(X, \mathcal{E}nd(\phi_*\mathcal{B})) \widetilde{\to} \mathrm{H}^1(Y, \mathcal{B} \otimes \phi^*((\phi_*\mathcal{B})^*))$ is surjective. We have a cartesian square

$$\begin{array}{ccc} \mathcal{O}(D_Y) \oplus \dfrac{\mathcal{B}}{\sigma_\phi^*\mathcal{B}}(D_Y) & \longrightarrow & \mathcal{O}(D_Y)/\mathcal{O} \oplus \mathcal{O}(D_Y)/\mathcal{O} \\ \uparrow & & \uparrow \\ \mathcal{B} \otimes \phi^*((\phi_*\mathcal{B})^*) & \longrightarrow & \mathcal{O}(D_Y)/\mathcal{O} \end{array}$$

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where the right vertical arrow is the diagonal map. This yields an exact sequence

$$0 \to \mathcal{O}_Y \to \mathcal{B} \otimes \phi^*((\phi_*\mathcal{B})^*) \to \frac{\mathcal{B}}{\sigma_\phi^*\mathcal{B}}(D_Y) \to 0.$$

We have $\mathrm{H}^1(Y, \mathcal{B}(D_Y) \otimes \sigma_\phi^*\mathcal{B}^{-1}) = \mathrm{H}^0(Y, \phi^*\Omega \otimes \sigma_\phi^*\mathcal{B} \otimes \mathcal{B}^{-1})^* = 0$, because $\sigma_\phi^*\mathcal{B} \otimes \mathcal{B}^{-1} \in {}^0\mathrm{Bun}_{U_\phi}$. We are done. □

Our purpose is to study the $*$-restriction of $F_{\mathcal{S}}(\rho_{H\,!} K_{\tilde{E},\mathcal{X},\tilde{H}})$ under $\operatorname{Pic} Y \hookrightarrow {}^{rss}\mathcal{S}_P^r \subset \mathcal{S}_P$. For $d \in \mathbb{Z}$ set $\bar{d} = 2\deg\Omega - d$. The complex $F_{\mathcal{S}}(\rho_{H\,!} K_{\tilde{E},\mathcal{X},\tilde{H}}|_{\mathrm{Bun}_H^d})$ lies in $\mathrm{D}(\operatorname{Pic}^{\bar{d}} Y)$.

Define $\tilde{Y}$ by the following cartesian square.

$$\begin{array}{ccc} \tilde{Y} & \xrightarrow{\tilde{\pi}} & Y \\ \downarrow{\scriptstyle \tilde{\phi}} & & \downarrow{\scriptstyle \phi} \\ \tilde{X} & \xrightarrow{\pi} & X \end{array} \qquad (38)$$

Consider the exact sequence $1 \to \mathbb{G}_m \to \mathbb{G}_m \times \mathbb{G}_m \xrightarrow{N} \mathbb{G}_m \to 1$, where N is the product map, and the first map sends z to (z, z^{-1}). Twisting it by the $\mathbb{Z}/2\mathbb{Z}$-torsor $\tilde{X} \xrightarrow{\pi} X$ via the action permuting two factors of $\mathbb{G}_m \times \mathbb{G}_m$, we obtain an exact sequence $1 \to U_\pi \to \pi_*\mathbb{G}_m \xrightarrow{N} \mathbb{G}_m \to 1$ of group schemes on X; here N is the norm map. Consider the composition $U_\pi \to \pi_*\mathbb{G}_m \to \pi_*\tilde{\phi}_*\mathbb{G}_m \widetilde{\to} \phi_*\tilde{\pi}_*\mathbb{G}_m$. Define the group scheme R_ϕ on X by the exact sequence

$$1 \to U_\pi \to \phi_*\tilde{\pi}_*\mathbb{G}_m \to R_\phi \to 1. \qquad (39)$$

The corresponding map $\operatorname{Pic}\tilde{Y} \to \mathrm{Bun}_{R_\phi}$ is smooth and surjective.

Let ${}_\phi\mathrm{GL}_2$ be the group scheme of automorphisms of $\phi_*\mathcal{O}_Y$, this is an inner form of GL_2. We denote by the same symbol ${}_\phi\mathrm{GL}_2$ its restriction to $\tilde{X}$. We have a natural map $\tilde{\phi}_*\mathbb{G}_m \to {}_\phi\mathrm{GL}_2$ of group schemes on $\tilde{X}$. Let ${}_\phi\tilde{H}$ be the group scheme on X included into a morphism of exact sequences

$$\begin{array}{ccccccccc} 1 & \longrightarrow & U_\pi & \longrightarrow & \pi_*\tilde{\phi}_*\mathbb{G}_m & \longrightarrow & R_\phi & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & U_\pi & \longrightarrow & \pi_*({}_\phi\mathrm{GL}_2) & \longrightarrow & {}_\phi\tilde{H} & \longrightarrow & 1 \end{array}$$

Since $\mathrm{Bun}_{{}_\phi\tilde{H}} \widetilde{\to} \mathrm{Bun}_{\tilde{H}}$ canonically, we obtain a morphism $\mathfrak{q}_{R_\phi} : \mathrm{Bun}_{R_\phi} \to \mathrm{Bun}_{\tilde{H}}$.

For $\mathcal{L} \in \operatorname{Pic}\tilde{X}$ we have $\phi^*N(\mathcal{L}) \widetilde{\to} N(\tilde{\phi}^*\mathcal{L})$ canonically. Consider the map $\phi_*\tilde{\pi}_*\mathbb{G}_m \xrightarrow{\phi_*N} \phi_*\mathbb{G}_m$ induced by the norm $\tilde{\pi}_*\mathbb{G}_m \xrightarrow{N} \mathbb{G}_m$. It is easy to check that $U_\pi \subset \operatorname{Ker}\phi_*N$, so we obtain a map $R_\phi \to \phi_*\mathbb{G}_m$. Let $\mathfrak{p}_{R_\phi}$ denote the composition of the extension of scalars $\mathrm{Bun}_{R_\phi} \to \operatorname{Pic} Y$ with the automorphism $\epsilon : \operatorname{Pic} Y \widetilde{\to} \operatorname{Pic} Y$ sending $\mathcal{B}$ to $\mathcal{B}^\star = \mathcal{B}^* \otimes \Omega_Y$. So, the following diagram commutes.

$$\begin{array}{ccccc} \operatorname{Pic}\tilde{Y} & \longrightarrow & \mathrm{Bun}_{R_\phi} & \xrightarrow{\mathfrak{q}_{R_\phi}} & \mathrm{Bun}_{\tilde{H}} \\ \downarrow{\scriptstyle N} & & \downarrow{\scriptstyle \mathfrak{p}_{R_\phi}} & & \\ \operatorname{Pic} Y & \xrightarrow{\epsilon} & \operatorname{Pic} Y & & \end{array} \qquad (40)$$

When $(\mathcal{E}_\phi, D_X)$ run through RCov^r, the group schemes R_ϕ are naturally organized into a group scheme R over $\mathrm{RCov}^r \times X$. Let Bun_R denote the stack over RCov^r associating to a scheme S the following category: a map $S \to \mathrm{RCov}^r$, and a R_S-torsor on $S \times X$, where R_S is the restriction of R under $S \times X \to \mathrm{RCov}^r \times X$.

Diagrams (40) naturally form a family

$$\mathrm{Bun}_{\tilde{H}} \xleftarrow{\mathfrak{q}_R} \mathrm{Bun}_R \xrightarrow{\mathfrak{p}_R} {}^{rss}\mathcal{S}_P^r.$$

PROPOSITION 9. *The restriction of* $F_{\mathcal{S}}(\rho_{H\,!}K_{\tilde{E},\chi,\tilde{H}})$ *to the open substack* ${}^{rss}\mathcal{S}_P^r \subset \mathcal{S}_P$ *is canonically isomorphic to*

$$(\mathfrak{p}_R)_!\, \mathfrak{q}_R^* K_{\tilde{E},\chi,\tilde{H}}[\dim.\mathrm{rel}(\mathfrak{q}_R)].$$

In particular, for a k*-point of* RCov^r *given by* $\phi : Y \to X$*, the* $*$*-restriction identifies canonically*

$$F_{\mathcal{S}}(\rho_{H\,!}K_{\tilde{E},\chi,\tilde{H}})|_{\mathrm{Pic}\, Y} \widetilde{\to} (\mathfrak{p}_{R_\phi})_!\, \mathfrak{q}_{R_\phi}^* K_{\tilde{E},\chi,\tilde{H}}[\dim.\mathrm{rel}(\mathfrak{q}_R)].$$

Proof. Define a map $\zeta_\phi : \mathrm{Pic}\,\tilde{Y} \to \mathcal{V}_{\tilde{H},P}$ as follows. Given $\mathcal{B} \in \mathrm{Pic}\,\tilde{Y}$, let $W = \tilde{\phi}_*\mathcal{B}$ and $(V, \mathcal{C}, \mathrm{Sym}^2 V \to \mathcal{C}, \gamma) \in \mathrm{Bun}_{\tilde{H}}$ be the corresponding $\tilde{H}$-torsor. We have $\sigma^*\tilde{\phi}_*\mathcal{B} \widetilde{\to} \tilde{\phi}_*\sigma^*\mathcal{B}$, so there is a natural map

$$\pi^*V \widetilde{\to} W \otimes \sigma^*W \to \tilde{\phi}_*\tilde{\pi}^* N_Y(\mathcal{B}) \widetilde{\to} \pi^*\phi_* N_Y(\mathcal{B}),$$

where $N_Y : \mathrm{Pic}\,\tilde{Y} \to \mathrm{Pic}\,Y$ is the norm map. It descends to a map $V \to \phi_* N_Y(\mathcal{B})$. So, for $L = \phi_*(N_Y(\mathcal{B})^\star)$ we obtain a map $t : V \to L^* \otimes \Omega$. By definition, ζ_ϕ sends $\mathcal{B}$ to $(V, \mathcal{C}, \mathrm{Sym}^2 V \to \mathcal{C}, \gamma) \in \mathrm{Bun}_{\tilde{H}}$, $L \in \mathrm{Bun}_2$, and $t : V \to L^* \otimes \Omega$.

Denote by $\tilde{\phi}_1 : \mathrm{Pic}\,\tilde{Y} \to \mathrm{Bun}_{2,\tilde{X}}$ the map sending $\mathcal{B}$ to $\tilde{\phi}_*\mathcal{B}$. We have the following commutative diagram.

$$\begin{array}{ccccc}
\mathrm{Bun}_{2,\tilde{X}} & \xleftarrow{\tilde{\phi}_1} & \mathrm{Pic}\,\tilde{Y} & \xrightarrow{\epsilon\circ N} & \mathrm{Pic}\,Y \\
\downarrow & & \downarrow{\scriptstyle \zeta_\phi} & & \downarrow \\
\mathrm{Bun}_{\tilde{H}} & \longleftarrow & \mathcal{V}_{\tilde{H},P} & \longrightarrow & \mathcal{S}_P
\end{array}$$

It extends naturally to the following diagram.

$$\begin{array}{ccccc}
 & & \mathrm{Pic}\,\tilde{Y} & & \\
 & \swarrow{\scriptstyle \zeta_\phi} & \downarrow & \searrow{\scriptstyle \epsilon\circ N} & \\
\mathcal{V}_{\tilde{H},P} & \longleftarrow & \mathrm{Bun}_{R_\phi} & \xrightarrow{\mathfrak{p}_{R_\phi}} & \mathrm{Pic}\,Y \\
 & \searrow & & \swarrow & \\
 & & \mathcal{S}_P & &
\end{array}$$

As ϕ varies in RCov^r these diagrams form the following family.

$$\begin{array}{ccccc}
 & & \mathrm{Bun}_R & \xrightarrow{\mathfrak{p}_R} & {}^{rss}\mathcal{S}_P^r \\
 & \swarrow{\scriptstyle \mathfrak{q}_R} & \downarrow & & \downarrow \\
\mathrm{Bun}_{\tilde{H}} & \longleftarrow & \mathcal{V}_{\tilde{H},P} & \longrightarrow & \mathcal{S}_P
\end{array} \tag{41}$$

Our assertion is reduced to the following lemma. □

LEMMA 15. *The square in (41) is cartesian.*

Our proof of Lemma 15 uses the following elementary observation.

SUBLEMMA 1. *Let* K *be a field of characteristic different from two. Let* V_i *be two-dimensional* K*-vector spaces. Let* $\mathcal{B}, \mathcal{B}'$ *be one-dimensional* K*-vector spaces. Equip* $\mathcal{B} \oplus \mathcal{B}'$ *with the quadratic form*

$$s : \mathrm{Sym}^2(\mathcal{B} \oplus \mathcal{B}') \widetilde{\to} \mathcal{B}^2 \oplus \mathcal{B}'^2 \oplus \mathcal{B} \otimes \mathcal{B}' \overset{(0,0,\mathrm{id})}{\to} \mathcal{B} \otimes \mathcal{B}'.$$

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Assume we are given a map $t : \mathcal{B}\oplus\mathcal{B}' \to V_1\otimes V_2$ such that there is the following commutative diagram.

$$\begin{array}{ccc} \mathrm{Sym}^2(\mathcal{B}\oplus\mathcal{B}') & \xrightarrow{t\otimes t} & \mathrm{Sym}^2(V_1\otimes V_2) \\ \downarrow s & & \downarrow \\ \mathcal{B}\otimes\mathcal{B}' & \xrightarrow{\sim} & \det V_1\otimes\det V_2 \end{array}$$

Then there exist a unique decomposition into a direct sum of one-dimensional subspaces $V_i \widetilde{\to} U_i\oplus U_i'$ and unique isomorphisms $\mathcal{B}\widetilde{\to} U_1\otimes U_2$, $\mathcal{B}'\widetilde{\to} U_1'\otimes U_2'$ under which t identifies with a natural inclusion

$$(U_1\otimes U_2)\oplus(U_1'\otimes U_2')\hookrightarrow V_1\otimes V_2. \qquad \square$$

Proof of Lemma 15. Consider a point of $\mathcal{V}_{\tilde H,P}$ given by $\mathcal{F}_{\tilde H} = (V,\mathcal{C},\mathrm{Sym}^2 V\to\mathcal{C},\gamma)\in \mathrm{Bun}_{\tilde H}$, $L\in\mathrm{Bun}_2$ and $t: V\to L^*\otimes\Omega$. Assume that its image in $\mathcal{S}_P$ is identified with a point $(\phi: Y\to X,\ \mathcal{B}\in\mathrm{Pic}\,Y)$ of ${}^{rss}\mathcal{S}_P^r$. So, we are given an isomorphism $L\widetilde{\to}\phi_*\mathcal{B}$ and the following diagram commutes.

$$\begin{array}{ccccc} \mathrm{Sym}^2 V & \xrightarrow{t\otimes t} & \mathrm{Sym}^2(L^*\otimes\Omega) & & \\ \uparrow & & \uparrow & & \\ \mathcal{C} & \xrightarrow{\sim} & N(\mathcal{B})^{-1}\otimes\Omega^2 & \xrightarrow{\sim} & N(\mathcal{B}^\star)(-D_X) \end{array} \qquad (42)$$

Since $L^*\otimes\Omega\widetilde{\to}\phi_*\mathcal{B}^\star$, we view the datum of t as $t:\phi^*V\to\mathcal{B}^\star$. We have a commutative diagram

$$\begin{array}{ccc} \phi^*V & \xrightarrow{(t,\sigma_\phi^* t)} & \mathcal{B}^\star\oplus\sigma_\phi^*\mathcal{B}^\star \\ \downarrow & \nearrow & \\ \phi^*\phi_*\mathcal{B}^\star & & \end{array} \qquad (43)$$

where

$$\phi^*\phi_*\mathcal{B}^\star = \{b\in\mathcal{B}^\star\oplus\sigma_\phi^*\mathcal{B}^\star \mid \text{the image of } b \text{ in } (\mathcal{B}^\star\oplus\sigma_\phi^*\mathcal{B}^\star)|_{D_Y} \text{ lies in the diagonal } \mathcal{B}^\star|_{D_Y}\}.$$

Pick a lifting of $\mathcal{F}_{\tilde H}\in\mathrm{Bun}_{\tilde H}$ to a point $W\in\mathrm{Bun}_{2,\tilde X}$. We obtain $\pi^*V\widetilde{\to} W\otimes\sigma^*W$. From (43) we obtain a map

$$\tilde\phi^*(W\otimes\sigma^*W)\xrightarrow{(t,\sigma_\phi^* t)}\tilde\pi^*(\mathcal{B}^\star\oplus\sigma_\phi^*\mathcal{B}^\star)$$

whose tensor square fits into the following commutative diagram.

$$\begin{array}{ccc} \tilde\phi^*\,\mathrm{Sym}^2(W\otimes\sigma^*W) & \longrightarrow & \tilde\pi^*(\mathcal{B}^{\star 2}\oplus\sigma_\phi^*\mathcal{B}^{\star 2}\oplus\mathcal{B}^\star\otimes\sigma_\phi^*\mathcal{B}^\star) \\ \uparrow & & \uparrow (0,0,1) \\ \tilde\phi^*(\det W\otimes\sigma^*\det W) & \xrightarrow{\sim} & \tilde\pi^*(\mathcal{B}^\star\otimes\sigma_\phi^*\mathcal{B}^\star(-2D_Y)) \end{array}$$

By abuse of notation, we also write σ and σ_ϕ for the involutions of $\tilde Y$ obtained by base change in the square (38).

Note that any surjection $\tilde\phi^*W\to\mathcal{L}$, where $\mathcal{L}$ is a line bundle on $\tilde Y$, gives rise to a map $\xi_{\mathcal{L}}: V\to\phi_*N_Y(\mathcal{L})$. Indeed, the composition

$$\pi^*V\widetilde{\to}W\otimes\sigma^*W\to\tilde\phi_*\mathcal{L}\otimes\sigma^*\tilde\phi_*\mathcal{L}\to\tilde\phi_*(\mathcal{L}\otimes\sigma^*\mathcal{L})\widetilde{\to}\tilde\phi_*\tilde\pi^*N_Y(\mathcal{L})$$

descends to a map $\xi_{\mathcal{L}}: V\to\phi_*N_Y(\mathcal{L})$.

By Sublemma 1, there is a unique rank-one subbundle $W_1\subset\tilde\phi^*W$, for which we set $\mathcal{L} = (\tilde\phi^*W)/W_1$, and a unique σ-invariant inclusion of coherent sheaves $\mathcal{L}\otimes\sigma^*\mathcal{L}\hookrightarrow\tilde\pi^*\mathcal{B}^\star$ with the following properties. The latter inclusion gives rise to an inclusion $N_Y(\mathcal{L}) \hookrightarrow \mathcal{B}^\star$, and the composition

$$V \xrightarrow{\xi_{\mathcal{L}}} \phi_* N_Y(\mathcal{L}) \to \phi_* \mathcal{B}^\star \tag{44}$$

equals t.

Taking symmetric squares in (44), we obtain a commutative diagram

$$\begin{array}{ccccccc} \mathrm{Sym}^2 V & \longrightarrow & \mathrm{Sym}^2(\phi_* N_Y(\mathcal{L})) & \hookrightarrow & \mathrm{Sym}^2(\phi_*\mathcal{B}^\star) & \longrightarrow & N(\mathcal{B}^\star) \\ \uparrow & & \uparrow & & & \nearrow & \\ \mathcal{C} & & (N_X N_Y(\mathcal{L}))(-D_X) & \hookrightarrow & N_X(\mathcal{B}^\star)(-D_X) & & \end{array} \tag{45}$$

in which the middle square is cartesian (and all three of the vertical arrows are subbundles). Using (42) we conclude that there is a unique isomorphism $\eta : \mathcal{C} \widetilde{\to} (N_X N_Y(\mathcal{L}))(-D_X)$ making (45) commute, and the inclusion $N_Y(\mathcal{L}) \hookrightarrow \mathcal{B}^\star$ is actually an isomorphism.

Let us show that the natural map $W \to \tilde{\phi}_*\mathcal{L}$ is an isomorphism. We have inclusions

$$\tilde{\phi}^* W \hookrightarrow \tilde{\phi}^*\tilde{\phi}_*\mathcal{L} \hookrightarrow \mathcal{L} \oplus \sigma_\phi^*\mathcal{L},$$

whose determinants yield $\tilde{\phi}^* \det W \hookrightarrow \det(\tilde{\phi}^*\tilde{\phi}_*\mathcal{L}) \widetilde{\to} (\mathcal{L}\otimes\sigma_\phi^*\mathcal{L})(-\tilde{\pi}^* D_Y)$. Symmetrizing with respect to the action of σ, one obtains inclusions

$$\tilde{\phi}^*\pi^*\mathcal{C} \widetilde{\to} \tilde{\phi}^*(\det W \otimes \sigma^* \det W) \hookrightarrow (\mathcal{L}\otimes\sigma^*\mathcal{L})\otimes\sigma_\phi^*(\mathcal{L}\otimes\sigma^*\mathcal{L})(-2\tilde{\pi}^* D_Y) \widetilde{\to} (\tilde{\pi}^*\phi^* N_X N_Y(\mathcal{L}))(-2\tilde{\pi}^* D_Y)$$

whose composition is an isomorphism (equal to restriction of η). So, $W \widetilde{\to} \tilde{\phi}_*\mathcal{L}$ is an isomorphism.

Viewing $\mathcal{L}$ as a $\phi_*\tilde{\pi}_*\mathbb{G}_m$-torsor on X, let $\mathcal{F}_{R_\phi}$ be the R_ϕ-torsor on X obtained from it by extension of scalars (39). Then $\mathfrak{q}_{R_\phi}(\mathcal{F}_{R_\phi}) \widetilde{\to} \mathcal{F}_{\tilde{H}}$ equips $\mathcal{F}_{\tilde{H}}$ with a R_ϕ-structure that does not depend on a choice of a lifting of $\mathcal{F}_{\tilde{H}}$ to a $\pi_*\,\mathrm{GL}_2$-torsor. We are done. □

Remark 9. Consider the case of $\tilde{X} \xrightarrow{\pi} X$ split. We have an exact sequence $0 \to \mathbb{Z}/2\mathbb{Z} \to \pi_1(\tilde{H}) \to \mathbb{Z} \to 0$. The $*$-restriction

$$F_{\mathcal{S}}(\rho_{H\,!} K_{\tilde{E},\chi,\tilde{H}})|_{\mathrm{Pic}^{\bar{d}} Y}$$

is naturally a direct sum of two complexes indexed by $\theta \in \pi_1(\tilde{H})$ whose image in $\mathbb{Z}$ is d. If Y is connected then $\mathfrak{q}_{R_\phi} : \mathrm{Bun}_{R_\pi} \to \mathrm{Bun}_{\tilde{H}}$ induces a bijection at the level of connected components $\pi_0(\mathrm{Bun}_{R_\phi}) \widetilde{\to} \pi_0(\mathrm{Bun}_{\tilde{H}}) \widetilde{\to} \pi_1(\tilde{H})$.

6.1.2 Recall the diagram

$$\mathrm{Bun}_2 \xleftarrow{\mathfrak{q}_1} {}^{rss}\mathcal{S}_P^r \xrightarrow{\mathfrak{p}_1} \mathrm{RCov}^r$$

introduced in § 6.1.1. In this section we prove the following acyclicity result.

Theorem 3. *Let E be a rank-two local system on X and let $K \in \mathrm{D}(\mathrm{Bun}_2)$ be a Hecke eigensheaf with eigenvalue E. Then $\mathfrak{q}_1^* K$ is ULA with respect to $\mathfrak{p}_1 : {}^{rss}\mathcal{S}_P^r \xrightarrow{\mathfrak{p}_1} \mathrm{RCov}^r$.*

Proof. The proof proceeds in three steps.

Step 1. The difficulty comes from the fact that $\mathfrak{q}_1 \times \mathfrak{p}_1 : {}^{rss}\mathcal{S}_P^r \to \mathrm{Bun}_2 \times \mathrm{RCov}^r$ is not smooth (for $g \geqslant 1$), we come around it using the Hecke property of K. Namely, for $d \geqslant 0$ consider the diagram

$$X^{(d)} \times \mathrm{Bun}_2 \xleftarrow{\mathrm{supp}\times p'} \mathrm{Mod}_2^d \xrightarrow{p} \mathrm{Bun}_2,$$

where Mod_2^d is the stack classifying a lower modification $(L \subset L')$ of rank-two vector bundles on X with $\deg(L'/L) = d$, the map p (respectively, p') sends $(L \subset L')$ to L (respectively, L'). The map supp sends this point to $\operatorname{div}(L'/L)$. As in [FGV02, § 9.5] one shows that

$$(\operatorname{supp} \times p')_! p^* K[d] \widetilde{\to} (E^*)^{(d)} \boxtimes K,$$

this is the only property of K that we actually use.

Define temporarily the stack $\mathcal{X}$ and the maps $p_{\mathcal{X}}, p'_{\mathcal{X}}$ by the following diagram, where the square is cartesian.

$$\begin{array}{ccc}
 & & \mathrm{Bun}_2 \times \mathrm{RCov}^r \\
 & \overset{p_{\mathcal{X}}}{\nearrow} & \uparrow{\scriptstyle p \times \mathrm{id}} \\
\mathcal{X} & \longrightarrow & \mathrm{Mod}_2^d \times \mathrm{RCov}^r \\
\downarrow{\scriptstyle p'_{\mathcal{X}}} & & \downarrow{\scriptstyle \operatorname{supp} \times p' \times \mathrm{id}} \\
X^{(d)} \times {}^{rss}\mathcal{S}_P^r & \xrightarrow{\mathrm{id} \times \mathfrak{q}_1 \times \mathfrak{p}_1} & X^{(d)} \times \mathrm{Bun}_2 \times \mathrm{RCov}^r
\end{array}$$

The above property of K yields an isomorphism

$$(p'_{\mathcal{X}})_! p^*_{\mathcal{X}} (K \boxtimes \bar{\mathbb{Q}}_\ell)[d] \widetilde{\to} (E^*)^{(d)} \boxtimes \mathfrak{q}_1^* K. \tag{46}$$

Step 2. Let us show that for $d > 4g-4$ the map $p_{\mathcal{X}} : \mathcal{X} \to \mathrm{Bun}_2 \times \mathrm{RCov}^r$ is smooth. The projection $\mathcal{X} \to {}^{rss}\mathcal{S}_P^r$ is smooth of relative dimension $2d$. Since ${}^{rss}\mathcal{S}_P^r$ is smooth, $\mathcal{X}$ is also smooth.

Fix a k-point of RCov^r given by a two-sheeted covering $\phi : Y \to X$. The corresponding objects D_Y, $\mathcal{E}_\phi$, and σ_ϕ are as in § 6.1.1. Let $p_{\mathcal{X},\phi} : \mathcal{X}_\phi \to \mathrm{Bun}_2$ be obtained from $p_{\mathcal{X}}$ by the base change $\mathrm{Bun}_2 \times \operatorname{Spec} k \to \mathrm{Bun}_2 \times \mathrm{RCov}^r$. The stack $\mathcal{X}_\phi$ classifies $L \in \mathrm{Bun}_2$ and $\mathcal{B} \in \operatorname{Pic} Y$ with an inclusion of coherent sheaves $L \subset \phi_* \mathcal{B}$ such that $\operatorname{div}((\phi_*\mathcal{B})/L)$ is of degree d. It suffices to show that $p_{\mathcal{X},\phi}$ is smooth.

Write $\overline{\mathrm{Bun}}_{B,Y}$ for the stack classifying $V_1 \in \operatorname{Pic} Y$, $V \in \mathrm{Bun}_{2,Y}$, and an inclusion of coherent sheaves $V_1 \subset V$. Let ${}^0\overline{\mathrm{Bun}}_{B,Y}$ be the open substack given by $\mathrm{H}^1(Y, V_1^* \otimes (V/V_1)) = 0$. One checks that the projection ${}^0\overline{\mathrm{Bun}}_{B,Y} \to \mathrm{Bun}_{2,Y}$ is smooth. Set

$$\mathcal{Y} = \mathrm{Bun}_2 \times_{\mathrm{Bun}_{2,Y}} {}^0\overline{\mathrm{Bun}}_{B,Y},$$

where the map $\mathrm{Bun}_2 \to \mathrm{Bun}_{2,Y}$ sends L to $\phi^* L^*$. So, the projection $\mathcal{Y} \to \mathrm{Bun}_2$ is smooth.

We have an open immersion $j : \mathcal{X}_\phi \hookrightarrow \mathrm{Bun}_2 \times_{\mathrm{Bun}_{2,Y}} \overline{\mathrm{Bun}}_{B,Y}$ sending $(L \subset \phi_*\mathcal{B})$ to $L \in \mathrm{Bun}_2$, $V_1 = \mathcal{B}^{-1}$ with the induced inclusion $\mathcal{B}^{-1} \hookrightarrow \phi^* L^*$. It suffices to show that the image of j is contained in $\mathcal{Y}$.

Let $(L \subset \phi_*\mathcal{B})$ be a k-point of $\mathcal{X}_\phi$ with $D = \operatorname{div}((\phi_*\mathcal{B})/L)$. Note that $(\det L)(D) \widetilde{\to} \mathcal{E}_\phi \otimes N(\mathcal{B})$. Define an effective divisor D' on Y and $L_1 \in \operatorname{Pic} Y$ by the exact sequence

$$0 \to L_1 \to \phi^* L \to \mathcal{B}(-D') \to 0$$

Then $L \subset \phi_*(\mathcal{B}(-D'))$, and taking the determinants we obtain $\det L \subset \mathcal{E}_\phi \otimes N(\mathcal{B})(-\phi_* D')$, so $D \geqslant \phi_* D'$. We must show that $\mathrm{H}^1(Y, \mathcal{B} \otimes L_1^*) = 0$.

We have $L_1 \otimes \mathcal{B}(-D') \widetilde{\to} \phi^* \det L \widetilde{\to} \mathcal{B} \otimes \sigma_\phi^* \mathcal{B}(-D_Y - \phi^* D)$, because $\phi^* \mathcal{E}_\phi \widetilde{\to} \mathcal{O}(-D_Y)$. Our assertion follows from the fact that

$$\mathcal{B}^* \otimes L_1 \otimes \Omega_Y \widetilde{\to} \frac{\sigma_\phi^* \mathcal{B}}{\mathcal{B}} \otimes \phi^* \Omega(D' - \phi^* D)$$

is of degree $4g - 4 - 2d + \deg D' \leqslant 4g - 4 - d < 0$.

Step 3. Assume $d > 4g - 4$. Since $K \boxtimes \bar{\mathbb{Q}}_\ell$ is ULA with respect to the projection $\mathrm{Bun}_2 \times \mathrm{RCov}^r \to \mathrm{RCov}^r$, it follows that $p^*_{\mathcal{X}}(K \boxtimes \bar{\mathbb{Q}}_\ell)$ is ULA over RCov^r. Since $p'_{\mathcal{X}}$ is proper,

$$(p'_{\mathcal{X}})_! p^*_{\mathcal{X}} (K \boxtimes \bar{\mathbb{Q}}_\ell)$$

is ULA over RCov^r (cf. [BG02, § 5.1.2]). Using (46), we learn that $(E^*)^{(d)} \boxtimes \mathfrak{q}_1^* K$ is ULA over RCov^r. So, the restriction of $(E^*)^{(d)} \boxtimes \mathfrak{q}_1^* K$ to ${}^{rss}X^{(d)} \times {}^{rss}\mathcal{S}_P^r$ is also ULA over RCov^r. Since $E^{(d)}$ is a local system over ${}^{rss}X^{(d)}$, and the ULA property is local in the smooth topology of the source, our assertion follows. □

Consider a k-point of RCov^r given by $\phi : Y \to X$ as in § 6.1.1. Recall that $\phi_1 : \mathrm{Pic}\, Y \to \mathrm{Bun}_2$ sends $\mathcal{B}$ to $\phi_* \mathcal{B}$. Using [BG02, Property 4 of § 5.1.2] and Lemma 13 we obtain the following.

COROLLARY 6. *Let E be a rank-two local system on X and let K be a E-Hecke eigensheaf on* Bun_2*. Then*

$$\mathbb{D}(\phi_1^* K[\dim.\mathrm{rel}(\phi_1)]) \widetilde{\to} \phi_1^* \mathbb{D}(K)[\dim.\mathrm{rel}(\phi_1)].$$

In addition, if K is perverse, then for $r > 4g - 4$ the sheaf $\phi_1^* K[\dim.\mathrm{rel}(\phi_1)]$ is perverse.

Remark 10. (i) Let E be an irreducible rank-two local system on X and let Aut_E be the corresponding automorphic sheaf on Bun_2 normalized as in [FGV02]. Then for any r the complex $\phi_1^* \mathrm{Aut}_E$ is a direct sum of (possibly shifted) perverse sheaves. Indeed, take $d > 4g - 4$ and apply the decomposition theorem for the (shifted) perverse sheaf $p_{\mathcal{X},\phi}^* \mathrm{Aut}_E$ and the proper map $p'_{\mathcal{X},\phi} : \mathcal{X}_\phi \to X^{(d)} \times \mathrm{Pic}\, Y$ as in the proof of Theorem 3.

(ii) The map $\phi_1 : \mathrm{Pic}\, Y \to \mathrm{Bun}_2$ is not flat, because its fibres have different dimensions (this is related to the fact that the dimension of the scheme of automorphisms of $L \in \mathrm{Bun}_2$ varies).

6.2 We may view σ and σ_ϕ as (commuting) automorphisms of $\tilde{Y}$. Let Z be the quotient of $\tilde{Y}$ by the involution $\sigma \circ \sigma_\phi$, so we get two-sheeted coverings $\tilde{Y} \xrightarrow{\alpha} Z \xrightarrow{\beta} X$. Note that Z is smooth, and α is unramified. Let D_Z be the ramification divisor of β, then $\beta_* D_Z = D_X$. Let σ_β be the nontrivial automorphism of Z over X.

Another way is to say that we let $\mathcal{E}_\beta = \mathcal{E} \otimes \mathcal{E}_\phi$, it is equipped with $\mathcal{E}_\beta^2 \widetilde{\to} \mathcal{O}(-D_X)$. Then $Z \widetilde{\to} \mathrm{Spec}(\mathcal{O}_X \oplus \mathcal{E}_\beta)$, the structure of an $\mathcal{O}_X$-algebra on $(\mathcal{O}_X \oplus \mathcal{E}_\beta)$ is given by $\mathcal{E}_\beta^2 \hookrightarrow \mathcal{O}_X$. Let $\mathcal{E}_{0,\phi}$ be the σ_ϕ-anti-invariants in $\phi_* \bar{\mathbb{Q}}_\ell$. Then we have $\beta_* \bar{\mathbb{Q}}_\ell \widetilde{\to} \bar{\mathbb{Q}}_\ell \oplus \mathcal{E}_{0,\beta}$ with

$$\mathcal{E}_{0,\beta} \widetilde{\to} \mathcal{E}_{0,\phi} \otimes \mathcal{E}_0.$$

Let $\mathrm{Pic}(Y, Z)$ be the stack classifying $\mathcal{B}_1 \in \mathrm{Pic}\, Y, \mathcal{B}_2 \in \mathrm{Pic}\, Z$, an isomorphism of line bundles $N(\mathcal{B}_1) \widetilde{\to} N(\mathcal{B}_2)$ on X, and its refinement $\gamma_{12} : \mathcal{B}_1|_{D_Y} \widetilde{\to} \mathcal{B}_2|_{D_Z}$ over $D_Y \widetilde{\to} D_Z$. This means that γ_{12}^2 coincides with

$$N(\mathcal{B}_1)|_{D_X} \widetilde{\to} N(\mathcal{B}_2)|_{D_X}$$

We have used the fact that β and ϕ yield isomorphisms of (reduced) schemes $D_Z \widetilde{\to} D_X \widetilde{\to} D_Y$.

LEMMA 16. *The map* $\mathrm{Pic}\, \tilde{Y} \to \mathrm{Pic}(Y, Z)$ *sending $\mathcal{B}$ to $(N_Y(\mathcal{B}), N_Z(\mathcal{B}))$ yields an isomorphism* $\mathrm{Bun}_{R_\phi} \widetilde{\to} \mathrm{Pic}(Y, Z)$.

Proof. Denote by $\tilde{R}_\phi$ the preimage of the diagonal $\mathbb{G}_m \hookrightarrow \mathbb{G}_m \times \mathbb{G}_m$ under the homomorphism

$$\phi_* \mathbb{G}_m \times \beta_* \mathbb{G}_m \xrightarrow{N \times N} \mathbb{G}_m \times \mathbb{G}_m$$

of group schemes on X. The product of norms yields a homomorphism $\phi_* \tilde{\pi}_* \mathbb{G}_m \to \tilde{R}_\phi$ of group schemes on X, and U_π lies in its kernel. The induced map $R_\phi \to \tilde{R}_\phi$ is an isomorphism over $X - D_X$, but not everywhere (if ϕ is ramified). The group scheme $\tilde{R}_\phi|_{D_X}$ over D_X has several connected components, and $R_\phi|_{D_X}$ is its component of unity. Our assertion follows. □

The map $\mathfrak{p}_{R_\phi} : \mathrm{Bun}_{R_\phi} \to \mathrm{Pic}\, Y$ sends $(\mathcal{B}_1, \mathcal{B}_2, \gamma_{12}, N(\mathcal{B}_1) \widetilde{\to} N(\mathcal{B}_2))$ to $\mathcal{B}_1^\star$. The following square is cartesian.

$$\begin{array}{ccc} \tilde{Y} & \xrightarrow{\tilde{\phi}} & \tilde{X} \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \pi} \\ Z & \xrightarrow{\beta} & X \end{array}$$

The map $\rho_H \circ \mathfrak{q}_{R_\phi} : \mathrm{Pic}(Y, Z) \to \mathrm{Bun}_H$ sends $(\mathcal{B}_1, \mathcal{B}_2, \gamma_{12}, N(\mathcal{B}_1) \widetilde{\to} N(\mathcal{B}_2))$ as above to the collection $(V, \mathcal{C}, \mathrm{Sym}^2 V \to \mathcal{C})$, where $V \subset \phi_*\mathcal{B}_1 \oplus \beta_*\mathcal{B}_2$ is the lower modification defined by the cartesian square

$$\begin{array}{ccccc} \phi_*\mathcal{B}_1 \oplus \beta_*\mathcal{B}_2 & \longrightarrow & (\phi_*\mathcal{B}_1 \oplus \beta_*\mathcal{B}_2)|_{D_X} & \longrightarrow & \mathcal{B}_1|_{D_Y} \oplus \mathcal{B}_2|_{D_Z} \\ \uparrow & & & & \uparrow{\scriptstyle \mathrm{id} + \gamma_{12}} \\ V & & \longrightarrow & & \mathcal{B}_1|_{D_Y} \end{array} \tag{47}$$

$\mathcal{C} = N(\mathcal{B}_1)(-D_X)$, and the quadratic form $\mathrm{Sym}^2 V \to \mathcal{C}$ is the restriction of the difference of forms on $\mathcal{B}_i$

$$\mathrm{Sym}^2(\phi_*\mathcal{B}_1 \oplus \beta_*\mathcal{B}_2) \to N(\mathcal{B}_1).$$

Denote by ZY the gluing of Z and Y along the isomorphism $D_Z \widetilde{\to} D_Y$. If D_X is empty, then ZY is the disjoint union of Z and Y. The projection $\varrho : ZY \to X$ is a 4-sheeted covering.

For a point $(\mathcal{B}_1, \mathcal{B}_2, \gamma_{12}, N(\mathcal{B}_1) \widetilde{\to} N(\mathcal{B}_2)) \in \mathrm{Pic}(Y, Z)$ denote by $\mathcal{B}_{12}$ the line bundle on ZY obtained by gluing $\mathcal{B}_1$ and $\mathcal{B}_2$ along γ_{12}. Then V from diagram (47) is nothing but $V \widetilde{\to} \varrho_*\mathcal{B}_{12}$. Let K_{ZY} denote the dualizing complex on ZY then $K_{ZY}[-1]$ is the gluing of $\Omega_Y(D_Y)$ and $\Omega_Z(D_Z)$ via

$$\Omega_Y(D_Y)|_{D_Y} \widetilde{\to} \mathcal{O}_{D_Y} \widetilde{\to} \mathcal{O}_{D_Z} \widetilde{\to} \Omega_Z(D_Z)|_{D_Z}.$$

Set $(\mathcal{B}_{12})^\star = \mathcal{H}om(\mathcal{B}_{12}, K_{ZY})[-1]$, this is the gluing of $\mathcal{B}_1^* \otimes \Omega_Y(D_Y)$ and $\mathcal{B}_2^* \otimes \Omega_Z(D_Z)$ along the isomorphism

$$\gamma_{12}^* : \mathcal{B}_2^* \otimes \Omega_Z(D_Z)|_{D_Z} \widetilde{\to} \mathcal{B}_1^* \otimes \Omega_Y(D_Y)|_{D_Y}.$$

Then $(\varrho_*\mathcal{B}_{12})^\star \widetilde{\to} \varrho_*(\mathcal{B}_{12}^\star)$ canonically. If ϕ is unramified, then ZY is smooth, and the definition of $(\mathcal{B}_{12})^\star$ coincides with that of § 1.1.

6.3.1 In §§ 6.3.1–6.3.4 we assume $n = 1$.

Fix a two-sheeted covering $\phi : Y \to X$ as in § 6.1.1. For a given rank-one local system $\mathcal{J}$ on Y we want to calculate

$$\mathrm{R}\Gamma(\mathrm{Pic}\, Y, A\mathcal{J} \otimes (\mathfrak{p}_{R_\phi})_!\, \mathfrak{q}_{R_\phi}^* K_{\tilde{E},\chi,\tilde{H}}). \tag{48}$$

Assume that we are given an isomorphism $\chi \widetilde{\to} NJ$ of local systems on X. Then the complex that we integrate will descend under $e_\phi : \mathrm{Pic}\, Y \to \mathrm{Bun}_{U_\phi}$; we will actually integrate over Bun_{U_ϕ}.

First, consider the situation when E is an irreducible rank-two local system on X and $\tilde{E} = \pi^* E$. Let Aut_{E^*} denote the corresponding automorphic sheaf on Bun_2 normalized as in [FGV02]. The following result is a calculation of

$$(\mathfrak{p}_{R_\phi})_!\, \mathfrak{q}_{R_\phi}^* F_{\tilde{H}}(\mathrm{Aut}_{E^*})[\dim.\mathrm{rel}(\mathfrak{q}_{R_\phi})], \tag{49}$$

under these assumptions.

Theorem 4. *Assume that $Y \xrightarrow{\phi} X$ is unramified and nonsplit, and the coverings $\tilde{X} \xrightarrow{\pi} X$ and $Y \xrightarrow{\phi} X$ are not isomorphic over X. Then the complex (49) is isomorphic to*

$$\bigoplus_{d \geqslant 0} A(\mathcal{E}_{0,\beta} \otimes \det E)_\Omega \otimes \mathrm{mult}_!^{d,d_1}(A(\mathcal{E}_0 \otimes \det E^*) \boxtimes (\phi^* E^*)^{(d)})[d + g - 1],$$

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where d_1 is a function of a connected component of $\operatorname{Pic} Y$ *given by* $2d_1 + d = \deg \mathcal{B}$ *for* $\mathcal{B} \in \operatorname{Pic} Y$ *(the sum is over* $d \geqslant 0$ *such that* $d_1 \in \mathbb{Z}$*). Here*

$$\operatorname{mult}^{d,d_1} : \operatorname{Pic}^{d_1} X \times Y^{(d)} \to \operatorname{Pic}^{2d_1+d} Y$$

is the map introduced in § 4.4.1*.*

Remark 11. One may extend Theorem 4 to the case of $Y \xrightarrow{\phi} X$ split. To do so, first extend Theorem 2 to the case of split $\tilde{X}$, then the argument of § 6.3.2 will go through. We leave this to the interested reader.

6.3.2 Define $\operatorname{Bun}_{G,R_\phi}$ by the following commutative following diagram, where the square is cartesian.

$$\begin{array}{ccccc}
 & & \operatorname{Bun}_{G,R_\phi} & \longrightarrow & \operatorname{Bun}_{G,\tilde{H}} & \xrightarrow{\mathfrak{p}} & \operatorname{Bun}_G \\
 & & \downarrow & & \downarrow{\scriptstyle \mathfrak{q}} & & \\
\operatorname{Pic} Y & \xleftarrow{\mathfrak{p}_{R_\phi}} & \operatorname{Bun}_{R_\phi} & \xrightarrow{\mathfrak{q}_{R_\phi}} & \operatorname{Bun}_{\tilde{H}} & &
\end{array}$$

Then $\operatorname{Bun}_{G,R_\phi}$ classifies $M \in \operatorname{Bun}_2$ for which we set $\mathcal{A} = \det M$ and $\mathcal{C} = \Omega \otimes \mathcal{A}^{-1}$, $\mathcal{B}_1 \in \operatorname{Pic} Y$, $\mathcal{B}_2 \in \operatorname{Pic} Z$, isomorphisms $N(\mathcal{B}_1) \widetilde{\to} N(\mathcal{B}_2) \widetilde{\to} \mathcal{C}(D_X)$ and its refinement $\gamma_{12} : \mathcal{B}_1|_{D_Y} \widetilde{\to} \mathcal{B}_2|_{D_Z}$.

Proof of Theorem 4. Let $\tilde{H}_\beta$ (respectively, $\tilde{H}_\phi$) denote the group scheme on X obtained as a twisting of $\mathrm{G}\mathbb{O}_2^0$ by the $\mathbb{Z}/2\mathbb{Z}$-torsor $\beta : Z \to X$ (respectively, $\phi : Y \to X$) as in § 3.1.

We have a commutative diagram

$$\begin{array}{ccccc}
\operatorname{Bun}_{G,R_\phi} & \xrightarrow{f} & \operatorname{Bun}_{G,\tilde{H}_\beta} \times_{\operatorname{Bun}_G} \operatorname{Bun}_{G,\tilde{H}_\phi} & \xrightarrow{\tilde{\tau}\times\tilde{\tau}} & \widetilde{\operatorname{Bun}}_{G_2} \times \widetilde{\operatorname{Bun}}_{G_2} \\
\downarrow & & & & \downarrow{\scriptstyle \tilde{\pi}_2} \\
\operatorname{Bun}_{G,\tilde{H}} & & \xrightarrow{\qquad\qquad \tilde{\tau} \qquad\qquad} & & \widetilde{\operatorname{Bun}}_{G_4}
\end{array}$$

where $\tilde{\pi}_2$ is the map defined in § 3.5 for $X \sqcup X$. Here f is the isomorphism sending the above point of $\operatorname{Bun}_{G,R_\phi}$ to $(\mathcal{B}_2, M, N(\mathcal{B}_2) \otimes \mathcal{A} \widetilde{\to} \Omega) \in \operatorname{Bun}_{G,\tilde{H}_\beta}$, $(\mathcal{B}_1, M, N(\mathcal{B}_1) \otimes \mathcal{A} \widetilde{\to} \Omega) \in \operatorname{Bun}_{G,\tilde{H}_\phi}$.

By Proposition 3,

$$\tilde{\pi}_2^* \operatorname{Aut}[\dim.\mathrm{rel}] \widetilde{\to} \operatorname{Aut} \boxtimes \operatorname{Aut},$$

where $\dim.\mathrm{rel} = 2 \dim \operatorname{Bun}_{G_2} - \dim \operatorname{Bun}_{G_4}$. We must calculate the direct image under the composition of projections

$$\operatorname{Bun}_{G,\tilde{H}_\beta} \times_{\operatorname{Bun}_G} \operatorname{Bun}_{G,\tilde{H}_\phi} \to \operatorname{Bun}_{G,\tilde{H}_\phi} \to \operatorname{Bun}_{\tilde{H}_\phi}.$$

To calculate the direct image with respect to the first map we use Proposition 6 applied to the functor $F_G^{\tilde{H}_\beta} : \mathrm{D}(\operatorname{Bun}_{\tilde{H}_\beta}) \to \mathrm{D}(\operatorname{Bun}_G)$. It yields an isomorphism

$$F_G^{\tilde{H}_\beta}(\bar{\mathbb{Q}}_\ell)[\dim \operatorname{Bun}_{\tilde{H}_\beta}] \widetilde{\to} \operatorname{Aut}_{\mathcal{E}_{0,\beta} \oplus \bar{\mathbb{Q}}_\ell} \otimes (A\mathcal{E}_{0,\beta})_\Omega.$$

So, (49) identifies with

$$(A\mathcal{E}_{0,\beta})_\Omega \otimes \epsilon_! F_{\tilde{H}_\phi}^G(\operatorname{Aut}_{E^*} \otimes \operatorname{Aut}_{\mathcal{E}_{0,\beta} \oplus \bar{\mathbb{Q}}_\ell})[-\dim \operatorname{Bun}_G].$$

Applying Theorem 2 for $F_{\tilde{H}_\phi}$, one identifies (49) with the direct sum

$$\bigoplus_{d \geqslant 0} A(\mathcal{E}_{0,\beta} \otimes \det E)_\Omega \otimes \operatorname{mult}_!^{d,d_1}(A(\mathcal{E}_0 \otimes \det E^*) \boxtimes (\pi^* E^*)^{(d)})[d + g - 1],$$

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where d_1 is a function of a connected component of $\operatorname{Pic} Y$ given by $2d_1 + d = \deg \mathcal{B}$ for $\mathcal{B} \in \operatorname{Pic} Y$. The sum is over $d \geqslant 0$ such that $d_1 \in \mathbb{Z}$. □

6.3.3 *Geometric Waldspurger periods.* In this section we assume that $\tilde{X}$ is split and $\phi : Y \to X$ is nonsplit. From Theorem 4 combined with Proposition 8 one derives the following.

Corollary 7. *Assume that $\phi : Y \to X$ is nonramified. For an irreducible rank-two local system E on X we have*

$$(\mathfrak{p}_{R_\phi})_! \mathfrak{q}^*_{R_\phi} K_{\pi^* E, \det E, \tilde{H}}[\dim.\operatorname{rel}(\mathfrak{q}_{R_\phi})] \widetilde{\to} \bigoplus_{d \geqslant 0} (A\mathcal{E}_{0,\phi})_\Omega \otimes \operatorname{mult}^{d,d_1}_! (A(\det E^*) \boxtimes (\phi^* E^*)^{(d)})[d+g-1], \tag{50}$$

where d_1 is a function of a connected component of $\operatorname{Pic} Y$ given by $2d_1 + d = \deg \mathcal{B}$ for $\mathcal{B} \in \operatorname{Pic} Y$ (the sum is over $d \geqslant 0$ such that $d_1 \in \mathbb{Z}$).

Remark 12. By Remark 9, (50) is naturally a direct sum of two complexes indexed by those $\theta \in \pi_1(\tilde{H})$ whose image in $\mathbb{Z}$ is $2\deg\Omega - \deg \mathcal{B}$, $\mathcal{B} \in \operatorname{Pic} Y$. However, the right-hand side of (50) seems not to be the refinement of this decomposition (cf. Remark 6).

Recall the exact sequence $1 \to \mathbb{G}_m \to \phi_* \mathbb{G}_m \to U_\phi \to 1$ on X (cf. Appendix A.1). The corresponding extension of scalars map $e_\phi : \operatorname{Pic} Y \to \operatorname{Bun}_{U_\phi}$ sends $\mathcal{B}$ to $\mathcal{B}^{-1} \otimes \sigma^*_\phi \mathcal{B}$. If $\phi : Y \to X$ is unramified, then U_ϕ is also the kernel of the norm map $\phi_* \mathbb{G}_m \to \mathbb{G}_m$.

For $a \in \mathbb{Z}/2\mathbb{Z}$ we write $\operatorname{Bun}^a_{U_\phi}$ for the connected component of $\operatorname{Bun}_{U_\phi}$ corresponding to a, so $\operatorname{Bun}^0_{U_\phi}$ is the connected component of unity (cf. Appendix A.1). Let $\phi_1 : \operatorname{Pic} Y \to \operatorname{Bun}_2$ be the map sending $\mathcal{B}$ to $\phi_* B$.

Definition 10. Let $\mathcal{J}$ be a rank-one local system on Y. Let $K \in \mathrm{D}(\operatorname{Bun}_2)$ be a complex with central character $N(\mathcal{J})$. Then the complex $A\mathcal{J}^{-1} \otimes \phi_1^* K$ is equipped with natural descent data for the map $e_\phi : \operatorname{Pic} Y \to \operatorname{Bun}_{U_\phi}$. Assume that the following holds:[4]

(C_W) $\mathcal{K}_K$ is a complex on $\operatorname{Bun}_{U_\phi}$ equipped with

$$e^*_\phi \mathcal{K}_K[\dim.\operatorname{rel}(e_\phi)] \widetilde{\to} A\mathcal{J}^{-1} \otimes \phi_1^* K[\dim.\operatorname{rel}(\phi_1)].$$

For $a \in \mathbb{Z}/2\mathbb{Z}$ the *Waldspurger period* of K is

$$\mathrm{WP}^a(K, \mathcal{J}) = \mathrm{R}\Gamma_c(\operatorname{Bun}^a_{U_\phi}, \mathcal{K}_K).$$

Let $m_{\phi,d} : Y^{(d)} \to \operatorname{Bun}_{U_\phi}$ be the map sending D to $\mathcal{O}(D - \sigma^*_\phi D)$ with natural trivializations $N(\mathcal{O}(D - \sigma^*_\phi D)) \widetilde{\to} \mathcal{O}$ and $\mathcal{O}(D - \sigma^*_\phi D)|_{D_Y} \widetilde{\to} \mathcal{O}_{D_Y}$. The map $m_{\phi,d}$ is proper.

Let $\operatorname{mult}_\phi : \operatorname{Bun}_{U_\phi} \times \operatorname{Bun}_{U_\phi} \to \operatorname{Bun}_{U_\phi}$ denote the multiplication map ($\operatorname{Bun}_{U_\phi}$ has a natural structure of a group stack). If ϕ is ramified, then mult_ϕ is proper.

Theorem 5. *Assume that $\phi : Y \to X$ is unramified. Let E be an irreducible rank-two local system on X and let $\mathcal{J}$ be a rank-one local system on Y equipped with $\det E \widetilde{\to} N(\mathcal{J})$. The condition (C_W) is satisfied for Aut_E giving rise to $\mathcal{K}_E := \mathcal{K}_{\operatorname{Aut}_E}$. The complex $\mathcal{K}_E$ is a direct sum of (possibly shifted) perverse sheaves. We have*

$$\operatorname{mult}_{\phi !}(\mathcal{K}_E \boxtimes \mathcal{K}_E) \widetilde{\to} \bigoplus_{d \geqslant 0} (A\mathcal{E}_{0,\phi} \otimes A(N\mathcal{J})^*)_\Omega \otimes (m_{\phi,d})_! (\mathcal{J} \otimes \phi^* E^*)^{(d)}[d].$$

[4] Although each perverse cohomology of the latter complex descends with respect to e_ϕ, we ignore whether the same is true for the complex itself, as the fibres of e_ϕ are not contractible.

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In particular, for $a \in \mathbb{Z}/2\mathbb{Z}$ there are isomorphisms

$$\bigoplus_{\substack{a_1+a_2=a,\\ a_i\in\mathbb{Z}/2\mathbb{Z}}} \mathrm{WP}^{a_1}(\mathrm{Aut}_E, \mathcal{J}) \otimes \mathrm{WP}^{a_2}(\mathrm{Aut}_E, \mathcal{J}) \widetilde{\to} (A\mathcal{E}_{0,\phi} \otimes A(N\mathcal{J})^*)_\Omega \otimes \Bigg(\bigoplus_{\substack{d\geqslant 0,\\ a=d \bmod 2}} \mathrm{R}\Gamma(Y^{(d)}, (\mathcal{J}\otimes \phi^* E^*)^{(d)})[d]\Bigg).$$

If ϕ^*E is irreducible, then the latter complex is a vector space (placed in cohomological degree zero).

Proof. Under our assumptions the sequence (39) fits as the low row in the diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathbb{G}_m \times \mathbb{G}_m & \longrightarrow & \phi_*\mathbb{G}_m \times \phi_*\mathbb{G}_m & \longrightarrow & U_\phi \times U_\phi & \longrightarrow & 1\\
 & & \uparrow & & \uparrow \mathrm{id} & & \uparrow & & \\
1 & \longrightarrow & \mathbb{G}_m & \longrightarrow & \phi_*\mathbb{G}_m \times \phi_*\mathbb{G}_m & \longrightarrow & R_\phi & \longrightarrow & 1
\end{array}$$

where the left vertical arrow sends z to (z, z^{-1}). Let $\kappa_\phi : \mathrm{Bun}_{R_\phi} \to \mathrm{Bun}_{U_\phi} \times \mathrm{Bun}_{U_\phi}$ denote the corresponding extension of scalars map. The following diagram is cartesian.

$$\begin{array}{ccc}
\mathrm{Bun}_{R_\phi} & \xrightarrow{\mathfrak{p}_{R_\phi}} & \mathrm{Pic}\, Y\\
\downarrow \kappa_\phi & & \downarrow e_\phi\circ\epsilon\\
\mathrm{Bun}_{U_\phi} \times \mathrm{Bun}_{U_\phi} & \xrightarrow{\mathrm{mult}_\phi} & \mathrm{Bun}_{U_\phi}
\end{array}$$

By Remark 10, the condition (C_W) is satisfied, and we obtain

$$(A\mathcal{J})^{-1}_{\Omega_Y} \otimes \mathfrak{p}^*_{R_\phi} A\mathcal{J} \otimes \mathfrak{q}^*_{R_\phi} K_{\pi^*E, \det E, \tilde{H}}[\dim.\mathrm{rel}(\mathfrak{q}_{R_\phi})] \widetilde{\to} \kappa^*_\phi(\mathcal{K}_E \boxtimes \mathcal{K}_E)[\dim.\mathrm{rel}(\kappa_\phi)].$$

By Corollary 7, we obtain

$$\begin{aligned}
&(e_\phi\circ\epsilon)^* \,\mathrm{mult}_{\phi !}(\mathcal{K}_E \boxtimes \mathcal{K}_E)[\dim.\mathrm{rel}(\kappa_\phi)]\\
&\quad \widetilde{\to} (A\mathcal{E}_{0,\phi} \otimes A(N\mathcal{J})^*)_\Omega \otimes \Bigg(\bigoplus_{d\geqslant 0} A\mathcal{J} \otimes \mathrm{mult}^{d,d_1}_!(A(\det E^*) \boxtimes (\phi^* E^*)^{(d)})[d+g-1]\Bigg)\\
&\quad \widetilde{\to} (A\mathcal{E}_{0,\phi} \otimes A(N\mathcal{J})^*)_\Omega \otimes \Bigg(\bigoplus_{d\geqslant 0} \mathrm{mult}^{d,d_1}_!(\bar{\mathbb{Q}}_\ell \boxtimes (\mathcal{J}\otimes\phi^* E^*)^{(d)})[d+g-1]\Bigg)
\end{aligned}$$

where d_1 is a function of a connected component of $\mathrm{Pic}\, Y$ given by $2d_1 + d = \deg \mathcal{B}$ for $\mathcal{B} \in \mathrm{Pic}\, Y$ (the sum is over $d \geqslant 0$ such that $d_1 \in \mathbb{Z}$).

The following square is cartesian

$$\begin{array}{ccc}
\mathrm{Pic}^{d_1} X \times Y^{(d)} & \xrightarrow{\mathrm{mult}^{d,d_1}} & \mathrm{Pic}^{2d_1+d}\, Y\\
\downarrow & & \downarrow e_\phi\circ\epsilon\\
Y^{(d)} & \xrightarrow{m_{\phi,d}} & \mathrm{Bun}_{U_\phi}
\end{array}$$

where the left vertical arrow is the projection. Since $\dim.\mathrm{rel}(\kappa_\phi) = g-1$, we obtain an isomorphism

$$(e_\phi\circ\epsilon)^* \,\mathrm{mult}_{\phi !}(\mathcal{K}_E \boxtimes \mathcal{K}_E) \widetilde{\to} (A\mathcal{E}_{0,\phi} \otimes A(N\mathcal{J})^*)_\Omega \otimes (e_\phi\circ\epsilon)^* \Bigg(\bigoplus_{d\geqslant 0} (m_{\phi,d})_!(\mathcal{J}\otimes\phi^* E^*)^{(d)}[d]\Bigg)$$

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compatible with the descent data for $e_\phi \circ \epsilon$. So,

$$\mathrm{mult}_{\phi !}(\mathcal{K}_E \boxtimes \mathcal{K}_E) \widetilde{\to} (A\mathcal{E}_{0,\phi} \otimes A(N\mathcal{J})^*)_\Omega \otimes \left(\bigoplus_{d \geqslant 0} (m_{\phi,d})_!(\mathcal{J} \otimes \phi^* E^*)^{(d)}[d] \right),$$

the sum over all $d \geqslant 0$. For d even (respectively, odd) $m_{\phi,d}$ maps $Y^{(d)}$ to $\mathrm{Bun}^0_{U_\phi}$ (respectively, to $\mathrm{Bun}^1_{U_\phi}$).

If $\phi^* E$ is irreducible, then $\mathrm{R}\Gamma(Y^{(d)}, (\mathcal{J} \otimes \phi^* E^*)^{(d)})[d] \widetilde{\to} \bigwedge^d V$ with $V = \mathrm{H}^1(Y, \mathcal{J} \otimes \phi^* E^*)$. The last statement follows. □

6.3.4 In this section $\phi : Y \to X$ is allowed to be ramified. Let us calculate the geometric Waldspurger periods of Eisenstein series on Bun_2. Let E_1, E_2 be rank-one local systems on X and let $\mathcal{J}$ be a rank-one local system on Y equipped with $N(\mathcal{J}) \widetilde{\to} E_1 \otimes E_2$. Recall the complex $\mathrm{Aut}_{E_1 \oplus E_2}$ on Bun_2 (cf. § 4.3). Let

$$\tilde{m}_{\phi,d} : Y^{(d)} \to \mathrm{Bun}_{U_\phi}$$

be the map sending D to $\mathcal{O}(\sigma_\phi^* D - D)$ with canonical trivialization $N(\mathcal{O}(\sigma_\phi^* D - D)) \widetilde{\to} \mathcal{O}_X$.

Proposition 10. *The condition (C_W) is satisfied for $\mathrm{Aut}_{E_1 \oplus E_2}$. For the corresponding complex $\mathcal{K}_{E_1 \oplus E_2} := \mathcal{K}_{\mathrm{Aut}_{E_1 \oplus E_2}}$ we have*

$$\mathcal{K}_{E_1 \oplus E_2} \widetilde{\to} \bigoplus_{d \geqslant 0} (AE_2)_{\mathcal{E}_\phi \otimes \Omega^{-1}} \otimes (\tilde{m}_{\phi,d})_!(\mathcal{J}^{-1} \otimes \phi^* E_2)^{(d)}[d]. \tag{51}$$

In particular,

$$\mathrm{mult}_{\phi\, !}(\mathcal{K}_{E_1 \oplus E_2} \boxtimes \mathcal{K}_{E_1 \oplus E_2}) \widetilde{\to} \bigoplus_{d \geqslant 0} A(E_1 \otimes E_2)_{\mathcal{E}_\phi \otimes \Omega^{-1}} \otimes (\tilde{m}_{\phi,d})_!(\mathcal{J}^{-1} \otimes \phi^*(E_1 \oplus E_2))^{(d)}[d].$$

Proof. We have a cartesian square

$$\begin{array}{ccc} \bigsqcup_{d \geqslant 0} \mathrm{Pic}\, X \times Y^{(d)} & \xrightarrow{\mathrm{mult}} & \mathrm{Pic}\, Y \\ \downarrow {\scriptstyle \phi_{1,P}} & & \downarrow {\scriptstyle \phi_1} \\ \overline{\mathrm{Bun}}_P & \xrightarrow{\bar{\mathfrak{p}}_P} & \mathrm{Bun}_2 \end{array}$$

where $\phi_{1,P}$ sends (L, D) to $(L \subset M)$, $M = L \otimes \phi_* \mathcal{O}(D)$. The map $\mathrm{mult}^{d,d_1} : \mathrm{Pic}^{d_1} X \times Y^{(d)} \to \mathrm{Pic}^{2d_1 + d}$ sends (L, D) to $(\phi^* L)(D)$. So, we have

$$\phi_1^* \mathrm{Aut}_{E_1 \oplus E_2} \otimes A\mathcal{J}^{-1}[\dim.\mathrm{rel}(\phi_1)] \widetilde{\to} \bigoplus_{d \geqslant 0} (AE_2)_{\mathcal{E}_\phi \otimes \Omega^{-1}} \otimes \mathrm{mult}^{d,d_1}_!(\bar{\mathbb{Q}}_\ell \boxtimes (\mathcal{J}^{-1} \otimes \phi^* E_2)^{(d)})[d + g - 1],$$

where d_1 is a function of a connected component of $\mathrm{Pic}\, Y$ given by $2d_1 + d = \deg \mathcal{B}$, $\mathcal{B} \in \mathrm{Pic}\, Y$ (and the sum is over $d \geqslant 0$ such that $d_1 \in \mathbb{Z}$). We have used that $\dim.\mathrm{rel}(\phi_1) = 2(1 - g) + \frac{1}{2} \deg D_X$.

The following square is cartesian

$$\begin{array}{ccc} \mathrm{Pic}^{d_1} X \times Y^{(d)} & \xrightarrow{\mathrm{mult}^{d,d_1}} & \mathrm{Pic}^{2d_1 + d} Y \\ \downarrow & & \downarrow {\scriptstyle e_\phi} \\ Y^{(d)} & \xrightarrow{\tilde{m}_{\phi,d}} & \mathrm{Bun}_{U_\phi} \end{array}$$

where the left vertical arrow is the projection. This yields an isomorphism

$$\phi_1^* \mathrm{Aut}_{E_1 \oplus E_2} \otimes A\mathcal{J}^{-1}[\dim.\mathrm{rel}(\phi_1)] \widetilde{\to} \bigoplus_{d \geqslant 0} (AE_2)_{\mathcal{E}_\phi \otimes \Omega^{-1}} \otimes e_\phi^* (\tilde{m}_{\phi,d})_!(\mathcal{J}^{-1} \otimes \phi^* E_2)^{(d)})[d + g - 1].$$

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Since $\dim.\mathrm{rel}(e_\phi) = g-1$, the first assertion follows.

To get the second assertion, recall that for rank-one local systems V_i on Y we have

$$(V_1 \oplus V_2)^{(d)} \widetilde{\to} \bigoplus_{0=k}^{d} (\mathrm{sym}_{k,d-k})_!(V_1^{(k)} \boxtimes V_2^{(d-k)}),$$

where $\mathrm{sym}_{k,d-k} : Y^{(k)} \times Y^{(d-k)} \to Y^{(d)}$ is the sum of divisors. □

Remark 13. (i) Recall that the Eisenstein series $\mathrm{Aut}_{E_1 \oplus E_2}$ above is called regular if E_1 and E_2 are not isomorphic (cf. [BG02, § 2.1.7]). Under these assumptions $(\tilde{m}_{\phi,d})_!(\mathcal{J}^{-1} \otimes \phi^* E_2)^{(d)} = 0$ for $d > 2g_Y - 2$ (so, the sum in (51) is actually by $0 \leqslant d \leqslant 2g_Y - 2$).

Indeed, $\tilde{m}_{\phi,d}$ decomposes as $Y^{(d)} \xrightarrow{s} \mathrm{Pic}\, Y \xrightarrow{e_\phi} \mathrm{Bun}_{U_\phi}$, where s sends D to $\mathcal{O}(D)$. If $d > 2g_Y - 2$, then s is a vector bundle of rank $d + 1 - g_Y$ over $\mathrm{Pic}\, Y$ with zero section removed. For a rank-one local system $\tilde{E}$ on Y we have $s_! \tilde{E}^{(d)} \widetilde{\to} A\tilde{E} \otimes s_! \bar{\mathbb{Q}}_\ell$. Further, $e_\phi : \mathrm{Pic}\, Y \to \mathrm{Bun}_{U_\phi}$ is a homomorphism of group stacks, each fibre of e_ϕ identifies with $\mathrm{Pic}\, X$. So, if $N(\tilde{E})$ is nontrivial, then $(e_\phi)_! A\tilde{E} = 0$.

(ii) If E is a rank-two local system on X, $\mathcal{J}$ is a rank-one local system on Y equipped with $\det E \widetilde{\to} N(\mathcal{J})$, then we have $\mathcal{J}^{-1} \otimes \phi^* E \widetilde{\to} \sigma_\phi^* \mathcal{J} \otimes \phi^* E^*$ canonically. Write also $\sigma_\phi : Y^{(d)} \to Y^{(d)}$ for the map sending D to $\sigma_\phi^* D$. Then the composition $Y^{(d)} \xrightarrow{\sigma_\phi} Y^{(d)} \xrightarrow{m_{\phi,d}} \mathrm{Bun}_{U_\phi}$ equals $\tilde{m}_{\phi,d}$. So,

$$(\tilde{m}_{\phi,d})_!(\mathcal{J}^{-1} \otimes \phi^* E)^{(d)} \widetilde{\to} (m_{\phi,d})_!(\mathcal{J} \otimes \phi^* E^*)^{(d)}$$

canonically. Thus, Theorem 5 and Proposition 10 are consistent.

Theorem 5 and Proposition 10 suggest the following conjecture, which is a theorem if one of the following holds:

(i) E is irreducible and ϕ is nonramified;

(ii) E is a direct sum of two rank-one local systems.

CONJECTURE 3 (Waldspurger periods). Let E be a rank-two local system on X. Let $K \in \mathrm{D}(\mathrm{Bun}_2)$ be an automorphic sheaf with eigenvalue E. Let $\phi : Y \to X$ be a (possibly ramified) degree-two covering and let $\mathcal{J}$ be a rank-one local system on Y. Assume that condition (C_W) is satisfied for $\mathcal{J}$ and K giving rise to $\mathcal{K}_K \in \mathrm{D}(\mathrm{Bun}_{U_\phi})$. Then for a suitable normalization of K there exists an isomorphism

$$\mathrm{mult}_{\phi!}(\mathcal{K}_K \boxtimes \mathcal{K}_K) \widetilde{\to} \bigoplus_{d \geqslant 0} (m_{\phi,d})_!(\mathcal{J} \otimes \phi^* E^*)^{(d)}[d].$$

6.3.5 *Geometric Bessel periods.* In this section we assume that $\phi : Y \to X$ is nonsplit. Assume $n = 2$, so $G = \mathbb{G}\mathrm{Sp}_4$.

DEFINITION 11. Let $K \in \mathrm{D}(\mathrm{Bun}_G)$ be a complex with central character χ^{-1}. Let $\mathcal{J}$ be a rank-one local system on Y equipped with $N(\mathcal{J}) \widetilde{\to} \chi$. For the inclusion $\mathrm{Pic}\, Y \hookrightarrow {}^{rss}\mathcal{S}_P \subset \mathcal{S}_P$ the $*$-restriction $A\mathcal{J} \otimes \mathrm{Four}_\psi(\nu_P^* K)|_{\mathrm{Pic}\, Y}$ is equipped with natural descent data for $e_\phi : \mathrm{Pic}\, Y \to \mathrm{Bun}_{U_\phi}$. Assume that the following holds:

(C_B) $\mathcal{K}_K$ is a complex on Bun_{U_ϕ} equipped with

$$e_\phi^* \mathcal{K}_K[\dim.\mathrm{rel}(e_\phi)] \widetilde{\to} A\mathcal{J} \otimes \mathrm{Four}_\psi(\nu_P^* K)[\dim.\mathrm{rel}(\nu_P)]|_{\mathrm{Pic}\, Y.}$$

For $a \in \mathbb{Z}/2\mathbb{Z}$ the *Bessel period* of K is

$$\mathrm{BP}^a(K, \mathcal{J}) = \mathrm{R}\Gamma_c(\mathrm{Bun}^a_{U_\phi}, \mathcal{K}_K).$$

Assume that $\tilde{X}$ is connected. Let $\tilde{E}$ be an irreducible rank-two local system on $\tilde{X}$ and let χ be a rank-one local system on X equipped with $\pi^*\chi \widetilde{\to} \det \tilde{E}$. Recall the complex $K_{\tilde{E},\chi,\tilde{H}}$ defined in § 5.1. Recall the map $\tilde{m}_{\phi,d} : Y^{(d)} \to \mathrm{Bun}_{U_\phi}$ introduced in § 6.3.4.

Theorem 6. *Assume that $\phi : Y \to X$ is nonramified. Let $\mathcal{J}$ be a rank-one local system on Y equipped with $N(\mathcal{J}) \widetilde{\to} \chi$. The condition (C_B) is satisfied for $\mathcal{J}$ and $K := F_G(\rho_{H\,!} K_{\tilde{E},\chi,\tilde{H}})$ giving rise to $\mathcal{K}_K \in \mathrm{D}(\mathrm{Bun}_{U_\phi})$. We have*

$$(\mathrm{mult}_\phi)_!(\mathcal{K}_K \boxtimes \mathcal{K}_K) \widetilde{\to} \bigoplus_{d \geqslant 0} (\tilde{m}_{\phi,d})_!(\mathcal{J} \otimes \phi^*(\pi_* \tilde{E}^*))^{(d)}[d].$$

In particular, for $a \in \mathbb{Z}/2\mathbb{Z}$ there are isomorphisms

$$\bigoplus_{\substack{a_1+a_2=a,\\ a_i \in \mathbb{Z}/2\mathbb{Z}}} \mathrm{BP}^{a_1}(K,\mathcal{J}) \otimes \mathrm{BP}^{a_2}(K,\mathcal{J}) \widetilde{\to} \bigoplus_{\substack{d \geqslant 0,\\ a = d \bmod 2}} \mathrm{R}\Gamma(Y^{(d)}, (\mathcal{J} \otimes \phi^*(\pi_* \tilde{E}^*))^{(d)})[d].$$

Proof. According to Proposition 9 and Corollary 1, we must find a complex $\mathcal{K}_K \in \mathrm{D}(\mathrm{Bun}_{U_\phi})$ together with an isomorphism

$$e_\phi^* \mathcal{K}_K[\dim.\mathrm{rel}(e_\phi)] \widetilde{\to} A\mathcal{J} \otimes (\mathfrak{p}_{R_\phi})_! \mathfrak{q}_{R_\phi}^* K_{\tilde{E},\chi,\tilde{H}}[\dim.\mathrm{rel}(\mathfrak{q}_{R_\phi})]. \qquad (52)$$

Let us show that R_ϕ fits into the following cartesian square.

$$\begin{array}{ccc} R_\phi & \longrightarrow & \pi_* U_{\tilde{\phi}} \\ \downarrow{\scriptstyle N_Y} & & \downarrow{\scriptstyle N_Y} \\ \phi_* \mathbb{G}_m & \longrightarrow & U_\phi \end{array} \qquad (53)$$

Indeed, (39) fits into the following commutative diagram.

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{G}_m & \longrightarrow & \phi_*\mathbb{G}_m & \longrightarrow & U_\phi & \longrightarrow & 1 \\ & & \uparrow{\scriptstyle N} & & \uparrow{\scriptstyle N_Y} & & \uparrow{\scriptstyle N_Y} & & \\ 1 & \longrightarrow & \pi_*\mathbb{G}_m & \longrightarrow & \pi_*\tilde{\phi}_*\mathbb{G}_m & \longrightarrow & \pi_* U_{\tilde{\phi}} & \longrightarrow & 1 \\ & & \uparrow & & \uparrow{\scriptstyle \mathrm{id}} & & \uparrow & & \\ 1 & \longrightarrow & U_\pi & \longrightarrow & \pi_*\tilde{\phi}_*\mathbb{G}_m & \longrightarrow & R_\phi & \longrightarrow & 1 \\ & & \downarrow & & \downarrow{\scriptstyle N_Y} & & \downarrow{\scriptstyle N_Y} & & \\ & & 1 & \longrightarrow & \phi_*\mathbb{G}_m & \xrightarrow{\mathrm{id}} & \phi_*\mathbb{G}_m & \longrightarrow & 1 \end{array}$$

The latter diagram together with the exact sequence $1 \to \mathbb{G}_m \to \phi_*\mathbb{G}_m \to U_\phi \to 1$ yield (53).

Let $\kappa_\phi : \mathrm{Bun}_{R_\phi} \to \mathrm{Bun}_{U_{\tilde{\phi}}}$ be the extension of scalars map given by the upper row in (53). The composition $\mathrm{Pic}\,\tilde{Y} \to \mathrm{Bun}_{R_\phi} \xrightarrow{\kappa_\phi} \mathrm{Bun}_{U_{\tilde{\phi}}}$ is the map $e_{\tilde{\phi}}$ sending $\mathcal{B}$ to $\sigma_\phi^* \mathcal{B} \otimes \mathcal{B}^{-1}$. We get a cartesian square

$$\begin{array}{ccc} \mathrm{Bun}_{R_\phi} & \xrightarrow{\kappa_\phi} & \mathrm{Bun}_{U_{\tilde{\phi}}} \\ \downarrow{\scriptstyle \mathfrak{p}_{R_\phi}} & & \downarrow{\scriptstyle \tilde{N}_Y} \\ \mathrm{Pic}\,Y & \xrightarrow{e_\phi} & \mathrm{Bun}_{U_\phi} \end{array}$$

where $\tilde{N}_Y(\mathcal{B}) := N_Y(\mathcal{B})^{-1}$ for $\mathcal{B} \in \mathrm{Pic}\,\tilde{Y}$.

Consider the commutative diagram

$$\begin{array}{ccccc}\operatorname{Pic}\tilde{Y} & \longrightarrow & \operatorname{Bun}_{R_\phi} & \xrightarrow{\kappa_\phi} & \operatorname{Bun}_{U_{\tilde{\phi}}}\\ \downarrow{\scriptstyle \tilde{\phi}_1} & & \downarrow{\scriptstyle \mathfrak{q}_{R_\phi}} & & \\ \operatorname{Bun}_{2,\tilde{X}} & \longrightarrow & \operatorname{Bun}_{\tilde{H}} & & \end{array}$$

where $\tilde{\phi}_1$ sends $\mathcal{B}$ to $\tilde{\phi}_*\mathcal{B}$.

By Remark 10, the condition (C_W) is satisfied for $\operatorname{Aut}_{\tilde{E}}$, the covering $\tilde{\phi}:\tilde{Y}\to\tilde{X}$, and the local system $\tilde{\pi}^*\mathcal{J}$. So, there is a complex $\mathcal{K}_{\tilde{E}}\in \mathrm{D}(\operatorname{Bun}_{U_{\tilde{\phi}}})$ equipped with

$$e_{\tilde{\phi}}^*\mathcal{K}_{\tilde{E}}[\dim.\operatorname{rel}(e_{\tilde{\phi}})]\,\tilde{\to}\, A(\pi^*\mathcal{J})^{-1}\otimes\tilde{\phi}_1^*\operatorname{Aut}_{\tilde{E}}[\dim.\operatorname{rel}(\tilde{\phi}_1)].$$

Set $\tilde{\mathcal{K}}=\mathcal{K}_{\tilde{E}}\otimes(A\mathcal{J})_{\Omega_Y}$, it is equipped with an isomorphism

$$\mathfrak{p}_{R_\phi}^*A\mathcal{J}\otimes\mathfrak{q}_{R_\phi}^*K_{\tilde{E},\chi,\tilde{H}}[\dim.\operatorname{rel}(\mathfrak{q}_{R_\phi})]\,\tilde{\to}\,\kappa_\phi^*\tilde{\mathcal{K}}[\dim.\operatorname{rel}(\kappa_\phi)]$$

over $\operatorname{Bun}_{R_\phi}$. Set $\mathcal{K}_K=\tilde{N}_{Y!}(\tilde{\mathcal{K}})$, it is equipped with an isomorphism (52). We have the following commutative diagram.

$$\begin{array}{ccc}\operatorname{Bun}_{U_{\tilde{\phi}}}\times\operatorname{Bun}_{U_{\tilde{\phi}}} & \xrightarrow{\operatorname{mult}_{\tilde{\phi}}} & \operatorname{Bun}_{U_{\tilde{\phi}}}\\ \downarrow{\scriptstyle \tilde{N}_Y\times\tilde{N}_Y} & & \downarrow{\scriptstyle \tilde{N}_Y}\\ \operatorname{Bun}_{U_\phi}\times\operatorname{Bun}_{U_\phi} & \xrightarrow{\operatorname{mult}_\phi} & \operatorname{Bun}_{U_\phi}\end{array}$$

By Theorem 5, we have

$$(\operatorname{mult}_{\tilde{\phi}})_!(\tilde{\mathcal{K}}\boxtimes\tilde{\mathcal{K}})\,\tilde{\to}\,\bigoplus_{d\geqslant 0}(A\mathcal{E}_{0,\tilde{\phi}})_{\Omega_{\tilde{X}}}\otimes(m_{\tilde{\phi},d})_!(\tilde{\pi}^*\mathcal{J}\otimes\tilde{\phi}^*\tilde{E}^*)^{(d)}[d],$$

where $m_{\tilde{\phi},d}:\tilde{Y}^{(d)}\to\operatorname{Bun}_{U_{\tilde{\phi}}}$ sends D to $\mathcal{O}(D-\sigma_{\tilde{\phi}}^*D)$ with natural trivialization

$$N_{\tilde{X}}(\mathcal{O}(D-\sigma_{\tilde{\phi}}^*D))\,\tilde{\to}\,\mathcal{O}_{\tilde{X}}.$$

Since $\mathcal{E}_{0,\tilde{\phi}}\,\tilde{\to}\,\pi^*\mathcal{E}_{0,\phi}$, we obtain $(A\mathcal{E}_{0,\tilde{\phi}})_{\Omega_{\tilde{X}}}\,\tilde{\to}\,A(N\mathcal{E}_{0,\tilde{\phi}})_\Omega\,\tilde{\to}\,\bar{\mathbb{Q}}_\ell$. So,

$$(\operatorname{mult}_\phi)_!(\mathcal{K}_K\boxtimes\mathcal{K}_K)\,\tilde{\to}\,(\tilde{N}_Y)_!(\operatorname{mult}_{\tilde{\phi}})_!(\tilde{\mathcal{K}}\boxtimes\tilde{\mathcal{K}})\,\tilde{\to}\,\bigoplus_{d\geqslant 0}(\tilde{N}_Y)_!(m_{\tilde{\phi},d})_!(\tilde{\pi}^*\mathcal{J}\otimes\tilde{\phi}^*\tilde{E}^*)^{(d)}[d].$$

The following diagram commutes.

$$\begin{array}{ccc}\tilde{Y}^{(d)} & \xrightarrow{m_{\tilde{\phi},d}} & \operatorname{Bun}_{U_{\tilde{\phi}}}\\ \downarrow{\scriptstyle \tilde{\pi}} & & \downarrow{\scriptstyle \tilde{N}_Y}\\ Y^{(d)} & \xrightarrow{\tilde{m}_{\phi,d}} & \operatorname{Bun}_{U_\phi}\end{array}$$

Our assertion follows. □

Theorem 6 combined with Conjecture 2 suggest the following.

Conjecture 4 (Bessel periods). For $G=\mathrm{G}\mathbb{S}\mathrm{p}_4$ let $E_{\check{G}}$ be a $\check{G}$-local system on X viewed as a pair (E,χ^{-1}), where E (respectively, χ) is a rank-four (respectively, rank-one) local system on X equipped with a symplectic form $\bigwedge^2 E\to\chi^{-1}$. Let K be an automorphic sheaf on Bun_G with eigenvalue $E_{\check{G}}$ (in particular, the central character of K is χ^{-1}).

Let $\phi:Y\to X$ be a (possibly ramified) degree-two covering. Let $\mathcal{J}$ be a rank-one local system on Y equipped with $N(\mathcal{J})\,\tilde{\to}\,\chi$. Assume that condition (C_B) is satisfied for $\mathcal{J}$ and K. If the

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corresponding complex $\mathcal{K}_K \in \mathrm{D}(\mathrm{Bun}_{U_\phi})$ is nonzero then

$$(\mathrm{mult}_\phi)_!(\mathcal{K}_K \boxtimes \mathcal{K}_K) \widetilde{\to} \bigoplus_{d\geqslant 0} (\tilde{m}_{\phi,d})_!(\mathcal{J}\otimes\phi^*E)^{(d)}[d]$$

for a suitable normalization of K. In particular, for $a \in \mathbb{Z}/2\mathbb{Z}$ there are isomorphisms

$$\bigoplus_{\substack{a_1+a_2=a,\\ a_i\in\mathbb{Z}/2\mathbb{Z}}} \mathrm{BP}^{a_1}(K,\mathcal{J})\otimes \mathrm{BP}^{a_2}(K,\mathcal{J}) \widetilde{\to} \bigoplus_{\substack{d\geqslant 0,\\ a=d \bmod 2}} \mathrm{R}\Gamma(Y^{(d)}, (\mathcal{J}\otimes\phi^*E)^{(d)})[d]. \qquad (54)$$

Remark 14. Under the assumptions of Conjecture 4 we have $\mathrm{H}^2(Y,\mathcal{J}\otimes\phi^*E)^* \widetilde{\to} \mathrm{H}^0(Y,\mathcal{J}\otimes\phi^*E)$. Consider the case $\mathrm{H}^i(Y,\mathcal{J}\otimes\phi^*E)=0$ for $i=0,2$. Then $\mathrm{BP}(K,\mathcal{J})\otimes\mathrm{BP}(K,\mathcal{J})$ identifies with the vector space (placed in degree zero)

$$\bigoplus_{i\geqslant 0} \bigwedge^i V, \qquad (55)$$

where $V=\mathrm{H}^1(Y,\mathcal{J}\otimes\phi^*E)$. The symplectic form on E induces a map

$$\mathrm{H}^2(Y,\mathcal{J}\otimes\sigma_\phi^*\mathcal{J}\otimes\phi^*(E\otimes E)) \to \mathrm{H}^2(Y,\bar{\mathbb{Q}}_\ell) \widetilde{\to} \bar{\mathbb{Q}}_\ell. \qquad (56)$$

Since the cup-product

$$V\otimes V \widetilde{\to} \mathrm{H}^1(Y,\mathcal{J}\otimes\phi^*E)\otimes \mathrm{H}^1(Y,\sigma_\phi^*\mathcal{J}\otimes\phi^*E) \to \mathrm{H}^2(Y,\mathcal{J}\otimes\sigma_\phi^*\mathcal{J}\otimes\phi^*(E\otimes E))$$

is anti-symmetric, composing it with (56) one obtains a nondegenerate symmetric form $\mathrm{Sym}^2 V \to \bar{\mathbb{Q}}_\ell$ on V. We have $\dim V = 8(g_Y-1)$, where g_Y is the genus of Y.

Let $\mathbb{S}\mathrm{pin}(V)$ denote the simply connected covering of $\mathbb{SO}(V)$. Let Γ_α and Γ_β be the half-spin representations of $\mathbb{S}\mathrm{pin}(V)$, here α and β are the corresponding fundamental weights of $\mathbb{S}\mathrm{pin}(V)$ (cf. [FH91, 19.2, p. 287]). Then

$$\Gamma_\alpha\otimes\Gamma_\alpha\oplus\Gamma_\beta\otimes\Gamma_\beta \widetilde{\to} \bigwedge^0 V\oplus\bigwedge^2 V\oplus\bigwedge^4 V\oplus\cdots$$

and

$$\Gamma_\alpha\otimes\Gamma_\beta\oplus\Gamma_\beta\otimes\Gamma_\alpha \widetilde{\to} \bigwedge^1 V\oplus\bigwedge^3 V\oplus\bigwedge^5 V\oplus\cdots.$$

Conjecture 5 (Bessel periods refined). Under the assumptions of Conjecture 4 consider the case $\mathrm{H}^i(Y,\mathcal{J}\otimes\phi^*E)=0$ for $i=0,2$. Set $V=\mathrm{H}^1(Y,\mathcal{J}\otimes\phi^*E)$. Then there is a numbering α_a ($a\in\mathbb{Z}/2\mathbb{Z}$) of the half-spin fundamental weights of $\mathbb{S}\mathrm{pin}(V)$ and isomorphisms for $a\in\mathbb{Z}/2\mathbb{Z}$

$$\mathrm{BP}^a(K,\mathcal{J}) \widetilde{\to} \Gamma_{\alpha_a},$$

where Γ_{α_a} is the irreducible (half-spin) representation of $\mathbb{S}\mathrm{pin}(V)$ with highest weight α_a.

7. The case $H = \mathrm{G}O_6$

7.1 In this section we assume that $m=3$ and $\tilde{X}$ is split, so $\mathbb{H}=\tilde{H}=\mathrm{G}\mathbb{O}_6^0$. We have an exact sequence $1\to\mu_2\to\mathrm{GL}_4\to\mathrm{G}\mathbb{O}_6^0\to 1$ of group schemes over $\mathrm{Spec}\, k$. By abuse of notation, we write $\rho : \mathrm{Bun}_4\to\mathrm{Bun}_{\tilde{H}}$ for the corresponding extension of scalars map. It sends $W\in\mathrm{Bun}_4$ to

$$\Big(V=\bigwedge^2 W, \mathcal{C}=\det W, \mathrm{Sym}^2 V \xrightarrow{h} \mathcal{C}, \gamma\Big).$$

Here h is the symmetric form induced by the exterior product $\bigwedge^2 W\otimes\bigwedge^2 W\to\det W$, and $\gamma : \det V \widetilde{\to} \mathcal{C}^3$ is a compatible trivialization. The connected components of $\mathrm{Bun}_{\tilde{H}}$ are indexed by $\pi_1(\tilde{H})$. We have an exact sequence $0\to\pi_1(\mathrm{GL}_4)\to\pi_1(\tilde{H})\to\mathbb{Z}/2\mathbb{Z}\to 0$, and the image of ρ is ${}_0\mathrm{Bun}_{\tilde{H}} := \bigsqcup_{a\in\pi_1(\mathrm{GL}_4)}\mathrm{Bun}_{\tilde{H}}^a$.

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Recall that

$$\check{\mathbb{H}} \widetilde{\to} \mathrm{GSpin}_6 \widetilde{\to} \{(c,b) \in \mathbb{G}_m \times \mathrm{GL}_4 \mid \det b = c^2\}.$$

Consider a $\check{\mathbb{H}}$-local system on X given by a collection: local systems E and χ on X of ranks four and one, respectively, and an isomorphism $\det E \widetilde{\to} \chi^2$. Assume that E is irreducible on X. Let Aut_E denote the corresponding automorphic sheaf on Bun_4 (cf. Definition 8). Then Aut_E is equipped with natural descent data with respect to $\rho : \mathrm{Bun}_4 \to \mathrm{Bun}_{\tilde{H}}$, so gives rise to a perverse sheaf $K_{E,\chi,\tilde{H}}$ on ${}_0\mathrm{Bun}_{\tilde{H}}$.

Lemma 17. *The sheaf $K_{E,\chi,\tilde{H}}$ extends naturally to a perverse sheaf (still denoted by the same symbol) over $\mathrm{Bun}_{\tilde{H}}$ with central character χ.*

Proof. Let $\mathrm{act} : \mathrm{Pic}\, X \times \mathrm{Bun}_{\tilde{H}} \to \mathrm{Bun}_{\tilde{H}}$ be the action map sending $\mathcal{L} \in \mathrm{Pic}\, X$ and $(V, \mathcal{C}, \mathrm{Sym}^2 \to \mathcal{C}) \in \mathrm{Bun}_{\tilde{H}}$ to $(V \otimes \mathcal{L}, \mathcal{C} \otimes \mathcal{L})$. Let ${}_1\mathrm{Bun}_{\tilde{H}} \subset \mathrm{Bun}_{\tilde{H}}$ denote the complement to ${}_0\mathrm{Bun}_{\tilde{H}}$. Then act sends $\mathrm{Pic}^k X \times {}_a\mathrm{Bun}_{\tilde{H}}$ to ${}_b\mathrm{Bun}_{\tilde{H}}$, where $b = a + k \mod 2$. The perverse sheaf

$$A\chi \boxtimes K_{E,\chi,\tilde{H}}[g-1]$$

on $\mathrm{Pic}^1 X \times {}_0\mathrm{Bun}_{\tilde{H}}$ is equipped with natural descent data for $\mathrm{act} : \mathrm{Pic}^1 X \times {}_0\mathrm{Bun}_{\tilde{H}} \to {}_1\mathrm{Bun}_{\tilde{H}}$. This yields a perverse sheaf $K_{E,\chi,\tilde{H}}$ on the whole of $\mathrm{Bun}_{\tilde{H}}$ equipped with $\mathrm{act}^* K_{E,\chi,\tilde{H}} \widetilde{\to} A\chi \boxtimes K_{E,\chi,\tilde{H}}$. Here $A\chi$ is the automorphic local system corresponding to χ. □

Assume $n = 2$, so $G = \mathrm{GSp}_4$. View a $\check{G}$-local system $E_{\check{G}}$ on X as a pair (E, χ), where E (respectively, χ) is a rank-four (respectively, rank-one) local system on X with symplectic form $\bigwedge^2 E \to \chi$. The symplectic form induces the isomorphism $\det E \widetilde{\to} \chi^2$, and (E,χ) identifies with the $\check{\mathbb{H}}$-local system $E_{\check{\mathbb{H}}}$ obtained from $E_{\check{G}}$ via the extension of scalars $\check{G} \hookrightarrow \check{\mathbb{H}}$.

Conjecture 6. We have the following.

(i) Let $E_{\check{G}}$ be a $\check{G}$-local system on X and $E_{\check{\mathbb{H}}}$ be the induced $\check{\mathbb{H}}$-local system on X given by (E,χ). There exists $K \in \mathrm{D}(\mathrm{Bun}_G)$ which is a $E_{\check{G}}$-Hecke eigensheaf satisfying $F_{\tilde{H}}(K) \widetilde{\to} K_{E^*,\chi^*,\tilde{H}}$.

(ii) Assume in addition that E is irreducible (as a local system of rank four). Then $K = F_G(K_{E^*,\chi^*,\tilde{H}})$ satisfies the properties of (i).

7.2 Recall the stack RCov^r introduced in § 6.1.1. Denote by $\mathrm{Bun}_{k,r}$ the following stack. For a scheme S, an S-point of $\mathrm{Bun}_{k,r}$ is a collection consisting of a map $S \to \mathrm{RCov}^r$ giving rise to a two-sheeted covering ${}_SY \to S \times X$, and a rank-k vector bundle on ${}_SY$. Let us precise that a map $S \to \mathrm{RCov}^r$ is given by a collection $(\mathcal{E}, \kappa, D)$, where $D \hookrightarrow S \times X$ is the preimage of the incidence divisor on ${}^{rss}X^{(r)} \times X$ under $S \times X \to {}^{rss}X^{(r)} \times X$, and $\mathcal{E}$ is a line bundle on $S \times X$ equipped with $\kappa : \mathcal{E}^2 \widetilde{\to} \mathcal{O}_{S\times X}(-D)$. Then $\mathcal{O}_{S \times X} \oplus \mathcal{E}$ is a $\mathcal{O}_{S\times X}$-algebra, and ${}_SY = \mathrm{Spec}(\mathcal{O}_{S\times X} \oplus \mathcal{E})$.

We simply think of $\mathrm{Bun}_{k,r}$ as the stack classifying $D_X \in {}^{rss}X^{(r)}$, a two-sheeted covering $\phi : Y \to X$ ramified exactly at D_X (with Y smooth), and a rank-k vector bundle U on Y.

Recall that we assume $n = 2$, so $G = \mathrm{GSp}_4$. Note that ${}^{rss}\mathcal{S}^r_P = \mathrm{Bun}_{1,r}$. We have a diagram

$$\mathrm{Bun}_4 \xleftarrow{\mathfrak{q}_2} \mathrm{Bun}_{2,r} \xrightarrow{\mathfrak{p}_2} {}^{rss}\mathcal{S}^r_P,$$

where $\mathfrak{q}_2$ (respectively, $\mathfrak{p}_2$) is the map sending $(\phi : Y \to X, U)$ as above to $W = \phi_* U$ (respectively, to the point $(\phi : Y \to X, (\det U)^{\star})$ of ${}^{rss}\mathcal{S}^r_P$). Extend it to a commutative diagram

$$\begin{array}{ccccc} \mathrm{Bun}_4 & \xleftarrow{\mathfrak{q}_2} & \mathrm{Bun}_{2,r} & & \\ \downarrow{\scriptstyle \rho} & & \downarrow & \searrow^{\mathfrak{p}_2} & \\ \mathrm{Bun}_{\tilde{H}} & \xleftarrow{\mathfrak{q}_R} & \mathrm{Bun}_R & \xrightarrow{\mathfrak{p}_R} & {}^{rss}\mathcal{S}^r_P \end{array} \qquad (57)$$

defined as follows.

For a point of RCov^r given by $\phi : Y \to X$ let R_ϕ now denote the group scheme on X included into an exact sequence $1 \to \mu_2 \to \phi_* \mathrm{GL}_2 \to R_\phi \to 1$. Let ${}_\phi\mathrm{GL}_4$ be the group scheme of automorphisms of $\phi_*(\mathcal{O}_Y^2)$. Define ${}_\phi\tilde{H}$ by the following commutative diagram, where the rows are exact sequences.

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mu_2 & \longrightarrow & \phi_* \mathrm{GL}_2 & \longrightarrow & R_\phi & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mu_2 & \longrightarrow & {}_\phi\mathrm{GL}_4 & \longrightarrow & {}_\phi\tilde{H} & \longrightarrow & 1
\end{array}$$

Since ${}_\phi\tilde{H}$ is an inner form of $\tilde{H}$, $\mathrm{Bun}_{\tilde{H}} \widetilde{\to} \mathrm{Bun}_{{}_\phi\tilde{H}}$ canonically. Let $\mathfrak{q}_{R_\phi} : \mathrm{Bun}_{R_\phi} \to \mathrm{Bun}_{\tilde{H}}$ denote the corresponding extension of scalars map. Since μ_2 lies in the kernel of the determinant map $\phi_* \mathrm{GL}_2 \stackrel{\det}{\to} \phi_*\mathbb{G}_m$, it yields a map $R_\phi \to \phi_*\mathbb{G}_m$. Let $\mathfrak{p}_{R_\phi} : \mathrm{Bun}_{R_\phi} \to \mathrm{Pic}\, Y$ denote the composition of the corresponding extension of scalars map $\mathrm{Bun}_{R_\phi} \to \mathrm{Pic}\, Y$ with the automorphism $\epsilon : \mathrm{Pic}\, Y \widetilde{\to} \mathrm{Pic}\, Y$ sending $\mathcal{B}$ to $\mathcal{B}^\star$. As ϕ runs through RCov^r, the group schemes R_ϕ organize into a group scheme R over $X \times \mathrm{RCov}^r$, and the diagrams

$$\mathrm{Bun}_{\tilde{H}} \xleftarrow{\mathfrak{q}_{R_\phi}} \mathrm{Bun}_{R_\phi} \xrightarrow{\mathfrak{p}_{R_\phi}} \mathrm{Pic}\, Y$$

form a family giving rise to (57). Recall the functor $F_{\mathcal{S}}$ introduced in § 3.3.2.

Proposition 11. *For $K \in \mathrm{D}(\mathrm{Bun}_{\tilde{H}})$ there is a functorial isomorphism*

$$F_{\mathcal{S}}(K)|_{{}^{rss}\mathcal{S}^r_P} \widetilde{\to} (\mathfrak{p}_R)_! \mathfrak{q}_R^* K[\dim.\mathrm{rel}(\mathfrak{q}_R)].$$

Proof. The proof proceeds in two steps.

Step 1. Define the stack $\mathcal{V}_{4,P}$ and the maps $\tilde{\mathfrak{q}}$, $\tilde{\mathfrak{p}}$ by the diagram

$$\begin{array}{ccccc}
\mathrm{Bun}_4 & \xleftarrow{\tilde{\mathfrak{q}}} & \mathcal{V}_{4,P} & & \\
\downarrow{\scriptstyle \rho} & & \downarrow & \searrow{\scriptstyle \tilde{\mathfrak{p}}} & \\
\mathrm{Bun}_{\tilde{H}} & \xleftarrow{\mathfrak{q}_{\mathcal{V}}} & \mathcal{V}_{\tilde{H},P} & \xrightarrow{\mathfrak{p}_{\mathcal{V}}} & \mathcal{S}_P
\end{array}$$

where the square is cartesian. The stack $\mathcal{V}_{4,P}$ classifies $L \in \mathrm{Bun}_2$, $W \in \mathrm{Bun}_4$, and a map $t : L \otimes V \to \Omega$ with $V = \bigwedge^2 W$.

Let $\zeta_2 : \mathrm{Bun}_{2,r} \to \mathcal{V}_{4,P}$ be the map sending $(\phi : Y \to X, U)$ to $(W = \phi_* U, L = \phi_*((\bigwedge^2 U)^\star), t)$, where $t : V \to L^* \otimes \Omega$ is the following map. We have $L^* \otimes \Omega \widetilde{\to} \phi_*(\bigwedge^2 U)$. The exterior square of the natural map $\phi^*\phi_* U \to U$ is a map $\phi^*(\bigwedge^2 W) \to \bigwedge^2 U$, by adjointness it yields a map $t : V \to \phi_*(\bigwedge^2 U)$.

We have the following commutative diagram.

$$\begin{array}{ccccc}
 & & \mathrm{Bun}_{2,r} & \xrightarrow{\mathfrak{p}_2} & {}^{rss}\mathcal{S}^r_P \\
 & \swarrow{\scriptstyle \mathfrak{q}_2} & \downarrow{\scriptstyle \zeta_2} & & \downarrow \\
\mathrm{Bun}_4 & \xleftarrow{\tilde{\mathfrak{q}}} & \mathcal{V}_{4,P} & \xrightarrow{\tilde{\mathfrak{p}}} & \mathcal{S}_P
\end{array} \qquad (58)$$

Let us show that the square in this diagram is cartesian. To do so, consider a k-point (L, W, t) of $\mathcal{V}_{4,P}$ whose image under $\tilde{\mathfrak{p}}$ is given by a k-point $(\phi : Y \to X, \mathcal{B}, D_X)$ of ${}^{rss}\mathcal{S}^r_P$. So, we are

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given isomorphisms $L \widetilde{\to} \phi_* \mathcal{B}$, $\mathcal{C} \widetilde{\to} \det W$, and $N(\mathcal{B}) \widetilde{\to} \Omega^2 \otimes \mathcal{C}^{-1}$ such that the following diagram commutes.

$$\begin{CD} \operatorname{Sym}^2 V @>{t\otimes t}>> \operatorname{Sym}^2(L^* \otimes \Omega) \\ @AAA @AAA \\ \mathcal{C} @>{\sim}>> N(\mathcal{B}^{-1}) \otimes \Omega^2 @>{\sim}>> N(\mathcal{B}^\star)(-D_X) \end{CD} \tag{59}$$

Write σ_ϕ for the nontrivial automorphism of Y over X, let D_Y be the ramification divisor of $\phi : Y \to X$, so $D_X = \phi_*(D_Y)$.

The map $t : V \to L^* \otimes \Omega \widetilde{\to} \phi_*(\mathcal{B}^\star)$ can be seen as $t : \phi^* V \to \mathcal{B}^\star$. The latter map is nonzero, because the symmetric form on L is generically nondegenerate. Applying ϕ^* to (59), we obtain the following commutative diagram.

$$\begin{CD} \phi^* \operatorname{Sym}^2 V @>>> (\mathcal{B}^\star)^2 \oplus (\sigma_\phi^* \mathcal{B}^\star)^2 \oplus (\mathcal{B}^\star \otimes \sigma_\phi^* \mathcal{B}^\star) \\ @AAA @AA{(0,0,1)}A \\ \phi^* \det W @>{\sim}>> (\mathcal{B}^\star \otimes \sigma_\phi^* \mathcal{B}^\star)(-2D_Y) \end{CD}$$

The transpose $\mathcal{B} \otimes \Omega^{-1} \to \phi^* V^*$ to t is an isotropic subsheaf in $\phi^* V^*$. So, there is a rank-two vector bundle U on Y and a surjection $\phi^* W \xrightarrow{v} U$ such that t factors as a composition

$$\phi^* V \to \bigwedge^2 U \overset{u}{\hookrightarrow} \mathcal{B}^\star.$$

We are going to check that $\bigwedge^2 U \overset{u}{\hookrightarrow} \mathcal{B}^\star$ is actually and isomorphism, and the map $W \xrightarrow{v} \phi_* U$ is also an isomorphism.

Indeed, the maps $V \to \phi_*(\bigwedge^2 U) \to \phi_*(\mathcal{B}^\star)$ yield a commutative diagram

$$\begin{CD} \operatorname{Sym}^2(V^* \otimes \Omega) @<<< \operatorname{Sym}^2(\phi_*((\bigwedge^2 U)^\star)) @<<< \operatorname{Sym}^2 L \\ @VVV @VVV @VVV \\ \mathcal{C}^{-1} \otimes \Omega^2 @<<< N((\bigwedge^2 U)^\star) @<<< N(\mathcal{B}) \end{CD}$$

and the composition of maps in the bottom row is an isomorphism. It follows that the transpose $\mathcal{B} \to (\bigwedge^2 U)^\star$ to u is an isomorphism.

Now consider the diagram

$$V \overset{\bigwedge^2 v}{\to} \bigwedge^2 (\phi_* U) \to \phi_*\Big(\bigwedge 2U\Big),$$

where the second map is induced by the natural map $\phi^* \phi_* U \to U$. Applying symmetric squares, one obtains the following commutative diagram.

$$\begin{array}{ccccc} \mathcal{C} & \xrightarrow{\sim} & N(\bigwedge^2 U)(-D_X) & & \\ \downarrow & & & \searrow & \\ \operatorname{Sym}^2 V & \longrightarrow & \operatorname{Sym}^2(\bigwedge^2(\phi_* U)) & \longrightarrow & \operatorname{Sym}^2(\phi_*(\bigwedge^2 U)) \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{C} & \longrightarrow & \det(\phi_* U) & \overset{\xi}{\hookrightarrow} & N(\bigwedge^2 U) \end{array}$$

It is easy to see that ξ induces an isomorphism $\det(\phi_* U) \widetilde{\to} N(\bigwedge^2 U)(-D_X)$. Thus, $\det v : \det W \widetilde{\to} \mathcal{C} \widetilde{\to} \det(\phi_* U)$ is an isomorphism, so $v : W \to \phi_* U$ is an isomorphism.

Step 2. The diagram (58) gives rise to the following commutative diagram.

$$\begin{array}{ccccc} & & \mathrm{Bun}_R & \xrightarrow{\mathfrak{p}_R} & {}^{rss}\mathcal{S}^r_P \\ & \swarrow^{\mathfrak{q}_R} & \downarrow & & \downarrow \\ \mathrm{Bun}_{\tilde{H}} & \xleftarrow{\mathfrak{q}_{\mathcal{V}}} & \mathcal{V}_{\tilde{H},P} & \xrightarrow{\mathfrak{p}_{\mathcal{V}}} & \mathcal{S}_P \end{array}$$

We have to show that the square in this diagram is cartesian. By Step 1, this is true after the base change ${}_0\mathrm{Bun}_{\tilde{H}} \hookrightarrow \mathrm{Bun}_{\tilde{H}}$. For the components of ${}_1\mathrm{Bun}_{\tilde{H}}$ the argument is similar. □

Definition 12. Let $K \in \mathrm{D}(\mathrm{Bun}_{\tilde{H}})$ be a complex with central character χ^{-1}. Then $F_{\mathcal{S}}(K)$ has central character χ. Let $\mathcal{J}$ be a rank-one local system on Y equipped with $N(\mathcal{J}) \widetilde{\to} \chi^{-1}$. Then for $\mathrm{Pic}\, Y \hookrightarrow {}^{rss}\mathcal{S}^r_P \subset \mathcal{S}_P$ the $*$-restriction $A\mathcal{J} \otimes F_{\mathcal{S}}(K)|_{\mathrm{Pic}\, Y}$ is equipped with natural descent data for $e_\phi : \mathrm{Pic}\, Y \to \mathrm{Bun}_{U_\phi}$. Assume that the following holds:

(C_G) $\mathcal{K}_K$ is a complex on Bun_{U_ϕ} equipped with

$$e^*_\phi \mathcal{K}_K[\dim.\mathrm{rel}(e_\phi)] \widetilde{\to} A\mathcal{J} \otimes (\mathfrak{p}_{R_\phi})_! \mathfrak{q}^*_{R_\phi}(K)[\dim.\mathrm{rel}(\mathfrak{q}_{R_\phi})].$$

For $a \in \mathbb{Z}/2\mathbb{Z}$ the *generalized Waldspurger period* of K is

$$\mathrm{GWP}^a(K, \mathcal{J}) = \mathrm{R}\Gamma_c(\mathrm{Bun}^a_{U_\phi}, \mathcal{K}_K).$$

8. Towards a construction of automorphic sheaves on $\mathrm{Bun}_{\boldsymbol{G\mathbb{S}p_4}}$

8.1 In this section we assume that the ground field is k algebraically closed of characteristic zero and work with $\mathcal{D}$-modules instead of ℓ-adic sheaves. A local system on X is now a vector bundle E with connection $\nabla : E \to E \otimes \Omega$.

Lemma 18. *For a local system E on X there is a canonical $\mathbb{Z}/2\mathbb{Z}$-graded isomorphism*

$$\det \mathrm{R}\Gamma_{DR}(X, E) \widetilde{\to} \det \mathrm{R}\Gamma(X, \det E) \otimes \det \mathrm{R}\Gamma(X, \Omega \otimes \det E)^{-1}.$$

Proof. Let $\mathrm{DR}(E) = (E \stackrel{\nabla}{\to} E \otimes \Omega)$ be the de Rham complex of E placed in degrees zero and one. The exact triangle $\mathrm{DR}(E) \to E \to E \otimes \Omega$ yields

$$\det \mathrm{R}\Gamma_{DR}(X, E) \widetilde{\to} \det \mathrm{R}\Gamma(X, E) \otimes \det \mathrm{R}\Gamma(X, E \otimes \Omega)^{-1}.$$

Our assertion follows now from Lemma 1. □

Set $n = 2$, so $G = \mathrm{G}\mathbb{S}\mathrm{p}_4$. Write $\mathrm{LocSys}_{\check{G}}$ for the moduli stack of $\check{G}$-local systems on X. View a $\check{G}$-local system $E_{\check{G}}$ as a pair (E, χ), where E (respectively, χ) is a rank-four (respectively, rank-one) local system on X with symplectic form $\bigwedge^2 E \to \chi^{-1}$.

Let $\phi : Y \to X$ be a (possibly ramified) two-sheeted covering. Write $\mathrm{LocSys}_{Y,r}$ for the moduli stack of rank-r local systems on Y.

Let $\mathcal{M}_Y$ denote the stack classifying a $\check{G}$-local system $E_{\check{G}} = (E, \chi)$ on X and $\mathcal{J} \in \mathrm{LocSys}_{Y,1}$ equipped with $N(\mathcal{J}) \widetilde{\to} \chi$. The following is an immediate consequence of Lemma 18.

Lemma 19. *The ($\mathbb{Z}/2\mathbb{Z}$-graded) line bundle on $\mathcal{M}_Y$ with fibre $\det \mathrm{R}\Gamma_{DR}(Y, \mathcal{J} \otimes \phi^* E)$ at $(\mathcal{J}, E_{\check{G}})$ is canonically trivialized.*

Let ${}^0\mathcal{M}_Y \subset \mathcal{M}_Y$ denote the open substack given by the condition $\mathrm{H}^i_{DR}(Y, \mathcal{J} \otimes \phi^* E) = 0$ for $i = 0, 2$. We have a vector bundle V on ${}^0\mathcal{M}_Y$ whose fibre at $(\mathcal{J}, E_{\check{G}})$ is

$$\mathrm{H}^1_{DR}(Y, \mathcal{J} \otimes \phi^* E).$$

The rank of V is $c := 8(g_Y - 1)$, where g_Y is the genus of Y. As in Remark 14, one equips V with a nondegenerate symmetric form $\mathrm{Sym}^2 V \to \mathcal{O}$. Moreover, the trivialization $\det V \widetilde{\to} \mathcal{O}$ given by Lemma 19 is compatible with this symmetric form.

Conjecture 7. The $\mathbb{SO}_c$-torsor V lifts naturally to a $\mathbb{S}\mathrm{pin}_c$-torsor $\mathcal{F}$ on ${}^0\mathcal{M}_Y$.

8.2 Let $E_{\check{G}}$ be a $\check{G}$-local system on X viewed as a pair (E, χ), where E and χ are local systems on X of ranks four and one respectively, and $\bigwedge^2 E \to \chi^{-1}$ is a symplectic form. Assume that E is irreducible. In this situation we propose the following conjectural construction of an automorphic sheaf $K_{E_{\check{G}}}$ on Bun_G corresponding to $E_{\check{G}}$.

Let $r \geqslant 0$, remind the stacks RCov^r and ${}^{rss}\mathcal{S}^r_P \subset \mathcal{S}^r_P$ (cf. § 6.1.1). A point of RCov^r is given by a divisor $D_X \in {}^{rss}X^{(r)}$ and a two-sheeted covering $\phi : Y \to X$ ramified exactly at D_X with Y smooth. Let $Y_{\mathrm{univ}} \to X \times \mathrm{RCov}^r$ denote the universal two-sheeted covering. For a morphism of stacks $\alpha : S \to \mathrm{RCov}^r$ denote by $Y_S \to X \times S$ the two-sheeted covering obtained from the universal covering by the base change $\mathrm{id} \times \alpha : X \times S \to X \times \mathrm{RCov}^r$.

Let $\mathcal{M}$ be the stack classifying collections: a point of RCov^r given by $D_X \in {}^{rss}X^{(r)}$, $Y \xrightarrow{\phi} X$, and a rank-one local system $\mathcal{J}$ on Y equipped with an isomorphism $N(\mathcal{J}) \widetilde{\to} \chi$. By definition, for a scheme S, an S-point of $\mathcal{M}$ is given by a map $S \to \mathrm{RCov}^r$ and a rank-one local system $\mathcal{J}$ (relative to S) over Y_S equipped with a trivialization of $N_{X\times S}(\mathcal{J})$.

Let V be the vector bundle on $\mathcal{M}$ whose fibre at the above point is $\mathrm{H}^1_{DR}(Y, \mathcal{J} \otimes \phi^* E)$. As in § 8.1, we equip it with a nondegenerate symmetric form and a compatible trivialization $\det V \widetilde{\to} \mathcal{O}$. Assuming that Conjecture 7 holds, we obtain a $\mathbb{S}\mathrm{pin}_c$-torsor on $\mathcal{M}$. For the half-spin fundamental weights α_a $(a \in \mathbb{Z}/2\mathbb{Z})$ of $\mathbb{S}\mathrm{pin}_c$ write V^{α_a} for the corresponding vector bundles on $\mathcal{M}$ induced from our $\mathbb{S}\mathrm{pin}_c$-torsor.

The projection $\mathcal{M} \to \mathrm{RCov}^r$ should be equipped with an integrable connection along RCov^r making $\mathcal{M}$ into a $\mathcal{D}_{\mathrm{RCov}^r}$-stack (in the sense of [BD04, § 2.3.1]). Then $\mathcal{M}$ carries a sheaf of algebras $\mathcal{O}_{\mathcal{M}}[\mathcal{D}_{\mathrm{RCov}^r}]$ (in the notation of [BD04, § 2.3.4]). We expect that V^{α_a} is naturally a module over $\mathcal{O}_{\mathcal{M}}[\mathcal{D}_{\mathrm{RCov}^r}]$.

Consider the two-sheeted covering $Y_{\mathcal{M}} \to X \times \mathcal{M}$ obtained from $Y_{\mathrm{univ}} \to X \times \mathrm{RCov}^r$ by the base change $\mathrm{id} \times \mathrm{pr} : X \times \mathcal{M} \to X \times \mathrm{RCov}^r$. Let $\mathcal{J}_{\mathrm{univ}}$ denote the universal local system (relative to $\mathcal{M}$) over $Y_{\mathcal{M}}$, its norm on $X \times \mathcal{M}$ is trivialized. Let $Y^{(d)}_{\mathcal{M}}$ denote the dth symmetric power of $Y_{\mathcal{M}}$ (relative to $\mathcal{M}$). That is, $Y^{(d)}_{\mathcal{M}}$ is the quotient of the dth power $Y_{\mathcal{M}} \times_{\mathcal{M}} \cdots \times_{\mathcal{M}} Y_{\mathcal{M}}$ by the symmetric group on d elements. Let $\mathcal{J}^{(d)}_{\mathrm{univ}}$ denote the corresponding local system (relative to $\mathcal{M}$) on $Y^{(d)}_{\mathcal{M}}$.

Recall that, for a scheme S, an S-point of ${}^{rss}\mathcal{S}^r_P$ is given by a map $S \to \mathrm{RCov}^r$ and an invertible sheaf $\mathcal{B}$ on Y_S. For $a \in \mathbb{Z}/2\mathbb{Z}$ write ${}^{rss}\mathcal{S}^r_{a,P}$ for the substack of ${}^{rss}\mathcal{S}^r_P$ given by $a = (\deg \mathcal{B}) \mod 2$.

An S-point of $Y^{(d)}_{\mathcal{M}}$ is given by the following collection: a map $S \to \mathrm{RCov}^r$, a rank-one local system $\mathcal{J}$ (relative to S) over Y_S with a trivialization of $N_{X\times S}(\mathcal{J})$, and an effective Cartier divisor D_S on Y_S flat over S of degree d. For $d \geqslant 0$ consider the Abel–Jacobi map

$$\mathrm{jac} : Y^{(d)}_{\mathcal{M}} \to \mathcal{M} \times_{\mathrm{RCov}^r} {}^{rss}\mathcal{S}^r_P$$

over $\mathcal{M}$, it is given by $\mathcal{B} = \mathcal{O}_{Y_S}(D_S)$.

There is a unique local system $\mathcal{P}$ (relative to $\mathcal{M}$) over $\mathcal{M} \times_{\mathrm{RCov}^r} {}^{rss}\mathcal{S}^r_P$ with the following properties. For any $d \geqslant 0$ one has $\mathrm{jac}^* \mathcal{P} \widetilde{\to} \mathcal{J}^{(d)}_{\mathrm{univ}}$ canonically, and $\mathcal{P}$ satisfies the usual automorphic property with respect to the group structure of $\mathrm{Pic}\, Y$. More precisely, $\mathcal{M} \times_{\mathrm{RCov}^r} {}^{rss}\mathcal{S}^r_P$ is a commutative group stack over $\mathcal{M}$, and the automorphic property of $\mathcal{P}$ is required for this group structure.

In more concrete terms, $\mathcal{M} \times_{\mathrm{RCov}^r} {}^{rss}\mathcal{S}^r_P$ classifies $D_X \in {}^{rss}X^{(r)}$, $\phi : Y \to X$, a rank-one local system $\mathcal{J}$ on Y with a trivialization of $N_X(\mathcal{J})$, and an invertible sheaf $\mathcal{B}$ on Y. Then the fibre of $\mathcal{P}$ at this point identifies with $(A\mathcal{J})_{\mathcal{B}}$.

Consider the diagram of projections

$$\mathcal{M} \xleftarrow{q_{\mathcal{M}}} \mathcal{M} \times_{\mathrm{RCov}^r} {}^{rss}\mathcal{S}^r_P \xrightarrow{q_{\mathcal{S}}} {}^{rss}\mathcal{S}^r_P.$$

For $a \in \mathbb{Z}/2\mathbb{Z}$ define a complex K_a on ${}^{rss}\mathcal{S}^r_{a,P}$ by

$$(q_{\mathcal{S}})_*(q^*_{\mathcal{M}} V^{\alpha_a} \otimes \mathcal{P})[\dim],$$

where the direct image with respect to $q_{\mathcal{S}}$ is understood in the (derived) quasi-coherent sense.

We expect that, for a suitable shift, K_a has a natural structure of a $\mathcal{D}$-module on ${}^{rss}\mathcal{S}^r_P$. Let then K be the $\mathcal{D}$-module on ${}^{rss}\mathcal{S}^r_P$ whose restriction to ${}^{rss}\mathcal{S}^r_{a,P}$ is K_a. Let $\tilde{K}$ denote the intermediate extension of K under ${}^{rss}\mathcal{S}^r_P \subset \mathcal{S}^r_P$.

Recall that $\mathcal{S}_P$ and Bun_P are dual (generalized) vector bundles over $\mathrm{Bun}_2 \times \mathrm{Pic}\, X$, let $\mathrm{Four}(\tilde{K})$ denote the Fourier transform of $\tilde{K}$. Recall the projection $\nu_P : \mathrm{Bun}_P \to \mathrm{Bun}_G$, let ${}^0\mathrm{Bun}_P \subset \mathrm{Bun}_P$ be the open substack, where ν_P is smooth. We expect that $\nu_P^* K_{E_{\check{G}}}$ identifies over ${}^0\mathrm{Bun}_P$ with $\mathrm{Four}(\tilde{K})$ as $\mathcal{D}$-modules.

Appendix A. Prym varieties

A.1 Let $\pi : \tilde{X} \to X$ be a two-sheeted covering ramified at some divisor D_π on X with $\deg D_\pi = d$. Let σ be the nontrivial automorphism of $\tilde{X}$ over X. Let $\mathcal{E}$ be the σ-anti-invariants in $\pi_*\mathcal{O}$, it is equipped with $\mathcal{E}^2 \widetilde{\to} \mathcal{O}(-D_\pi)$. Let $\mathcal{E}_0$ be the σ-anti-invariants in $\pi_*\bar{\mathbb{Q}}_\ell$, it is equipped with $\mathcal{E}_0^2 \widetilde{\to} \bar{\mathbb{Q}}_\ell$ over $X - D_\pi$.

The norm map $N : \mathrm{Pic}\, \tilde{X} \to \mathrm{Pic}\, X$ is given by $N(\mathcal{B}) = \mathcal{E}^{-1} \otimes \det(\pi_*\mathcal{B})$, this is a homomorphism of group stacks. We write $N_X(\mathcal{B}) = N(\mathcal{B})$ when we need to express the dependence on X. For $\mathcal{C} \in \mathrm{Pic}\, X$ we have canonically $N(\pi^*\mathcal{C}) \widetilde{\to} \mathcal{C}^2$.

Let $\tilde{E}$ be a rank-one local system on $\tilde{X}$. Then $\tilde{E} \otimes \sigma^*\tilde{E}$ is equipped with natural descent data for π, so there is a rank-one local system $N(\tilde{E})$ on X equipped with $\pi^* N(\tilde{E}) \widetilde{\to} \tilde{E} \otimes \sigma^*\tilde{E}$. (In the nonramified case we have $N(\tilde{E}) \widetilde{\to} \mathcal{E}_0 \otimes \det(\pi_*\tilde{E})$ canonically.) Recall that $A\tilde{E}$ denotes the automorphic local system on $\mathrm{Pic}\, \tilde{X}$ corresponding to $\tilde{E}$. The restriction of $A\tilde{E}$ under $\pi^* : \mathrm{Pic}\, X \to \mathrm{Pic}\, \tilde{X}$ identifies canonically with $AN(\tilde{E})$. For a rank-one local system E on X we have canonically $N^*(AE) \widetilde{\to} A(\pi^* E)$.

Write $\underline{\mathrm{Pic}}\, X$ for the Picard scheme of X, so we have a $\mathbb{G}_m$-gerbe $\mathrm{Pic}\, X \to \underline{\mathrm{Pic}}\, X$. Write $\mathrm{Pic}^r X$ for the connected component classifying line bundles of degree r. Let $\underline{P}$ denote the connected component of unity of $\mathrm{Ker}\, \underline{N}$, where $\underline{N} : \underline{\mathrm{Pic}}\, \tilde{X} \to \underline{\mathrm{Pic}}\, X$ is the norm map. This is the Prym variety [Mum74]. We need the following results proved in [Mum74]. Assume that $\tilde{X}$ is connected.

Case of ramified π. The group scheme $\mathrm{Ker}\, \underline{N}$ is connected, $\pi^* : \underline{\mathrm{Pic}}\, X \hookrightarrow \underline{\mathrm{Pic}}\, \tilde{X}$ is a closed immersion. For each r we have a surjection

$$\underline{\mathrm{Pic}}^{2r}\, \tilde{X} / \underline{\mathrm{Pic}}^r\, X \to \underline{P}$$

sending $\mathcal{B}$ to $\mathcal{B}^{-1} \otimes \sigma^*\mathcal{B}$. For $r = 0$ its kernel is a finite group isomorphic to $P_2/\phi(J_2)$ for some inclusion $\phi : J_2 \to P_2$. Here P_2 and J_2 are the groups of order two points of $\underline{P}$ and $\underline{\mathrm{Pic}}^0\, X$, respectively. Recall that $\dim \underline{\mathrm{Pic}}^0\, X = g$, $\dim \underline{\mathrm{Pic}}^0\, \tilde{X} = 2g + d/2 - 1$ and $\dim \underline{P} = g + d/2 - 1$. So,

$$J_2 \widetilde{\to} (\mathbb{Z}/2\mathbb{Z})^{2g} \quad \text{and} \quad P_2 \widetilde{\to} (\mathbb{Z}/2\mathbb{Z})^{2g+d-2}.$$

Case of unramified π. The group scheme $\mathrm{Ker}\, \underline{N}$ has two connected components, say $\mathrm{Ker}^r\, \underline{N}$ for $r \in \mathbb{Z}/2\mathbb{Z}$. We denote by $\mathrm{Ker}^0\, \underline{N}$ the connected component of unity. The kernel of $\pi^* : \underline{\mathrm{Pic}}\, X \to \underline{\mathrm{Pic}}\, \tilde{X}$ is $H_0 := \{\mathcal{O}, \mathcal{E}\}$, and we have an isomorphism $\underline{\mathrm{Pic}}^0\, \tilde{X} / (\underline{\mathrm{Pic}}^0\, X / H_0) \widetilde{\to} \underline{P}$ sending $\mathcal{B}$ to $\mathcal{B}^{-1} \otimes \sigma^*\mathcal{B}$.

The following is probably well known, but we have not found a proof of it, so we give one.

Lemma 20. *Assume that π is nonramified. Both $\underline{\mathrm{Pic}}\,\tilde X/(\underline{\mathrm{Pic}}\,X/H_0)$ and $\mathrm{Ker}\,\underline{N}$ have two connected components indexed by $\mathbb{Z}/2\mathbb{Z}$. The map $\delta : \underline{\mathrm{Pic}}\,\tilde X/(\underline{\mathrm{Pic}}\,X/H_0) \to \mathrm{Ker}\,\underline{N}$ sending $\mathcal{B}$ to $\mathcal{B}^{-1}\otimes\sigma^*\mathcal{B}$ is an isomorphism, so sends the odd connected component to the odd one.*

Proof. Consider the map $\pi_*\mathbb{G}_m \xrightarrow{\xi} \pi_*\mathbb{G}_m$ sending f to $\sigma(f)f^{-1}$. Let $\mathcal{N} : \pi_*\mathbb{G}_m \to \mathbb{G}_m$ be the norm map. The sequences

$$\pi_*\mathbb{G}_m \xrightarrow{\xi} \pi_*\mathbb{G}_m \xrightarrow{\mathcal{N}} \mathbb{G}_m \to 1$$

and

$$1 \to \mathbb{G}_m \to \pi_*\mathbb{G}_m \xrightarrow{\xi} \pi_*\mathbb{G}_m$$

are exact in étale topology (indeed, it suffices to check this after base change $\tilde X \xrightarrow{\pi} X$, which is easy). Taking the étale cohomology of X, we get an exact sequence

$$\mathrm{H}^1(X, \mathrm{Im}\,\xi) \to \underline{\mathrm{Pic}}\,\tilde X \xrightarrow{N} \underline{\mathrm{Pic}}\,X \to 1$$

and the map induced by ξ is δ. The map $\underline{\mathrm{Pic}}\,\tilde X \widetilde{\to} \mathrm{H}^1(X, \pi_*\mathbb{G}_m) \xrightarrow{\xi} \mathrm{H}^1(X, \mathrm{Im}\,\xi)$ is surjective, because $\mathrm{H}^2(X, \mathbb{G}_m) = 0$. □

Let $\tilde D_\pi \in \tilde X^{(d)}$ be the ramification divisor of π, so $D_\pi = \pi_*\tilde D_\pi$. Define the group scheme U_π on X by the exact sequence $1 \to \mathbb{G}_m \to \pi_*\mathbb{G}_m \to U_\pi \to 1$. The stack Bun_{U_π} classifies $\mathcal{B} \in \mathrm{Pic}\,\tilde X$ together with a trivialization

$$N(\mathcal{B}) \widetilde{\to} \mathcal{O}_X \tag{60}$$

and a compatible isomorphism $\gamma : \mathcal{B}|_{\tilde D_\pi} \widetilde{\to} \mathcal{O}_{\tilde D_\pi}$. This means that $\gamma^{\otimes 2} : N(\mathcal{B})|_{D_\pi} \widetilde{\to} \mathcal{O}_{D_\pi}$ is the isomorphism induced by (60). We have used that π induces an isomorphism of reduced divisors $\tilde D_\pi \widetilde{\to} D_\pi$. The corresponding extension of scalars map $e_\pi : \mathrm{Pic}\,\tilde X \to \mathrm{Bun}_{U_\pi}$ is smooth and surjective, it sends $\mathcal{B}$ to $\mathcal{C} = \mathcal{B}^{-1}\otimes\sigma^*\mathcal{B}$ with natural trivializations $\mathcal{C}|_{\tilde D_\pi} \widetilde{\to} \mathcal{O}_{\tilde D_\pi}$ and $N(\mathcal{C}) \widetilde{\to} \mathcal{O}_X$.

In both cases (π ramified or not) the stack Bun_{U_π} has two connected components indexed by $a \in \mathbb{Z}/2\mathbb{Z}$, here $\mathrm{Bun}^0_{U_\pi}$ is the connected component of unity. The image of $\mathrm{Pic}^r\,\tilde X$ under e_π equals $\mathrm{Bun}^a_{U_\pi}$ with $a = r \bmod 2$.

If π is ramified, then Bun_{U_π} is a scheme, and the projection $\mathrm{Bun}^a_{U_\pi} \to \underline{P}$ is a Galois covering with Galois group $(\mathbb{Z}/2\mathbb{Z})^{d-2}$.

Appendix B. Group schemes and Hecke operators

B.1.1 Let $\pi : \tilde X \to X$ be an étale Galois covering with Galois group $\Sigma = \mathrm{Aut}_X(\tilde X)$. The category of affine group schemes over X is canonically equivalent to the category of affine group schemes over $\tilde X$ equipped with an action of Σ.

Assume that $\mathbb{G}$ is an affine group scheme over $\mathrm{Spec}\,k$ (viewed as constant group scheme on $\tilde X$). Then an action of Σ on $\mathbb{G}$ is a datum of a homomorphism $\Sigma \to \mathrm{Aut}(\mathbb{G})$. The corresponding group scheme G over X is then obtained as the twisting of $\mathbb{G}$ by the Σ-torsor $\pi : \tilde X \to X$.

The action of Σ on $\mathbb{G}$ gives rise to the semi-direct product $\mathbb{G} \ltimes \Sigma$ included into an exact sequence $1 \to \mathbb{G} \to \mathbb{G} \ltimes \Sigma \to \Sigma \to 1$.

Let us describe the category of G-torsors on X. For $\sigma \in \Sigma$ and a $\mathbb{G}$-torsor $\mathcal{F}_\mathbb{G}$ on a scheme S denote by $\mathcal{F}^\sigma_\mathbb{G}$ the $\mathbb{G}$-torsor on S obtained from $\mathcal{F}_\mathbb{G}$ by the extension of scalars $\sigma : \mathbb{G} \to \mathbb{G}$.

Lemma 21. *Let S be a scheme. The category of G-torsors on $S \times X$ is canonically equivalent to the category of pairs $(\mathcal{F}_\mathbb{G}, \alpha)$, where $\mathcal{F}_\mathbb{G}$ is a $\mathbb{G}$-torsor on $S \times \tilde X$, and $\alpha = (\alpha_\sigma)_{\sigma\in\Sigma}$ is a collection of isomorphisms*

$$\alpha_\sigma : \sigma^*\mathcal{F}_\mathbb{G} \widetilde{\to} \mathcal{F}^\sigma_\mathbb{G}$$

such that for any $\sigma,\tau\in\Sigma$ the following diagram commutes.

$$\begin{array}{ccc} \tau^*\sigma^*\mathcal{F}_{\mathbb{G}} & \xrightarrow{\alpha_\sigma} & \tau^*(\mathcal{F}^\sigma_{\mathbb{G}}) = (\tau^*\mathcal{F}_{\mathbb{G}})^\sigma \\ \downarrow \alpha_{\sigma\tau} & & \downarrow \alpha_\tau \\ (\mathcal{F}_{\mathbb{G}})^{\sigma\tau} & = & (\mathcal{F}^\tau_{\mathbb{G}})^\sigma \end{array}$$

Proof. Let $\tilde{F}$ be an affine scheme over $\tilde{X}$. Assume that the action of Σ on $\tilde{X}$ is lifted to an action on $\tilde{F}$. Let F denote the affine scheme over X, the descent of $\tilde{F}$.

Assume that $\tilde{F}$ is, in addition, a $\mathbb{G}$-torsor. Then for F to be a G-torsor, the actions of $\mathbb{G}$ and of Σ on $\tilde{F}$ should come from an action of $\mathbb{G}\ltimes\Sigma$ on $\tilde{F}$. □

Example 1. Take $\mathbb{H}$ to be a group scheme over $\operatorname{Spec} k$, set $\mathbb{G}=\operatorname{Hom}(\Sigma,\mathbb{H})$ (the group structure on $\mathbb{G}$ comes from that of $\mathbb{H}$). Let Σ act on $\mathbb{G}$ via its action on Σ by translations. Then $G\widetilde{\to}\pi_*\mathbb{H}$.

B.1.2 Let $\mathbb{G}$ be a connected reductive group over $\operatorname{Spec} k$ equipped with a homomorphism $\Sigma\to\operatorname{Aut}(\mathbb{G})$. Let G be the twisting of $\mathbb{G}$ by the Σ-torsor $\pi:\tilde{X}\to X$. Assume that $\mathbb{T}\subset\mathbb{G}$ is a maximal torus invariant under Σ. Let Λ (respectively, $\check{\Lambda}$) be the coweights (respectively, weights) lattices of $\mathbb{T}$, so Σ acts on the corresponding root datum $(\Lambda,R,\check{\Lambda},\check{R})$. Here R (respectively $\check{R}$) are the coroots (respectively, roots) of $\mathbb{G}$.

Let $W=N_{\mathbb{G}}(\mathbb{T})/\mathbb{T}$ be the Weil group. Since Σ preserves both $\mathbb{T}$ and $N_{\mathbb{G}}(\mathbb{T})$, Σ acts on W by group automorphisms. For $\lambda\in\Lambda$, $w\in W$ and $\sigma\in\Sigma$ we have $\sigma(w\lambda)=(\sigma w)(\sigma\lambda)$, so Σ acts on the set Λ/W of dominant coweights of $\mathbb{G}$.

Write Bun_G for the stack of G-torsors on X. Let $\mathcal{H}_G$ denote the Hecke stack classifying $\tilde{x}\in\tilde{X}$, G-torsors $\mathcal{F}_G,\mathcal{F}'_G$ on X, and an isomorphism $\mathcal{F}_G\widetilde{\to}\mathcal{F}'_G|_{X-\pi(\tilde{x})}$. We have a diagram

$$\tilde{X}\times\operatorname{Bun}_G \xleftarrow{\operatorname{supp}\times p} \mathcal{H}_G \xrightarrow{p'} \operatorname{Bun}_G,$$

where p (respectively, p') sends the above point to $\mathcal{F}_G$ (respectively, to $\mathcal{F}'_G$).

A choice of a Borel subgroup in $\mathbb{G}$ containing $\mathbb{T}$ identifies Λ/W with the corresponding set of dominant coweights, hence yields a usual order on Λ/W. This order does not depend on a choice of such Borel subgroup.

Let D_x (respectively, $D_{\tilde{x}}$) denote the formal neighbourhood of $x\in X$ (respectively, of $\tilde{x}\in\tilde{X}$). The map π induces $D_{\tilde{x}}\widetilde{\to}D_x$ for $x=\pi(\tilde{x})$. For $\lambda\in\Lambda/W$ denote by $\overline{\mathcal{H}}^\lambda_G\hookrightarrow\mathcal{H}_G$ the closed substack given by the condition that $\mathcal{F}'_G|_{D_{\tilde{x}}}$ is in a position at most λ with respect to $\mathcal{F}_G|_{D_{\tilde{x}}}$, here we view them as $\mathbb{G}$-torsors using the canonical isomorphism $G|_{\tilde{X}}\widetilde{\to}\mathbb{G}\times X$. Given $\sigma\in\Sigma$, this condition is equivalent to requiring that $\mathcal{F}'_G|_{D_{\sigma\tilde{x}}}$ is in a position at most $\sigma\lambda$ with respect to $\mathcal{F}_G|_{D_{\sigma\tilde{x}}}$.

The Hecke functor

$$\mathrm{H}^\lambda_G:\mathrm{D}(\operatorname{Bun}_G)\to\mathrm{D}(\tilde{X}\times\operatorname{Bun}_G)$$

is given by

$$\mathrm{H}^\lambda_G(K)=(\operatorname{supp}\times p)_!((p')^*K\otimes\mathrm{IC}_{\overline{\mathcal{H}}^\lambda_G})[-\dim\operatorname{Bun}_G].$$

For each $\sigma\in\Sigma$ we have a commutative diagram

$$\begin{array}{ccccc} \tilde{X}\times\operatorname{Bun}_G & \xleftarrow{\operatorname{supp}\times p} & \overline{\mathcal{H}}^{\sigma^{-1}\lambda}_G & \xrightarrow{p'} & \operatorname{Bun}_G \\ \downarrow \sigma\times\mathrm{id} & & \downarrow & & \downarrow \mathrm{id} \\ \tilde{X}\times\operatorname{Bun}_G & \xleftarrow{\operatorname{supp}\times p} & \overline{\mathcal{H}}^{\lambda}_G & \xrightarrow{p'} & \operatorname{Bun}_G \end{array}$$

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where the vertical middle arrow is an isomorphism. This yields a compatible system of isomorphisms for $\sigma \in \Sigma$

$$(\sigma \times \mathrm{id})^* \circ \mathrm{H}^\lambda_G \widetilde{\to} \mathrm{H}^{\sigma^{-1}\lambda}_G. \tag{61}$$

Example 2. Given a homomorphism $\Sigma \to N_{\mathbb{G}}(\mathbb{T}) \subset \mathbb{G}$, consider the corresponding action of Σ on $\mathbb{G}$ by conjugation. Then G is an inner form of $\mathbb{G}$, and Σ acts trivially on Λ/W. Let $\mathcal{F}^1_{\mathbb{G}}$ be the $\mathbb{G}$-torsor on X obtained from the Σ-torsor $\tilde{X}$ by the extension of scalars via $\Sigma \to \mathbb{G}$. Then G identifies with the group scheme (over X) of automorphisms of the $\mathbb{G}$-torsor $\mathcal{F}^1_{\mathbb{G}}$. In this case we identify canonically $\mathrm{Bun}_{\mathbb{G}} \widetilde{\to} \mathrm{Bun}_G$ sending $\mathcal{F}_{\mathbb{G}}$ to the G-torsor $\mathrm{Isom}(\mathcal{F}_{\mathbb{G}}, \mathcal{F}^1_{\mathbb{G}})$. Then H^λ becomes the usual Hecke functor followed by restriction under $\pi \times \mathrm{id} : \tilde{X} \times \mathrm{Bun}_{\mathbb{G}} \to X \times \mathrm{Bun}_{\mathbb{G}}$.

B.1.3 The map $\mathrm{supp} \times p : \overline{\mathcal{H}}^\lambda_G \to \tilde{X} \times \mathrm{Bun}_G$ identifies with the twisted projection

$$(\tilde{X} \times \mathrm{Bun}_G) \tilde{\times}\, \overline{\mathrm{Gr}}^\lambda_{\mathbb{G}} \to \tilde{X} \times \mathrm{Bun}_G\,.$$

Similarly, $\mathrm{supp} \times p' : \overline{\mathcal{H}}^\lambda_G \to \tilde{X} \times \mathrm{Bun}_G$ identifies with $(\tilde{X} \times \mathrm{Bun}_G) \tilde{\times}\, \overline{\mathrm{Gr}}^{-w_0(\lambda)}_{\mathbb{G}} \to \tilde{X} \times \mathrm{Bun}_G$, where w_0 is the longest element of W. Note that $\mathrm{IC}_{\overline{\mathcal{H}}^\lambda_G}$ is ULA with respect to $\mathrm{supp} \times p'$. This implies that H^λ_G commutes with the Verdier duality.

B.1.4 Let $\check{\mathbb{G}}$ denote the Langlands dual group to $\mathbb{G}$, it comes equipped with a maximal torus $\check{\mathbb{T}}$. The group Σ acts naturally on the root datum $(\check{\Lambda}, \check{R}, \Lambda, R)$ of $(\check{\mathbb{G}}, \check{\mathbb{T}})$. Recall that we have an exact sequence

$$1 \to W \to \mathrm{Aut}(\check{\Lambda}, \check{R}, \Lambda, R) \to \mathbb{O}\mathrm{ut}(\mathbb{G}) \to 1,$$

where $\mathbb{O}\mathrm{ut}(\mathbb{G})$ is the group of exterior automorphisms of $\mathbb{G}$. Assume that we are given a lifting of

$$\Sigma \to \mathrm{Aut}(\check{\Lambda}, \check{R}, \Lambda, R)$$

to a homomorphism $\mu : \Sigma \to \mathrm{Aut}(\check{\mathbb{G}}, \check{\mathbb{T}})$. (Such a lifting exists under the additional assumption that the Σ-action on $(\mathbb{G}, \mathbb{T})$ preserves an epinglage of $\mathbb{G}$ containing $\mathbb{T}$.) This lifting is uniquely defined up to inner automorphisms by elements of $\check{\mathbb{T}}$.

Then we have the semi-direct product $G^L := \check{\mathbb{G}} \ltimes \Sigma$ included into an exact sequence $1 \to \check{\mathbb{G}} \to \check{\mathbb{G}} \ltimes \Sigma \to \Sigma \to 1$. This is a version of the L-group associated to G_F. Here G_F denotes the restriction of the group scheme G to the generic point $\mathrm{Spec}\, F \in X$ of X (cf. [Bor79]).

B.2 Let now $\mathbb{G}_1$ be another reductive connected group over $\mathrm{Spec}\, k$ equipped with an action $\Sigma \to \mathrm{Aut}(\mathbb{G}_1)$ and let G_1 be the group scheme on X obtained as the twisting of $\mathbb{G}_1$ by the Σ-torsor $\pi : \tilde{X} \to X$.

Assume that $\mathbb{G}_1$ satisfies the same conditions as $\mathbb{G}$ in Appendix B.1. (The subscript 1 denotes the corresponding objects for G_1.) So, we have a maximal torus $\mathbb{T}_1 \subset \mathbb{G}_1$ stable under Σ, and we assume that we are given a homomorphism $\mu_1 : \Sigma \to \mathrm{Aut}(\check{\mathbb{G}}_1)$ as above. Assume that we are given a Σ-equivariant homomorphism $\check{\mathbb{G}} \to \check{\mathbb{G}}_1$ sending $\check{\mathbb{T}}$ to $\check{\mathbb{T}}_1$. It yields a homomorphism $G^L \to G^L_1$.

The functoriality problem is to find a family of functors

$${}_SF : \mathrm{D}(S \times \mathrm{Bun}_G) \to \mathrm{D}(S \times \mathrm{Bun}_{G_1})$$

for each scheme S with the following property. Write $V_1^{\lambda_1}$ for the irreducible representation of $\check{\mathbb{G}}_1$ with highest weight $\lambda_1 \in \Lambda_1/W_1$. Similarly, V^λ denotes the irreducible representation of $\check{\mathbb{G}}$ with highest weight $\lambda \in \Lambda/W$ (this notion does not depend on a choice of a Borel subgroup in $\mathbb{G}$ containing $\mathbb{T}$). We would like to have for each $\lambda_1 \in \Lambda_1/W_1$ isomorphisms of functors

$$\mathrm{H}^{\lambda_1}_{G_1} \circ {}_SF \widetilde{\to} \bigoplus_\lambda {}_{\tilde{X} \times S}F \circ \mathrm{H}^\lambda_G \otimes \mathrm{Hom}_{\check{\mathbb{G}}}(V^\lambda, V_1^{\lambda_1})$$

from $D(S \times \mathrm{Bun}_G)$ to $D(\tilde{X} \times S \times \mathrm{Bun}_{G_1})$. It is required that these isomorphism are compatible with the action of Σ on both sides.

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References

Ada89 J. Adams, *L-functoriality for dual pairs*, Astérisque, vol. 171–172 (Société Mathématique de France, Paris, 1989), 85–129.

Bor79 A. Borel, *Automorphic L-functions*, Proceedings of Symposia in Pure Mathematics, vol. 33, part 2 (American Mathematical Society, Providence, RI, 1979), 27–61.

BD04 A. Beilinson and V. Drinfeld, *Chiral algebras*, AMS Colloquium Publications, vol. 51 (American Mathematical Society, Providence, RI, 2004).

BG02 A. Braverman and D. Gaitsgory, *Geometric Eisenstein series*, Invent. Math. **150** (2002), 287–384.

BFF97 D. Bump and S. Friedberg and M. Furusawa, *Explicit formulas for the Waldspurger and Bessel models*, Israel J. Math. **102** (1997), 125–177.

FGV02 E. Frenkel, D. Gaitsgory and K. Vilonen, *On the geometric Langlands conjecture*, J. Amer. Math. Soc. **15** (2002), 367–417.

FH91 W. Fulton and J. Harris, *Representation theory, a first course*, Graduate Texts in Mathematics, vol. 129 (Springer, Berlin, 1991).

Gai04 D. Gaitsgory, *On a vanishing conjecture appearing in the geometric Langlands correspondence*, Ann. of Math. (2) **160** (2004), 617–682.

Kud96 S. Kudla, *Notes on the local theta correspondence*, European School on Group Theory, September 1996, http://www.math.umd.edu/∼ssk/castle.pdf.

Lys02 S. Lysenko, *Local geometrized Rankin–Selberg method for* $\mathrm{GL}(n)$, Duke Math. J. **111** (2002), 451–493.

Lys06a S. Lysenko, *Moduli of metaplectic bundles on curves and Theta-sheaves*, Ann. Sci. École Norm. Sup. (4) **39** (2006), 415–466.

Lys06b S. Lysenko, *Whittaker and Bessel functors for* $\mathrm{G}\mathbb{S}\mathrm{p}_4$, Ann. Inst. Fourier (Grenoble) **56** (2006), 1505–1565.

Lys07 S. Lysenko, *Geometric theta-lifting for the dual pair* $\mathrm{S}\mathbb{O}_{2m}$, $\mathbb{S}\mathrm{p}_{2n}$, Preprint (2007), math.RT/0701170. Ann. of Math. (2), submitted.

MVW87 C. Moeglin, M.-F. Vigneras and J. L. Waldspurger, *Correspondence de Howe sur un corps p-adique*, Lecture Notes in Mathematics, vol. 1291 (Springer, Berlin, 1987).

Mum74 D. Mumford, *Prym varieties I, Contribution to analysis* (Academic Press, New York, 1974), 325–350.

Ral82 S. Rallis, *Langlands functoriality and the Weil representation*, Amer. J. Math. **104** (1982), 469–515.

Wal85 J.-L. Waldspurger, *Sur les valeurs de certaines fonctions L automorphes en leur centre de symetrie*, Compositio Math. **54** (1985), 173–242.

Wei65 A. Weil, *Sur la formule de Siegel dans la théorie des groupes classiques*, Acta Math. **113** (1965), 1–87.

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