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RESEARCH ARTICLE

Shokurov's conjecture on conic bundles with canonical singularities

Jingjun Han[1,2,3], Chen Jiang[4] and Yujie Luo[5]

[1]Shanghai Center for Mathematical Sciences, Fudan University, Shanghai, 200438, China;
E-mail: hanjingjun@fudan.edu.cn.
[2]Department of Mathematics, The University of Utah, Salt Lake City, UT 84112, USA; E-mail: jhan@math.utah.edu.
[3]Mathematical Sciences Research Institute, Berkeley, CA 94720, USA; E-mail: jhan@msri.org.
[4]Shanghai Center for Mathematical Sciences, Fudan University, Shanghai, 200438, China;
E-mail: chenjiang@fudan.edu.cn.
[5]Department of Mathematics, Johns Hopkins University, Baltimore, MD 21218, USA; E-mail: yluo32@jhu.edu.

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Abstract
A conic bundle is a contraction $X \to Z$ between normal varieties of relative dimension 1 such that $-K_X$ is relatively ample. We prove a conjecture of Shokurov that predicts that if $X \to Z$ is a conic bundle such that X has canonical singularities and Z is $\mathbb{Q}$-Gorenstein, then Z is always $\frac{1}{2}$-lc, and the multiplicities of the fibres over codimension 1 points are bounded from above by 2. Both values $\frac{1}{2}$ and 2 are sharp. This is achieved by solving a more general conjecture of Shokurov on singularities of bases of lc-trivial fibrations of relative dimension 1 with canonical singularities.

Contents

1. Introduction

We work over the field of complex numbers $\mathbb{C}$.

A $\mathbb{Q}$-*conic bundle* is a proper morphism $X \to Z$ from a 3-fold with only terminal singularities to a normal surface such that all fibres are connected and 1-dimensional, and $-K_X$ is relatively ample over Z. A conjecture of Iskovskikh predicts that the base surface Z has only canonical singularities, or equivalently Z is 1-lc. This conjecture has important applications to the rationality problem of conic bundles [21]. Mori and Prokhorov proved Iskovskikh's conjecture by showing that Z has only Du Val singularities of type A and giving a complete local classification of $\mathbb{Q}$-conic bundles over a singular base in [30, 31].

Motivated by Iskovskikh's conjecture, it is natural to study the singularities of the base surface Z when X has worse singularities: for example, canonical singularities. Such a contraction also appears naturally in the birational classification of 3-dimensional algebraic varieties. Indeed when $\rho(X/Z) = 1$, it is one of the three possible outcomes of the minimal model program for canonical 3-folds of negative Kodaira dimension. However, Z may no longer be 1-lc for such contractions. Shokurov conjectured that Z is always $\frac{1}{2}$-lc, and the value $\frac{1}{2}$ is optimal (see Remark 1.2). More generally, Shokurov's conjecture is expected to hold for conic bundles with canonical singularities in all dimensions.

Conjecture 1.1 (Shokurov, compare [38, 34]). Let $\pi : X \to Z$ be a contraction between normal varieties such that

1. $\dim X - \dim Z = 1$,
2. X is canonical,
3. K_Z is $\mathbb{Q}$-Cartier, and
4. $-K_X$ is ample over Z.

Then Z is $\frac{1}{2}$-lc.

Remark 1.2. 1. In Conjecture 1.1, the assumption in (4) can be replaced by '$-K_X$ is nef and big over Z', which can be reduced to Conjecture 1.1 by taking the anti-canonical model over Z.
2. In a private communication, Prokhorov shared his expectation that Z should be $\frac{1}{2}$-klt in Conjecture 1.1 motivated by [34, Example 10.6.1]. However, this is not always the case if $\dim X \geq 3$; see Example 1.3.

Example 1.3 (compare [34, Example 10.6.1]). Consider the following action of $\boldsymbol{\mu}_{4m}$ on $\mathbb{P}^1_x \times \mathbb{C}^2_{u,v}$:

$$(x; u, v) \mapsto (-x; \xi u, \xi^{2m-1} v),$$

where m is a positive integer and ξ is a primitive $4m$th root of unity. Let $X = (\mathbb{P}^1 \times \mathbb{C}^2)/\boldsymbol{\mu}_{4m}$, $Z = \mathbb{C}^2/\boldsymbol{\mu}_{4m}$ and $\pi : X \to Z$ the natural projection. Since $\boldsymbol{\mu}_{4m}$ acts freely in codimension 1, $-K_X$ is π-ample. Note that Z has an isolated cyclic quotient singularity of type $\frac{1}{4m}(1, 2m-1)$ at the origin $o \in Z$, and $\mathrm{mld}(Z \ni o) = \frac{1}{2}$ (see [4] for the computation of minimal log discrepancies of toric varieties). On the other hand, X is covered by 2 open affine charts $(x \neq 0)$ and $(x \neq \infty)$, and each chart is isomorphic to the affine toric variety $\mathbb{C}^3/\frac{1}{4m}(2m, 1, 2m-1)$, which is canonical (see [36, Theorem (4.11)]) and Gorenstein. Note that in this case, $\rho(X/Z) = 1$ and the singular locus of X is the whole fibre $\pi^{-1}(o)$, which is 1-dimensional. It is not clear yet whether there are such examples where X has isolated canonical singularities.

The main purpose of this paper is to give an affirmative answer to Shokurov's conjecture.

Theorem 1.4. *Conjecture 1.1 holds.*

Theorem 1.4 follows from a more general result; see Theorem 1.7. In order to state the result, we recall some background. Let $\pi : (X, B) \to Z$ be an lc-trivial fibration (see Definition 2.12: for example, $\pi : X \to Z$ is a contraction between normal varieties, and (X, B) is an lc pair with $K_X + B \sim_{\mathbb{R},Z} 0$). By the work of Kawamata [24, 25] and Ambro [3], we have the so-called *canonical bundle formula*

$$K_X + B \sim_{\mathbb{R}} \pi^*(K_Z + B_Z + M_Z),$$

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where B_Z is the *discriminant part* and M_Z is the *moduli part*; see Section 2.4 for more details. For inductive purposes, it is useful and important to study the relation between singularities of (X, B) and those of $(Z, B_Z + M_Z)$. In this context, Shokurov proposed the following conjecture. Recall that $\mathrm{mld}(X/Z \ni z, B)$ is the infimum of all the log discrepancies of prime divisors over X whose image on Z is $\bar{z}$ (see Definition 2.5).

Conjecture 1.5 (Shokurov, compare [2, Conjecture 1.2]). Let d be a positive integer and ϵ a positive real number. Then there is a positive real number $\delta = \delta(d, \epsilon)$ depending only on d, ϵ satisfying the following. Let $\pi : (X, B) \to Z$ be an lc-trivial fibration and $z \in Z$ a point of codimension ≥ 1 such that

1. $\dim X - \dim Z = d$,
2. $\mathrm{mld}(X/Z \ni z, B) \geq \epsilon$, and
3. the generic fibre of π is of Fano type.

Then we can choose $M_Z \geq 0$ representing the moduli part such that $(Z \ni z, B_Z + M_Z)$ is δ-lc.

Remark 1.6.
1. The formulation of Conjecture 1.5 here is stronger than that in the previous literature [2, 5], where a stronger assumption (2') that '(X, B) is an ϵ-lc pair' is required instead of the assumption in (2), and δ depends on $\dim X$ and ϵ instead of just $\dim X - \dim Z$ and ϵ. In our formulation, B can be non-effective and (X, B) can have non-klt centers over $Z \setminus \{z\}$.
2. Birkar [5] proved Conjecture 1.5 under assumption (2') for the following cases: (a) $(F, B|_F)$ belongs to a bounded family, where F is a general fibre of π, or (b) $\dim X = \dim Z + 1$. Hence, under assumption (2'), Conjecture 1.5 holds when the coefficients of $B|_F$ are bounded from below away from zero as a consequence of the Borisov–Alexeev–Borisov conjecture proved by Birkar [7, 8]. Very recently, Birkar and Y. Chen [10] proved Conjecture 1.5 under assumption (2') for toric morphisms between toric varieties. We refer the reader to [6, Theorems 1.9 and 2.5] for more related results.
3. Following ideas in [5], it is indicated by [11, Proposition 7.6] (see [13, Theorem 1.10] for an embryonic form) that Conjecture 1.5 might be a consequence of Shokurov's ϵ-lc complements conjecture. Moreover, following the proof of [5, Corollary 1.7], [11, Theorem 1.3] implies that Conjecture 1.5 holds for $\dim X = \dim Z + 1$.
4. It is worthwhile to mention that Conjecture 1.5 implies M^{c}Kernan's conjecture on Mori fibre spaces [2, Conjecture 1.1], which is closely related to Iskovskikh's conjecture. Alexeev and Borisov [2] proved M^{c}Kernan's conjecture for toric morphisms between toric varieties.

Our second main result gives the optimal value of $\delta(1, \epsilon) = \epsilon - \frac{1}{2}$ for any $\epsilon \geq 1$.

Theorem 1.7. *Let $\pi : (X, B) \to Z$ be an lc-trivial fibration and $z \in Z$ a codimension ≥ 1 point such that*

1. $\dim X - \dim Z = 1$, *the geometric generic fibre of π is a rational curve, and*
2. $\mathrm{mld}(X/Z \ni z, B) \geq 1$.

Then we can choose $M_Z \geq 0$ representing the moduli part such that

$$\mathrm{mld}(Z \ni z, B_Z + M_Z) \geq \mathrm{mld}(X/Z \ni z, B) - \frac{1}{2} \geq \frac{1}{2}.$$

The lower bound in Theorem 1.7 is optimal by Example 4.1.

As a corollary, we have the following global version of Theorem 1.7 with less technical notation involved.

Corollary 1.8. *Let (X, B) be a pair and $\pi : X \to Z$ a contraction between normal varieties such that*

1. $\dim X - \dim Z = 1$,
2. (X, B) *is canonical and B has no vertical irreducible component over Z,*
3. $K_X + B \sim_{\mathbb{R}, Z} 0$, *and*
4. *X is of Fano type over Z.*

Then we can choose $M_Z \geq 0$ representing the moduli part such that $(Z, B_Z + M_Z)$ is $\frac{1}{2}$-lc.

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Remark 1.9. 1. We remark that if $\dim X - \dim Z = 1$, then exceptional divisors over X cannot dominate Z, so the assumption in (2) in Corollary 1.8 is equivalent to the assumption that $\mathrm{mld}(X/Z \ni z, B) \geq 1$ for any codimension ≥ 1 point $z \in Z$.

2. Note that $\frac{1}{2}$ is the maximal accumulation point of the set of minimal log discrepancies in dimension 2 (see [1, Corollary 3.4], [37]). Thus it would be interesting if one could give a new proof of Iskovskikh's conjecture by applying Theorems 1.4 and 1.7 without using the classification of terminal singularities in dimension 3. In fact, we can apply Corollary 1.8 to show that in the setting of Iskovskikh's conjecture, Z is $\frac{1}{2}$-klt; see Corollary 4.5. Recall that in order to prove Iskovskikh's conjecture, it suffices to show that Z is $\frac{2}{3}$-klt (see [22, Lemma 5.1]), but our method could not achieve this. The reason is that in Corollary 4.5, there is no assumption on $\dim X$, but Prokhorov provides us with Example 4.7 showing that Corollary 4.5 cannot be improved if $\dim X \geq 4$.

Theorem 1.7 is a consequence of the following result, which gives a lower bound of certain log canonical thresholds for lc-trivial fibrations. We refer the reader to [11, Problem 7.18] for more discussions.

Theorem 1.10 (compare [38, Conjecture]). *Let $\pi : (X, B) \to Z$ be an lc-trivial fibration and $z \in Z$ a codimension 1 point such that*

1. $\dim X - \dim Z = 1$*, the geometric generic fibre of π is a rational curve, and*
2. $\mathrm{mld}(X/Z \ni z, B) \geq 1$.

Then

$$\mathrm{lct}(X/Z \ni z, B; \pi^*\overline{z}) \geq \mathrm{mld}(X/Z \ni z, B) - \frac{1}{2} \geq \frac{1}{2}.$$

In particular, if B is effective, then the multiplicity of each irreducible component of $\pi^{-1}(z)$ is bounded from above by 2.

The bounds in Theorem 1.10 are optimal by Example 4.1.

Y. Chen informed us that together with Birkar, they also got the lower bound $\frac{1}{2}$ in Theorem 1.10 for toric morphisms between toric varieties in an earlier version of [10]. As a related result, when $\dim X - \dim Z = 2$, Mori and Prokhorov [32] showed that any 3-dimensional terminal del Pezzo fibration has no fibres of multiplicity > 6.

It turns out that Theorem 1.10 can be reduced to a local problem on estimating the lower bound of the log canonical threshold of a smooth curve with respect to a canonical pair on a smooth surface germ; see Corollary 3.12. Here, we prove a general result as it might have broader applications in other topics in birational geometry (compare [28, Corollary 6.46]).

Theorem 1.11. *Let $(X \ni P, B)$ be a germ of surface pair such that X is smooth and $\mathrm{mult}_P B \leq 1$. Let C be a smooth curve at P such that $C \not\subseteq \mathrm{Supp}(B)$. Set $\mathrm{mult}_P B = m$, $(B \cdot C)_P = I$. Then* $\mathrm{lct}(X \ni P, B; C) \geq \min\{1, 1 + \frac{m}{I} - m\}$.

Example 3.10 shows that the lower bound in Theorem 1.11 is optimal (even in the case when $\mathrm{Supp}\, B$ is irreducible). It would be interesting to get an optimal lower bound of $\mathrm{lct}(X \ni P, B; C)$ if we do not assume that C is smooth in Theorem 1.11, as it might be related to alpha invariants; see [23, Lemmas 3.1, 3.2] for an attempt in this direction.

It would also be interesting to ask the following question.

Question 1.12. *When $\dim X = 3$, can one give a complete local classification of the extremal case in Conjecture 1.1 when Z is strictly $\frac{1}{2}$-lc? Or, more generally, can one give a complete local classification in Conjecture 1.1 when Z is singular?*

(Sketch of proofs.). *By applying [35, Theorem 8.1], we may reduce Theorem 1.7 to Theorem 1.10. Here, the sub-pair setting plays a key role, since it makes this reduction step simpler than that of the pair setting*

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(compare [5, Lemma 3.4, Proposition 3.5]), and it enables us to treat the case $\mathrm{mld}(X/Z \ni z, B) > 1$*. On the other hand, the sub-pair setting causes new technical difficulties in the proof of Theorem 1.10. By taking hyperplane sections of the base Z, we may reduce Theorem 1.10 to the case* $\dim X = 2$*. By an MMP argument, we may reduce Theorem 1.10 to the case when* $X \to Z$ *is a* $\mathbb{P}^1$*-bundle and* $B \geq 0$*, so the problem is reduced to a special case of Theorem 1.11 when* $\mathrm{mult}_P B \leq 1$ *and* $(B \cdot C)_P \leq 2$*. Since the conditions* $\mathrm{mult}_P B \leq 1$ *and* $(B \cdot C)_P \leq 2$ *do not behave well under blow-ups, one may encounter difficulties in applying the ideas in [1, 11, 18], which deal with the minimal log discrepancies for surfaces. The key idea is to consider* $\widehat{X}$*, the completion of X along P, and decompose B into irreducible components on* $\widehat{X}$*. By using the log canonical threshold polytope and applying the convexity of log canonical thresholds carefully, we may reduce Theorem 1.11 to the case when* $\mathrm{Supp}(B)$ *is irreducible on* $\widehat{X}$*. Here, recall that* $\mathrm{lct}(X \ni P, B; C) = \mathrm{lct}(\widehat{X} \ni P, B; C)$*. Finally, for this last case, following the ideas in [29], we may give a lower bound of* $\mathrm{lct}(\widehat{X} \ni P, B; C)$ *by using the first pair of Puiseux exponents of B. The proof of Theorem 1.11 is provided in Section 3, and the proofs of other main results in this paper are provided in Section 4.*

2. Preliminaries

In this section, we collect basic definitions and results. We adopt the standard notation and definitions in [27] and [9]. Recall that we work over the complex number field.

2.1. Divisors

Let $\mathbb{K}$ be either the rational number field $\mathbb{Q}$ or the real number field $\mathbb{R}$. Let X be a normal variety. A $\mathbb{K}$*-divisor* is a finite $\mathbb{K}$-linear combination $D = \sum d_i D_i$ of prime Weil divisors D_i, and d_i denotes the *coefficient* of D_i in D. A $\mathbb{K}$*-Cartier divisor* is a $\mathbb{K}$-linear combination of Cartier divisors.

We use $\sim_{\mathbb{K}}$ to denote the $\mathbb{K}$-linear equivalence between $\mathbb{K}$-divisors. For a projective morphism $X \to Z$, we use $\sim_{\mathbb{K},Z}$ to denote the relative $\mathbb{K}$-linear equivalence and $\equiv_Z$ to denote the relative numerical equivalence.

Definition 2.1 (compare [35]). Let X be a normal variety. Consider an infinite linear combination $\mathbf{D} := \sum_D d_D D$, where $d_D \in \mathbb{K}$ and the infinite sum runs over all divisorial valuations of the function field of X. For any birational model Y of X, the *trace* of $\mathbf{D}$ on Y is defined by $\mathbf{D}_Y := \sum_{\mathrm{codim}_Y D=1} d_D D$. Such a $\mathbf{D}$ is called a *b-*$\mathbb{K}$*-divisor* (or *b-divisor* for short when the base field is clear) if on each birational model Y of X, the trace $\mathbf{D}_Y$ is a $\mathbb{K}$-divisor, or equivalently, $\mathbf{D}_Y$ is a finite sum. If $d_D \neq 0$ in $\mathbf{D}$ for some D, D is called a *birational component* of $\mathbf{D}$.

Let D be a $\mathbb{K}$-Cartier divisor on X. The *Cartier closure* of D is the b-$\mathbb{K}$-divisor $\overline{D}$ whose trace on every birational model $f : Y \to X$ is f^*D.

A b-$\mathbb{K}$-divisor $\mathbf{D}$ is said to be *b-semi-ample* if there is a birational model X' over X such that $\mathbf{D}_{X'}$ is $\mathbb{K}$-Cartier and semi-ample, and $\mathbf{D} = \overline{\mathbf{D}_{X'}}$.

2.2. Pairs and singularities

Definition 2.2. Let $\pi : X \to Z$ be a morphism between varieties. We say that $\pi : X \to Z$ is a *contraction* if π is projective and $\pi_*\mathcal{O}_X = \mathcal{O}_Z$. In particular, π is surjective and has connected fibres.

Definition 2.3. Let $\pi : X \to Z$ be a contraction between normal varieties. For a prime divisor E on X, E is said to be *horizontal* over Z if E dominates Z, and E is said to be *vertical* over Z if E does not dominate Z. An $\mathbb{R}$-divisor on X is said to be *vertical* over Z if all its irreducible components are vertical over Z.

Definition 2.4 (compare [11, Definition 3.2]). A *sub-pair* (X, B) consists of a normal variety X and an $\mathbb{R}$-divisor B on X such that $K_X + B$ is $\mathbb{R}$-Cartier. We say that (X, B) is a *pair* if (X, B) is a sub-pair and B is effective.

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A *(relative) sub-pair* $(X/Z \ni z, B)$ consists of normal varieties X, Z, a contraction $\pi : X \to Z$, a scheme-theoretic point $z \in Z$ and an $\mathbb{R}$-divisor B on X such that $K_X + B$ is $\mathbb{R}$-Cartier over an open neighbourhood of z and $\dim z < \dim X$. We say that $(X/Z \ni z, B)$ is a *(relative) pair* if $(X/Z \ni z, B)$ is a sub-pair and B is effective. We say that a pair $(X/Z \ni z, B)$ is a *germ* near z if z is a closed point. When $Z = X$, $z = x$ and π is the identity map, we will use $(X \ni x, B)$ instead of $(X/Z \ni z, B)$ for simplicity. When $B = 0$, we will use X or $X/Z \ni z$ instead of $(X, 0)$ or $(X/Z \ni z, 0)$ for simplicity.

Definition 2.5. Let $(X/Z \ni z, B)$ be a sub-pair with contraction $\pi : X \to Z$ and E a prime divisor over X. Let $\phi : Y \to X$ be a proper birational morphism such that E is a divisor on Y and write $K_Y + B_Y = \phi^*(K_X + B)$. The *log discrepancy* of E with respect to (X, B) is defined to be $a(E, X, B) := 1 - \operatorname{mult}_E B_Y$, which is independent of the choice of Y.

Set

$$\mathfrak{D}(X/Z \ni z) := \{E \mid E \text{ is a prime divisor over } X, \pi(\operatorname{center}_X(E)) = \overline{z}\}.$$

The *minimal log discrepancy* of $(X/Z \ni z, B)$ is defined to be

$$\operatorname{mld}(X/Z \ni z, B) := \inf\{a(E, X, B) \mid E \in \mathfrak{D}(X/Z \ni z)\}.$$

By [11, Lemma 3.5], the infimum is a minimum if $(X/Z \ni z, B)$ is an lc sub-pair, and it can be computed on a log resolution $\phi : Y \to (X, B)$, where $\operatorname{Supp}(\phi^{-1}(\pi^{-1}(\overline{z}))) + \phi_*^{-1} \operatorname{Supp}(B) + \operatorname{Exc}(\phi)$ is a simple normal crossing divisor.

When $X = Z$, $z = x$, and π is the identity map, we use $\operatorname{mld}(X \ni x, B)$ instead of $\operatorname{mld}(X/Z \ni z, B)$ for simplicity.

Example 2.6. We emphasise that our definition of $\operatorname{mld}(X/Z \ni z, B)$ requires that $\pi(\operatorname{center}_X(E)) = \overline{z}$, so it only reflects the singularities in a neighbourhood of z. For example, if $X = Z = \mathbb{P}^2$, $B = L_1 + L_2$, $B' = \frac{1}{2}L_1 + 2L_2$, where L_1, L_2 are 2 distinct lines. Then

$$\operatorname{mld}(X/Z \ni z, B) = \begin{cases} 2 & \text{if } z \notin L_1 \cup L_2; \\ 1 & \text{if } z \in L_1 \cup L_2 \setminus L_1 \cap L_2; \\ 0 & \text{if } z \in L_1 \cap L_2, \end{cases}$$

and

$$\operatorname{mld}(X/Z \ni z, B') = \begin{cases} 2 & \text{if } z \notin L_1 \cup L_2; \\ \frac{3}{2} & \text{if } z \in L_1 \setminus L_1 \cap L_2; \\ -\infty & \text{if } z \in L_2. \end{cases}$$

Definition 2.7. Fix a non-negative real number ϵ. We say that the sub-pair $(X/Z \ni z, B)$ is *ϵ-lc* (respectively, *ϵ-klt*, *klt*, *lc*) if $\operatorname{mld}(X/Z \ni z, B) \geq \epsilon$ (respectively, $> \epsilon$, > 0, ≥ 0).

We say that (X, B) is ϵ-lc (respectively, ϵ-klt, klt, lc) if $(X \ni x, B)$ is so for any codimension ≥ 1 point $x \in X$; we say that (X, B) is *canonical* (respectively, *terminal*) if $a(E, X, B) \geq 1$ (respectively, $a(E, X, B) > 1$) for any exceptional prime divisor E over X. These coincide with the usual definitions (compare [27, Definition 2.34]).

The following lemma is well-known to experts, saying that being lc over $z \in Z$ is an open condition.

Lemma 2.8. *Let $(X/Z \ni z, B)$ be a sub-pair with contraction $\pi : X \to Z$, and fix a log resolution $f : Y \to (X, B)$ such that $f_*^{-1} \operatorname{Supp} B + f^{-1}\pi^{-1}(\overline{z})$ is a simple normal crossing divisor, and write $K_Y + B_Y = f^*(K_X + B)$. The following are equivalent:*

1. *$(X/Z \ni z, B)$ is lc.*
2. *For any prime divisor E on Y with $\pi(f(E)) \ni z$, $\operatorname{mult}_E B_Y \leq 1$.*
3. *There exists an open neighbourhood U of $z \in Z$ such that $(\pi^{-1}(U), B|_{\pi^{-1}(U)})$ is lc.*

Proof. By definition, (3) implies (2). By direct computations ([27, Corollary 2.31]), if (2) holds for the given log resolution Y, it holds for any log resolution. Thus (2) implies (3). It is obvious that (3) implies (1). It suffices to show that (1) implies (2).

Suppose that sub-pair $(X/Z \ni z, B)$ is lc. Assume to the contrary that there exists a prime divisor E such that $\operatorname{mult}_E B_Y > 1$ and $E \cap f^{-1}\pi^{-1}(z) \neq \emptyset$. Then by successively blowing up some components of the closure of $E \cap f^{-1}\pi^{-1}(z)$ several times, we can replace Y by a higher model such that there exists a prime divisor E' on Y with $\pi(f(E')) = \bar{z}$ and $\operatorname{mult}_{E'} B_Y > 1$ (compare [27, Corollary 2.31]), a contradiction. To be more precise, suppose that there exists a prime divisor D in $f^{-1}\pi^{-1}(\bar{z})$, which maps to $\bar{z}$ such that $D \cap E \neq \emptyset$, and suppose $\operatorname{mult}_D B_Y = d$ and $\operatorname{mult}_E B_Y = e > 1$; then we can blow up $D \cap E$ to get a new log resolution $f' : Y' \to X$ with $K_{Y'} + B_{Y'} = f'^*(K_X + B)$ such that there exists a prime divisor in $D' \cap f'^{-1}\pi^{-1}(\bar{z})$, which maps to $\bar{z}$ with $\operatorname{mult}_{D'} B_{Y'} = d + e - 1$. So, inductively, we can replace Y by a higher model to find the required E'. □

Definition 2.9. A *non-klt place* of a sub-pair (X, B) (respectively, $(X/Z \ni z, B)$) is a prime divisor E over X (respectively, $E \in \mathfrak{D}(X/Z \ni z)$) such that $a(E, X, B) \leq 0$, and a *non-klt center* is the center of a non-klt place on X.

2.3. *Log canonical thresholds*

Definition 2.10. Let $(X/Z \ni z, B)$ be an lc sub-pair with contraction $\pi : X \to Z$, and let $D \neq 0$ be an effective $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X such that $z \in \pi(\operatorname{Supp}(D))$. The *log canonical threshold* of D with respect to $(X/Z \ni z, B)$ is

$$\operatorname{lct}(X/Z \ni z, B; D) := \sup\{t \in \mathbb{R} \mid (X/Z \ni z, B + tD) \text{ is lc}\}.$$

When $z \in Z$ is a codimension 1 point, we may assume that $\bar{z}$ is a Cartier divisor on a neighbourhood U of $z \in Z$. Then we define

$$\operatorname{lct}(X/Z \ni z, B; \pi^*\bar{z}) := \sup\{t \in \mathbb{R} \mid (X/Z \ni z, B + t\pi^*\bar{z}) \text{ is lc over } U\},$$

and this definition does not depend on the choice of the neighbourhood U of $z \in Z$.

We may write $\operatorname{lct}(X/Z \ni z; D) := \operatorname{lct}(X/Z \ni z, 0; D)$ when $B = 0$. When $X = Z$, $z = x$ and π is the identity map, we may write $\operatorname{lct}(X \ni x, B; D) := \operatorname{lct}(X/Z \ni z, B; D)$.

Remark 2.11 ([26]). Keep the same setting as in Definition 2.10. Log canonical thresholds can be computed by a log resolution. In fact, take $g : X' \to X$ to be a log resolution of $(X, B + D)$, and write $K_{X'} + B' = g^*(K_X + B)$. Then

$$\operatorname{lct}(X/Z \ni z, B; D) = \min_E \frac{1 - \operatorname{mult}_E(B')}{\operatorname{mult}_E g^*D},$$

where the minimum runs over all prime divisors $E \subseteq \operatorname{Supp}(g^*D)$ such that $\pi(g(E)) \ni z$ (compare Lemma 2.8(3)).

2.4. *Canonical bundle formula*

The *discrepancy b-divisor* $\mathbf{A} = \mathbf{A}(X, B)$ of a sub-pair (X, B) is the b-divisor of X with the trace $\mathbf{A}_Y$ defined by the formula

$$\mathbf{A}_Y = K_Y - f^*(K_X + B),$$

for any proper birational morphism $f : Y \to X$ between normal varieties. Similarly, we define $\mathbf{A}^* = \mathbf{A}^*(X, B)$ by $\mathbf{A}^*_Y = \sum_{a_i > -1} a_i E_i$ for any proper birational morphism $f : Y \to X$ between normal

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varieties, where $\mathbf{A}_Y = \sum a_i E_i$. Note that $\mathbf{A}^*(X, B) = \mathbf{A}(X, B)$ if and only if (X, B) is klt. See [14, 2.3] for more details.

Definition 2.12 (compare [14, Definition 3.2], [20, Definition 3.1]). An *lc-trivial fibration* $\pi : (X, B) \to Z$ consists of a contraction $\pi : X \to Z$ between normal varieties and a sub-pair (X, B) satisfying the following properties:

1. (X, B) is lc over the generic point of Z,
2. $\operatorname{rank} \pi_* \mathcal{O}_X(\lceil \mathbf{A}^*(X, B) \rceil) = 1$, and
3. there exists an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor L on Z such that $K_X + B \sim_{\mathbb{R}} \pi^* L$.

Remark 2.13. Here, we discuss more details about the condition in (2). If B is effective on the generic fibre of π, then $\mathcal{O}_X(\lceil \mathbf{A}^*(X, B) \rceil) = \mathcal{O}_X$ over the generic point of Z, so in this case the condition in (2) holds. Conversely, if the geometric generic fibre of π is a rational curve, then $\operatorname{rank} \pi_* \mathcal{O}_X(\lceil \mathbf{A}^*(X, B) \rceil) = 1$ implies that B is effective on the generic fibre of π.

Let $\pi : (X, B) \to Z$ be an lc-trivial fibration. Then we may write $K_X + B \sim_{\mathbb{R}} \pi^* L$ for some $\mathbb{R}$-Cartier $\mathbb{R}$-divisor L. By the work of Kawamata [24, 25] and Ambro [3] (see [20, Section 3] for $\mathbb{R}$-divisors), we have the so-called *canonical bundle formula*

$$K_X + B \sim_{\mathbb{R}} \pi^*(K_Z + B_Z + M_Z),$$

where B_Z is defined by

$$B_Z := \sum_P (1 - \operatorname{lct}(X/Z \ni \eta_P, B; \pi^* P))P \tag{2.1}$$

and

$$M_Z := L - K_Z - B_Z. \tag{2.2}$$

Here, the sum runs over all prime divisors P on Z, η_P is the generic point of P and it is known that it is a finite sum. So B_Z is uniquely determined by (X, B), and M_Z is determined up to $\mathbb{R}$-linear equivalence. Here, B_Z is called the *discriminant part* and M_Z is called the *moduli part* of the canonical bundle formula. Recall that if B is effective, then B_Z is also effective.

In the following, we suppose that B is a $\mathbb{Q}$-divisor for simplicity. In fact, the canonical bundle formula satisfies certain functorial properties as follows. By [35, Remark 7.7] or [14, 3.4], there are b-divisors $\mathbf{B}$ and $\mathbf{M}$ of Z such that

- $\mathbf{B}_Z = B_Z$, $\mathbf{M}_Z = M_Z$, and
- for any birational contraction $g : Z' \to Z$, let X' be a resolution of the main component of $X \times_Z Z'$ with induced morphisms $g' : X' \to X$ and $\pi' : X' \to Z'$. Let $K_{X'} + B'$ be the *crepant pull back* of $K_X + B$—that is, $K_{X'} + B' = g'^*(K_X + B)$—then $\mathbf{B}_{Z'}$ (respectively, $\mathbf{M}_{Z'}$) is the discriminant part (respectively, the moduli part) of the canonical bundle formula of $K_{X'} + B'$ on Z' defined by equations (2.1) and (2.2):

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "\pi'"'] & X \arrow[d, "\pi"] \\ Z' \arrow[r, "g"'] & Z. \end{tikzcd}$$

The effective adjunction conjecture ([35, Conjecture 7.13]) predicts that $\mathbf{M}$ is b-semi-ample. It was confirmed in the case of relative dimension 1.

Theorem 2.14 ([35, Theorem 8.1]). *Keep the notation in this subsection. Assume that B is a $\mathbb{Q}$-divisor. If* $\dim X - \dim Z = 1$ *and the geometric generic fibre of π is a rational curve, then* $\mathbf{M}$ *is b-semi-ample.*

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Remark 2.15. Note that [35, Theorem 8.1] holds for an lc-trivial fibration $\pi : (X, B) \to Z$ under two additional assumptions:

1. B is effective over the generic point of Z [35, Assumption 7.1], and
2. there exists a $\mathbb{Q}$-divisor Θ on X such that $K_X + \Theta \sim_{\mathbb{Q},Z} 0$ and (X, Θ) is klt over the generic point of Z [35, Assumption 7.11].

Here, (i) is automatically satisfied by Remark 2.13. Also, (ii) is automatically satisfied by the following. Since the generic fibre X_η of π is a rational curve, we can find an effective $\mathbb{Q}$-divisor D_η on X_η such that $K_{X_\eta} + D_\eta \sim_{\mathbb{Q}} 0$ and (X_η, D_η) is klt. Denote D to be the closure of D_η on X; then $K_X + D \sim_{\mathbb{Q}} E$, where E is vertical over Z. Then we just take $\Theta = D - E$.

2.5. Contractions of Fano type

Definition 2.16 ([35]). Let $\pi : X \to Z$ be a contraction between normal varieties. We say that X is *of Fano type* over Z if one of the following equivalent conditions holds:

1. there exists a klt pair (X, B) such that $-(K_X + B)$ is ample over Z;
2. there exists a klt pair (X, B') such that $-(K_X + B')$ is nef and big over Z;
3. there exists a klt pair (X, B'') such that $K_X + B'' \equiv_Z 0$ and B'' is big over Z.

When Z is a point, we just say that X is of Fano type.

2.6. Formal surface germs

Let P be a smooth closed point on a surface X. By the Cohen structure theorem, $\widehat{\mathcal{O}}_{X,P} \cong \widehat{\mathcal{O}}_{\mathbb{C}^2,o} = \mathbb{C}[[x, y]]$. Denote by $\widehat{X}_P$ the completion of X along P. We will use $\widehat{X}$ instead of $\widehat{X}_P$ if P is clear from the context.

We call C a *Cartier divisor* on $\widehat{X}$ if C is defined by $(g = 0)$ for some $g \in \widehat{\mathcal{O}}_{X,P}$. We call B an $\mathbb{R}$*-divisor* (respectively, a $\mathbb{Q}$*-divisor*) on $\widehat{X}$ if $B = \sum_i b_i B_i$ for some Cartier divisors B_i on $\widehat{X}$ and $b_i \in \mathbb{R}$ (respectively $b_i \in \mathbb{Q}$).

Since the resolution of singularities is known for complete local rings ([39]), the definition of singularities of pairs and log canonical thresholds can be extended to the formal case (see [26] and [12]).

Definition 2.17. Let $(\widehat{X} \ni P, B = \sum_i b_i B_i)$ be an lc pair where $P \in X$ is a smooth formal surface germ and B_i is defined by $(f_i = 0)$ for some $f_i \in \widehat{\mathcal{O}}_{X,x}$. Let $C = \sum_i c_i C_i \neq 0$ be an effective $\mathbb{R}$-divisor, where C_i is defined by $(g_i = 0)$ for some $g_i \in \widehat{\mathcal{O}}_{X,x}$. Let $\phi : \widehat{Y} \to (\widehat{X}, B + C)$ be a log resolution ([39]); then

$$\operatorname{lct}(\widehat{X} \ni P, B; C) := \min_E \frac{1 + \operatorname{mult}_E K_{\widehat{Y}/\widehat{X}} - \sum_i b_i \operatorname{mult}_E(f_i)}{\sum_i c_i \operatorname{mult}_E(g_i)}, \tag{2.3}$$

where the minimum runs over all prime divisors E in $\operatorname{Supp}(\phi^* C)$ such that $P \in \phi(E)$. The definition does not depend on the choice of log resolutions. Here, mult_E means the vanishing order of a function along E or the coefficient of E in a divisor.

Remark 2.18. Let $(X \ni P, B)$ be a germ of an lc surface pair such that $P \in X$ is smooth, and let C be an effective $\mathbb{R}$-divisor near P. Consider $\widehat{X}$ (respectively B', C'), the completion of X (respectively B, C) along P. Since a log resolution of $(X \ni P, B + C)$ also gives a log resolution of $(\widehat{X}, B' + C')$, $\operatorname{lct}(\widehat{X} \ni P, B'; C') = \operatorname{lct}(X \ni P, B; C)$. In other words, in order to study the log canonical threshold of a smooth surface germ $(X \ni P, B)$, it is equivalent to study that of the corresponding smooth formal surface germ $(\widehat{X} \ni P, B')$.

Recall that log canonical thresholds satisfy convexity with respect to the coefficients.

Lemma 2.19 (compare [16, Lemma 3.8]). *Let $P \in X$ be a smooth surface germ or a smooth formal surface germ. Let $(X \ni P, B_i)$ be an lc pair for $1 \le i \le m$, $C \neq 0$ an effective $\mathbb{R}$-divisor on X and λ_i non-negative real numbers such that $\sum_{i=1}^m \lambda_i = 1$. Then*

$$\mathrm{lct}(X \ni P, \sum_{i=1}^m \lambda_i B_i; C) \ge \sum_{i=1}^m \lambda_i \,\mathrm{lct}(X \ni P, B_i; C).$$

3. Log canonical thresholds on a smooth surface germ

In this section, we study the lower bounds of log canonical thresholds on a smooth surface germ. The main goal of this section is to prove Theorem 1.11.

Recall the following result on computing log canonical thresholds of hypersurfaces.

Proposition 3.1 ([29, Proposition 2.1]). *Let B be a Cartier divisor in a neighbourhood of $o \in \widehat{\mathbb{C}^n}$ defined by $(f = 0)$, where $f \in \mathbb{C}[[x_1, \dots, x_n]]$. Assign rational weights $w(x_i)$ to the variables, and let $w(f)$ be the weighted multiplicity of f. Let*

$$f_w := \{\textit{sum of monomial terms appearing in } f \textit{ whose } w\textit{-weight are equal to } w(f)\}$$

denote the weighted homogeneous leading term of f. Take $b = \frac{\sum_{i=1}^n w(x_i)}{w(f)}$. If $(\widehat{\mathbb{C}^n}, b \cdot (f_w = 0))$ is lc outside o, then $\mathrm{lct}(\widehat{\mathbb{C}^n} \ni o; B) = b$.

To warm up, the following proposition is an application of Proposition 3.1.

Proposition 3.2. *Let B be a Cartier divisor in a neighbourhood of $o \in \widehat{\mathbb{C}^2}$ defined by $(f = 0)$, where $f = x^n(x^{m_1} + y^{m_2})^k$ for some positive integers k, n, m_1, m_2. Then*

$$\mathrm{lct}(\widehat{\mathbb{C}^2} \ni o; B) = \min\left\{\frac{m_1 + m_2}{km_1m_2 + nm_2}, \frac{1}{n}, \frac{1}{k}\right\}.$$

Proof. Consider C_1 defined by $(x = 0)$ and C_2 defined by $(x^{m_1} + y^{m_2} = 0)$; then $(C_1 \cdot C_2)_o = m_2$. Consider the weight $w = (m_2, m_1)$; then $f_w = f$ and $b = \frac{m_1+m_2}{km_1m_2+nm_2}$ as in Proposition 3.1.

If $b \le \min\{\frac{1}{n}, \frac{1}{k}\}$, then $(\widehat{\mathbb{C}^2}, b \cdot (f_w = 0))$ is lc outside o, and hence $\mathrm{lct}(\widehat{\mathbb{C}^2} \ni o; B) = b$ by Proposition 3.1. If $b > \frac{1}{n}$, then $n > km_2$. Then [27, Corollary 5.57] implies that $(\widehat{\mathbb{C}^2} \ni o, C_1 + \frac{k}{n}C_2)$ is lc. If $b > \frac{1}{k}$, then either $m_1 = 1$ or $m_2 = 1$. In either case, C_2 is smooth and $k > nm_2$. Then [27, Corollary 5.57] implies that $(\widehat{\mathbb{C}^2} \ni o, \frac{n}{k}C_1 + C_2)$ is lc. □

Definition 3.3 (compare [29, Definition 2.10]). Let $B = (f = 0)$ be an irreducible curve in a neighbourhood of $o \in \widehat{\mathbb{C}^2}$. If B is smooth, then we set $m = 1$ and $n = \infty$. Otherwise, the Puiseux expansion of B (under suitable local parameters x, y) is expressed as $x = t^m, y = \sum_{i=n}^\infty \alpha_i t^i$ for some local parameter t, where $m, n \in \mathbb{Z}_{\ge 2}$, $m < n$, and m does not divide n. Here, (m, n) is called the *first pair of Puiseux exponents of f*. Note that $m = \mathrm{mult}_o f$ is the multiplicity of f at $o \in \widehat{\mathbb{C}^2}$.

Example 3.4. If $n > m > 1$ and m, n are coprime, then the first pair of Puiseux exponents of $f = x^m + y^n$ is just (m, n).

The close relation between the first pair of Puiseux exponents and log canonical thresholds can be illustrated by the following result.

Theorem 3.5 ([29, Theorem 1.3]). *Let B be a Cartier divisor in a neighbourhood of $o \in \widehat{\mathbb{C}^2}$ defined by $(f = 0)$, where $f \in \mathbb{C}[[x, y]]$. Write $f = \prod_{j=1}^r f_j^{\alpha_j}$, where each f_j is irreducible. Write $B = \sum_j \alpha_j B_j$,*

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where B_j is defined by $(f_j = 0)$. Then $\mathrm{lct}(\widehat{\mathbb{C}^2} \ni o; B)$ *depends only on the first pairs of Puiseux exponents of f_j, $(B_i \cdot B_j)_o$, and α_j.*

Following the ideas in [29, Theorem 1.2], we have the following.

Proposition 3.6. *Let B be a Cartier divisor in a neighbourhood of $o \in \widehat{\mathbb{C}^2}$ defined by $(f = 0)$, where $f \in \mathbb{C}[[x, y]]$. Suppose that f is irreducible. Let* $\mathrm{mult}_o f = m$*, and let (m, n) be the first pair of Puiseux exponents of f. Let $C \neq B$ be a smooth curve passing through o and $(B \cdot C)_o = I$. Then for every positive real numbers s, t,*

$$\mathrm{lct}(\widehat{\mathbb{C}^2} \ni o; sB + tC) = \min\left\{\frac{m+n}{smn+tI}, \frac{m+I}{(sm+t)I}, \frac{1}{s}, \frac{1}{t}\right\}.$$

Remark 3.7. 1. By convention, if $(m, n) = (1, \infty)$, we set $\frac{1+\infty}{s\cdot\infty+tI} := \frac{1}{s}$.
2. In the case that $s = t = 1$, Proposition 3.6 is a special case of [29, Theorem 1.2]. We also remark that Proposition 3.6 might be indicated by more general results in [15], but the formulation there is complicated, and we give a simple proof in this special case for the reader's convenience.
3. Recall that under the setting of Proposition 3.6, by [29, Proof of Theorem 1.2, Case 2, Page 711–712],

$$I \in \left\{m, 2m, \dots, \left\lfloor \frac{n}{m} \right\rfloor m, n\right\}.$$

Proof. Set

$$c := \min\left\{\frac{m+n}{smn+tI}, \frac{m+I}{(sm+t)I}, \frac{1}{s}, \frac{1}{t}\right\}.$$

As being lc is a closed condition on coefficients, we may assume that $s, t \in \mathbb{Q}$. Possibly replacing s, t by a multiple, we may assume that s, t are integers.

If $m = 1$, then by Theorem 3.5, we may assume that $sB + tC$ is defined by $(x^s(x + y^I)^t = 0)$. Then the proposition follows from Proposition 3.2. In the following, we may assume that $m > 1$, and in particular, B is singular at o.

Suppose that $\frac{1}{s} \leq \frac{m+n}{smn+tI}$; then we have $m = 1$ (recall that $n > 1$), which is absurd.

Suppose that $\frac{1}{t} \leq \frac{m+I}{(sm+t)I}$; then $sI \leq t$. Then [27, Corollary 5.57] implies that $(\widehat{\mathbb{C}^2} \ni o, \frac{s}{t}B + C)$ is lc. Since $n \geq I \geq m$, we have $\frac{m+n}{smn+tI} \geq \frac{1}{t}$, and hence $\frac{1}{t} = c$.

So from now on, we may assume that

$$\frac{1}{s} > \frac{m+n}{smn+tI} \quad \text{and} \quad \frac{1}{t} > \frac{m+I}{(sm+t)I}, \tag{3.1}$$

in particular,

$$c = \min\left\{\frac{m+n}{smn+tI}, \frac{m+I}{(sm+t)I}\right\}.$$

If $I = n$, then by Theorem 3.5, we may assume that $sB + tC$ is defined by $((x^m + y^n)^s x^t = 0)$. Then by Proposition 3.2,

$$\mathrm{lct}(\widehat{\mathbb{C}^2} \ni o; sB + tC) = \min\left\{\frac{m+n}{smn+tn}, \frac{1}{s}, \frac{1}{t}\right\} = c.$$

If $I = pm$ for some $1 \leq p \leq \lfloor \frac{n}{m} \rfloor$, then by Theorem 3.5, we may assume that $sB + tC$ is defined by $(h = 0)$, where $h = (x^m + y^n)^s(x + y^p)^t$.

If $tp \le sm$, consider the weight $w = (n, m)$; then $h_w = y^{pt}(x^m + y^n)^s$ and $b = \frac{m+n}{smn+tI}$, as defined in Proposition 3.1. Moreover, $(\widehat{\mathbb{C}^2}, bh_w)$ is lc outside o as $b \le \frac{1}{pt}$ by $tp \le sm$ and $b < \frac{1}{s}$ by equation (3.1). Hence, by Proposition 3.1,

$$\operatorname{lct}(\widehat{\mathbb{C}^2} \ni o; sB + tC) = \frac{m+n}{smn+tI} = c.$$

If $tp > sm$, consider the weight $w' = (p, 1)$; then $h_{w'} = x^{ms}(x + y^p)^t$ and $b' = \frac{1+p}{(sm+t)p} = \frac{m+I}{(sm+t)I}$, as defined in Proposition 3.1. Moreover, $(\widehat{\mathbb{C}^2}, b'h_{w'})$ is lc outside o as $b' < \frac{1}{ms}$ by $tp > sm$ and $b' < \frac{1}{t}$ by equation (3.1). Hence, by Proposition 3.1,

$$\operatorname{lct}(\widehat{\mathbb{C}^2} \ni o; sB + tC) = \frac{m+I}{(sm+t)I} = c. \qquad \square$$

Corollary 3.8. *Let B be a Cartier divisor in a neighbourhood of $o \in \widehat{\mathbb{C}^2}$ defined by $(f = 0)$, where $f \in \mathbb{C}[[x, y]]$. Suppose that f is irreducible, $\operatorname{mult}_o f = m$, and let (m, n) be the first pair of Puiseux exponents of f. Let $C \ne B$ be a smooth curve passing through o, and $(B \cdot C)_o = I$. Let λ be a positive real number. Suppose that one of the following conditions holds:*

1. *$\lambda m \le 1$;*
2. *$n = I$ and $\lambda \le \min\{1, \frac{1}{m} + \frac{1}{I}\}$; or*
3. *$I \ne m$ and $\lambda I \le 2$.*

Then $(\widehat{\mathbb{C}^2} \ni o, \lambda B)$ is lc and

$$\operatorname{lct}(\widehat{\mathbb{C}^2} \ni o, \lambda B; C) \ge \min\Big\{1, 1 + \frac{m}{I} - \lambda m\Big\}.$$

Proof. First we remark that if (c) holds, then (a) holds. In fact, suppose that $\lambda I \le 2$ and $\lambda m > 1$; then $I < 2m$. Then by Remark 3.7(3), $I = m$, so we get a contradiction by assumption (c). So in the following, we only assume that (a) or (b) holds.

Here, note that under condition (a), $\lambda \le \min\{1, \frac{1}{m} + \frac{1}{I}\}$ automatically holds. So we always have $\lambda \le \min\{1, \frac{1}{I} + \frac{1}{m}\}$.

Set $t := \min\{1, 1 + \frac{m}{I} - \lambda m\} \ge 0$. The statement is equivalent to $\operatorname{lct}(\widehat{\mathbb{C}^2} \ni o; \lambda B + tC) \ge 1$. By Proposition 3.6, this is equivalent to showing that

1. $\frac{m+n}{\lambda mn+tI} \ge 1$,
2. $m + I \ge (\lambda m + t)I$,
3. $1 \ge \lambda$, and
4. $1 \ge t$.

Here, (2) and (4) follow from the definition of t, and (3) follows from $\lambda \le \min\{1, \frac{1}{I} + \frac{1}{m}\}$. It is enough to show (1).

If $m = 1$, then $n = \infty$, so (1) is equivalent to $\lambda = \lambda m \le 1$ under the convention in Lemma 3.7(1), which is already proved in (3).

In the following, we assume that $m \ge 2$. It suffices to prove that

$$m + n \ge \lambda mn + \Big(1 + \frac{m}{I} - \lambda m\Big)I,$$

which is equivalent to $(n - I)(1 - \lambda m) \ge 0$. Recall that $n \ge I$, so (1) holds if either $n = I$ or $\lambda m \le 1$ holds. This proves the conclusion for (a) and (b). $\square$

Remark 3.9. In applications, we only use Corollary 3.8 when condition (a) holds. The advantage of this corollary is that we can get rid of n in the first pair of Puiseux exponents of f, and the log canonical threshold can be estimated by only m and I. In practice, n is usually hard to control, while m and I can be controlled easily by geometric conditions.

The following example shows that both Theorem 1.11 and Corollary 3.8 are optimal.

Example 3.10. Consider two coprime positive integers m and I such that $m < I$. Take a positive real number λ such that $\lambda m \le 1 \le \lambda I$. Consider $(\mathbb{C}^2, \lambda B)$, where $B = (x^m + y^I = 0)$ and $C = (x = 0)$. Then $\operatorname{mult}_o \lambda B = \lambda m$, $(\lambda B \cdot C)_o = \lambda I$. A direct computation by Proposition 3.6 shows that $(\mathbb{C}^2 \ni o, \lambda B + (1 + \frac{m}{I} - \lambda m)C)$ is lc but $(\mathbb{C}^2 \ni o, \lambda B + (1 + \frac{m}{I} - \lambda m + \epsilon)C)$ is not lc for any $\epsilon > 0$. So in this case,

$$\operatorname{lct}(\mathbb{C}^2 \ni o, \lambda B; C) = 1 + \frac{m}{I} - \lambda m.$$

Now we may show Theorem 1.11, which could be regarded as an $\mathbb{R}$-divisor version of Corollary 3.8.

Proof of Theorem 1.11. Possibly approximating $\mathbb{R}$-coefficients with $\mathbb{Q}$-coefficients, we may assume that B is a $\mathbb{Q}$-divisor. Recall that $(B \cdot C)_P = I$ and $\operatorname{mult}_P B = m$.

If $I \le 1$, then $(X \ni P, B + C)$ is lc by [27, Corollary 5.57]. Hence we may assume that $I > 1$.

We may replace $P \in X$ by the formal neighbourhood $\widehat{X}$ of $P \in X$, which is isomorphic to the formal neighbourhood $o \in \widehat{\mathbb{C}^2}$. So from now on we may assume that $P \in X$ is just $o \in \widehat{\mathbb{C}^2}$. Write $B = \sum_{i=1}^n b_i B_i$, where $b_i \in (0, 1]$, and $\{B_i\}_{1 \le i \le n}$ are distinct irreducible curves on $\widehat{\mathbb{C}^2}$ passing through o.

If $n = 1$, then we are done by Corollary 3.8. So we may assume that $n \ge 2$.

Set $s := 1 + \frac{m}{I} - m$. The goal is to show that $(\widehat{\mathbb{C}^2} \ni o, B + sC)$ is lc. Consider the *log canonical threshold polytope* of the pair $(\widehat{\mathbb{C}^2} \ni o, sC)$ with respect to the divisors $B_1, \ldots, B_n$,

$$P(\widehat{\mathbb{C}^2} \ni o, sC; B_1, \ldots, B_n) := \left\{ (t_1, \ldots, t_n) \in \mathbb{R}^n_{\ge 0} \,\middle|\, \left(\widehat{\mathbb{C}^2} \ni o, sC + \sum_{i=1}^n t_i B_i \right) \text{ is lc} \right\}.$$

By Lemma 2.19, $P(\widehat{\mathbb{C}^2} \ni o, sC; B_1, \ldots, B_n)$ is a compact convex polytope in $\mathbb{R}^n$. It suffices to show that the convex polytope

$$\mathcal{P} := \left\{ (t_1, \ldots, t_n) \in \mathbb{R}^n_{\ge 0} \,\middle|\, \operatorname{mult}_o \sum_{i=1}^n t_i B_i = m, \; \sum_{i=1}^n t_i (B_i \cdot C)_o = I \right\}$$

is contained in $P(\widehat{\mathbb{C}^2} \ni o, sC; B_1, \ldots, B_n)$, here we remark that $\mathcal{P} \neq \emptyset$. By Lemma 3.11, all the vertices of $\mathcal{P}$ are contained in $\bigcup_{i \neq j} E_{i,j}$, where $E_{i,j} := \{(t_1, \cdots, t_n) \mid t_k = 0 \text{ for } k \neq i, j\}$. Hence it suffices to show that

$$E_{i,j} \cap \mathcal{P} \subseteq E_{i,j} \cap P(\widehat{\mathbb{C}^2} \ni o, sC; B_1, \ldots, B_n) \simeq P(\widehat{\mathbb{C}^2} \ni o, sC; B_i, B_j)$$

for all $1 \le i < j \le n$.

Without loss of generality, we may just consider the case $(i, j) = (1, 2)$. It suffices to show that any vertex point of $E_{1,2} \cap \mathcal{P}$ is contained in $P(\widehat{\mathbb{C}^2} \ni o, sC; B_1, B_2)$, where $E_{1,2}$ is identified with $\mathbb{R}^2$. Set $\operatorname{mult}_o B_i = m_i$, $(B_i \cdot C)_o = I_i \ge 1$ for $i = 1, 2$. Take (c_1, c_2) to be a vertex point of $E_{1,2} \cap \mathcal{P}$; then (c_1, c_2) satisfies the following equations:

$$m_1 c_1 + m_2 c_2 = m, \quad I_1 c_1 + I_2 c_2 = I. \tag{3.2}$$

Here, we recall that m_1, m_2, I_1, I_2 are positive integers, $m_1 \le I_1$, $m_2 \le I_2$ and $m \le 1 < I$.

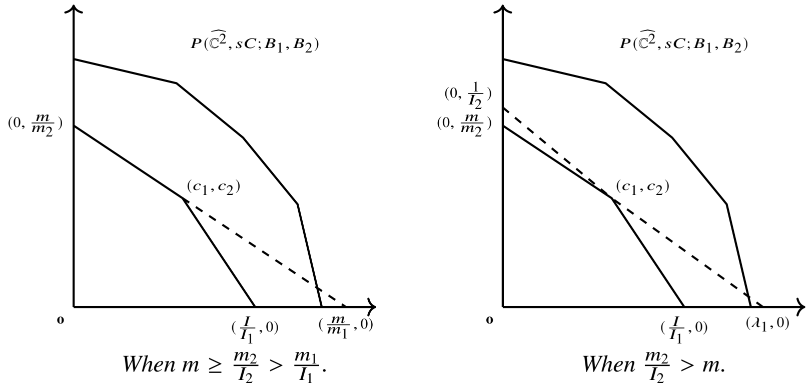

Figure 1. *Two cases.*

Suppose that either $c_1 = 0$ or $c_2 = 0$; then $(c_1, c_2) \in P(\widehat{\mathbb{C}^2} \ni o, sC; B_1, B_2)$ follows directly from Corollary 3.8.

Suppose that $c_1 > 0$ and $c_2 > 0$. Since (c_1, c_2) is a vertex of $E_{1,2} \cap \mathcal{P}$, it is the unique solution of equation (3.2). Thus $\frac{m_1}{I_1} \neq \frac{m_2}{I_2}$, and

$$\min\{\frac{m_1}{I_1}, \frac{m_2}{I_2}\} < \frac{m_1c_1 + m_2c_2}{I_1c_1 + I_2c_2} = \frac{m}{I} < \max\{\frac{m_1}{I_1}, \frac{m_2}{I_2}\}.$$

Without loss of generality, we may assume that $\frac{m_1}{I_1} < \frac{m}{I} < \frac{m_2}{I_2}$. See Figure 1.

If $m \geq \frac{m_2}{I_2} > \frac{m_1}{I_1}$, then we may write $c_1B_1 + c_2B_2 = \mu_1 \frac{m}{m_1} B_1 + \mu_2 \frac{m}{m_2} B_2$ for $\mu_1 = \frac{m_1c_1}{m}$ and $\mu_2 = \frac{m_2c_2}{m}$. Note that $\mu_1 + \mu_2 = 1$. By Corollary 3.8 and $m \leq 1$,

$$\operatorname{lct}(\widehat{\mathbb{C}^2} \ni o, \frac{m}{m_i} B_i; C) \geq \min\left\{1, 1 + \frac{m_i}{I_i} - m\right\} = 1 + \frac{m_i}{I_i} - m$$

for $i = 1, 2$. By Lemma 2.19 and the Cauchy–Schwarz inequality, we have

$$\begin{aligned}\operatorname{lct}(\widehat{\mathbb{C}^2} \ni o, c_1B_1 + c_2B_2; C) &\geq \mu_1 \operatorname{lct}(\widehat{\mathbb{C}^2} \ni o, \frac{m}{m_1} B_1; C) + \mu_2 \operatorname{lct}(\widehat{\mathbb{C}^2} \ni o, \frac{m}{m_2} B_2; C)\\ &\geq 1 - m + \mu_1 \frac{m_1}{I_1} + \mu_2 \frac{m_2}{I_2} = 1 - m + \frac{m_1^2 c_1}{I_1 m} + \frac{m_2^2 c_2}{I_2 m}\\ &\geq 1 - m + \frac{(m_1c_1 + m_2c_2)^2}{(I_1c_1 + I_2c_2)m} = 1 - m + \frac{m}{I} = s.\end{aligned}$$

Otherwise, $\frac{m_2}{I_2} > m$. We may write $c_1B_1 + c_2B_2 = \mu_1' \lambda_1 B_1 + \mu_2' \frac{1}{I_2} B_2$, where $\mu_2' = I_2c_2$, $\mu_1' = 1 - I_2c_2$, $\lambda_1 = \frac{c_1}{1 - I_2c_2}$. Note that $\mu_1' > 1 - \frac{m_2c_2}{m} > 0$, $\mu_1' + \mu_2' = 1$ and $\lambda_1 \leq \frac{c_1}{1 - \frac{m_2c_2}{m}} = \frac{m}{m_1} \leq \frac{1}{m_1}$. By Corollary 3.8, we have

$$\operatorname{lct}(\widehat{\mathbb{C}^2} \ni o, \lambda_1 B_1; C) \geq \min\left\{1, 1 + \frac{m_1}{I_1} - \lambda_1 m_1\right\} \quad \text{and} \quad \operatorname{lct}(\widehat{\mathbb{C}^2} \ni o, \frac{1}{I_2} B_2; C) \geq 1.$$

By Lemma 2.19, we have

$$\begin{aligned}\operatorname{lct}(\widehat{\mathbb{C}^2} \ni o, c_1B_1 + c_2B_2; C) &\geq \mu_1' \operatorname{lct}(\widehat{\mathbb{C}^2} \ni o, \lambda_1 B_1; C) + \mu_2' \operatorname{lct}(\widehat{\mathbb{C}^2} \ni o, \frac{1}{I_2}B_2; C)\\ &\geq \min\Big\{1, 1 + \mu_1' m_1(\frac{1}{I_1} - \lambda_1)\Big\} = \min\Big\{1, 1 + \frac{m_1}{I_1}(1 - I)\Big\}\\ &\geq \min\Big\{1, 1 + \frac{m}{I}(1 - I)\Big\} = s.\end{aligned}$$

Here, for the equality, we use the fact that

$$\mu_1'(\frac{1}{I_1} - \lambda_1) = \frac{1 - I_2c_2 - I_1c_1}{I_1} = \frac{1 - I}{I_1}.$$

In summary, we have showed that $(c_1, c_2) \in P(\widehat{\mathbb{C}^2} \ni o, sC; B_1, B_2)$, and the proof is completed. □

Lemma 3.11. *Let $b_j \geq 0$ and $\mathbf{n}_j \in \mathbb{R}^n_{>0}$ for $j = 1, 2$. Assume that $n \geq 2$; then*

$$\mathcal{P} := \{\mathbf{t} \in \mathbb{R}^n_{\geq 0} \mid \langle \mathbf{n_j}, \mathbf{t}\rangle = b_j,\ j = 1, 2\}$$

is a convex polytope, and all the vertices of $\mathcal{P}$ belong to $\bigcup_{1\leq i\neq j\leq n} E_{i,j}$, where

$$E_{i,j} := \{(t_1, \cdots, t_n) \in \mathbb{R}^n \mid t_k = 0 \textit{ for } k \neq i, j\}.$$

Proof. It is easy to check that $\mathcal{P}$, if non-empty, is a convex polytope of dimension at least $n - 2$. Note that each vertex of $\mathcal{P}$ belongs to at least $n - 2$ faces of $\mathcal{P}$. Since $\mathcal{P}$ has at most n faces $\{(t_1, \cdots, t_n) \in \mathbb{R}^n \mid t_i = 0\} \cap \mathcal{P}$ for $i = 1, 2, \ldots, n$, we conclude that each vertex of $\mathcal{P}$ belongs to $\bigcup_{1\leq i<j\leq n} E_{i,j}$. □

Corollary 3.12. *Let $(X \ni P, B)$ be a germ of surface pair such that X is smooth and* $\operatorname{mld}(X \ni P, B) \geq 1$. *Let C be a smooth curve at P such that $C \not\subseteq$* $\operatorname{Supp}(B)$ *and $(B \cdot C)_P \leq 2$. Then* $\operatorname{lct}(X \ni P, B; C) \geq \frac{1}{2}$.

Proof. Note that $\operatorname{mld}(X \ni P, B) \geq 1$ implies that $m := \operatorname{mult}_P B \leq 1$ (compare [18, Lemma 3.15]). By Theorem 1.11 and the fact that $I \leq 2$,

$$\operatorname{lct}(X \ni P, B; C) \geq \min\{1, 1 + \frac{m}{I} - m\} \geq 1 + \frac{m}{2} - m \geq \frac{1}{2}.$$ □

4. Proofs of the main theorems

4.1. Proof of Theorem 1.10

In this subsection, we give the proof of Theorem 1.10. We first treat the case when $\dim X = 2$.

Proof of Theorem 1.10 when $\dim X = 2$. Observe that when $\dim X = 2$, Z is a curve and z is a closed point. We split the proof into two steps.

Step 1. First we treat the case when X is smooth, $B \geq 0$.

In this case, we have $\operatorname{mld}(X/Z \ni z, B) = 1$. As the geometric generic fibre of π is a rational curve, we may run a K_X-MMP over Z and reach a minimal ruled surface $\pi' : X' \to Z$. Denote by $\phi : X \to X'$ the induced morphism and $B' = \phi_*B$. Since $K_X + B \sim_{\mathbb{R},Z} 0$, by the negativity lemma [9, Lemma 3.6.2], $\phi^*(K_{X'} + B') = K_X + B$. Thus, $K_{X'} + B' \sim_{\mathbb{R},Z} 0$, $\operatorname{mld}(X'/Z \ni z, B') = \operatorname{mld}(X/Z \ni z, B)$ and $\operatorname{lct}(X'/Z \ni z, B'; \pi'^*z) = \operatorname{lct}(X/Z \ni z, B; \pi^*z)$. Now $F := \pi'^*(z) \cong \mathbb{P}^1$ and $(K_{X'} + B') \cdot F = 0$. By the adjunction formula, $K_{X'} \cdot F = -2$. Hence $(B' \cdot F)_P \leq 2$ for any closed point $P \in F$. Recall that $\operatorname{mld}(X'/Z \ni z, B') = 1$ implies that $F \not\subseteq \operatorname{Supp}(B')$. By Corollary 3.12, $\operatorname{lct}(X' \ni P, B'; F) \geq \frac{1}{2}$ for any closed point $P \in F$, which implies that $\operatorname{lct}(X'/Z \ni z, B'; \pi'^*z) \geq \frac{1}{2}$. Hence $\operatorname{lct}(X/Z \ni z, B; \pi^*z) \geq \frac{1}{2}$.

Step 2. We treat the general case.

Write $\mathrm{mld}(X/Z \ni z, B) = 1+\epsilon$ for some $\epsilon \geq 0$. Let $f : W \to X$ be a log resolution of $(X, \mathrm{Supp}(B)+\pi^*z)$. We may write $K_W + B_W = f^*(K_X + B)$. Since $\mathrm{mld}(X/Z \ni z, B) = 1+\epsilon$, for any curve $C \subset \mathrm{Supp}(f^*\pi^*z)$, $\mathrm{mult}_C B_W \leq -\epsilon$. We can take $s \geq 0$ such that for any curve $C \subset \mathrm{Supp}(f^*\pi^*z)$, $\mathrm{mult}_C(B_W + sf^*\pi^*z) \leq 0$, and there exists a curve $C_0 \subset \mathrm{Supp}(f^*\pi^*z)$ with $\mathrm{mult}_{C_0}(B_W + sf^*\pi^*z) = 0$. By Lemma 2.8, possibly shrinking Z near z, we may assume that (X, B) is lc, so the coefficients of B_W are at most 1. Since $B_W + sf^*\pi^*z$ is a simple normal crossing divisor, by [11, Lemma 3.3], $\mathrm{mld}(W/Z \ni z, B_W + sf^*\pi^*z) = 1$. Note that $B_W + sf^*\pi^*z$ is not necessarily effective, so we cannot apply Step 1 directly.

We may write $B_W + sf^*\pi^*z = D - G$, where D and G are effective $\mathbb{R}$-divisors with no common components. Then

$$K_W + D = f^*(K_X + B + s\pi^*z) + G \sim_{\mathbb{R},Z} G.$$

By Remark 2.13, B is effective on the generic fibre of π, so $\mathrm{Supp}(G)$ does not dominate Z. Possibly shrinking Z near z, we may assume that $\mathrm{Supp}(G) \subset \mathrm{Supp}(f^*\pi^*z)$. By the construction, $C_0 \subset \mathrm{Supp}(f^*\pi^*z)$ but $C_0 \not\subset \mathrm{Supp}(G)$. Note that (W, D) is lc as the coefficients of D are at most 1.

If E is a curve on W that is contracted over Z with $(K_W + D) \cdot E < 0$, then $G \cdot E < 0$, and hence $E \subset \mathrm{Supp}(G)$. Then $E \not\subset \mathrm{Supp}(D)$ and $K_W \cdot E < 0$. This implies that any $(K_W + D)$-MMP over Z is also a K_W-MMP over Z, and it only contracts curves in $\mathrm{Supp}(G)$.

We may run a $(K_W + D)$-MMP over Z and reach a minimal model Y with induced maps $g : W \to Y$ and $h : Y \to Z$ such that $K_Y + D_Y \sim_{\mathbb{R},Z} G_Y$ is nef over Z, where D_Y and G_Y are the strict transforms of D and G on Y, respectively.

As this MMP is also a K_W-MMP, Y is a smooth surface. Recall that $C_0 \not\subseteq \mathrm{Supp}(G)$, so C_0 is not contracted by this MMP and $\mathrm{Supp}(G_Y) \subsetneq \mathrm{Supp}(h^*z)$. Hence $G_Y = 0$ as G_Y is nef over Z. Since $K_Y + D_Y = g_*(K_W + D - G) \sim_{\mathbb{R},Z} 0$, by the negativity lemma [9, Lemma 3.6.2],

$$g^*(K_Y + D_Y) = K_W + B_W + sf^*\pi^*z = f^*(K_X + B + s\pi^*z) \sim_{\mathbb{R},Z} 0.$$

Thus, $\mathrm{mld}(Y/Z \ni z, D_Y) = \mathrm{mld}(W/Z \ni z, B_W + sf^*\pi^*z) = 1$, and

$$\mathrm{lct}(Y/Z \ni z, D_Y; h^*z) = \mathrm{lct}(X/Z \ni z, B + s\pi^*z; \pi^*z) = \mathrm{lct}(X/Z \ni z, B; \pi^*z) - s.$$

Since X and Y are isomorphic over the generic point of Z, the geometric generic fibre of h is again a rational curve. So (Y, D_Y) satisfies the setting in Step 1. By Step 1, we get $\mathrm{lct}(Y/Z \ni z, D_Y; h^*z) \geq \frac{1}{2}$.

To conclude the proof, we need to give a lower bound for s. As Y is smooth, Y dominates a $\mathbb{P}^1$-bundle over Z. So there exists a curve C_1 on Y such that $C_1 \subset \mathrm{Supp}(h^*z)$ and $\mathrm{mult}_{C_1} h^*z = 1$. Denote C_1' to be the strict transform of C_1 on W; then $C_1' \subset \mathrm{Supp}(f^*\pi^*z)$ and $\mathrm{mult}_{C_1'} f^*\pi^*z = 1$. Note that $\mathrm{mult}_{C_1'}(B_W + sf^*\pi^*z) = \mathrm{mult}_{C_1}(D_Y) \geq 0$. On the other hand, $\mathrm{mult}_{C_1'}(B_W + sf^*\pi^*z) \leq 0$ by the definition of s. So $\mathrm{mult}_{C_1'}(B_W + sf^*\pi^*z) = 0$. As $\mathrm{mult}_{C_1'} B_W \leq -\epsilon$, we have $s \geq \epsilon$. Hence

$$\mathrm{lct}(X/Z \ni z, B; \pi^*z) = \mathrm{lct}(Y/Z \ni z, D_Y; h^*z) + s \geq \frac{1}{2} + \epsilon = \mathrm{mld}(X/Z \ni z, B) - \frac{1}{2}.$$

This concludes the proof. □

Next we give the proof of Theorem 1.10 by induction on dimensions.

Proof of Theorem 1.10. We prove the theorem by induction on the dimension of X. We have proved the case when $\dim X = 2$. Suppose that Theorem 1.10 holds when $\dim X = n$ for some integer $n \geq 2$; we will show that the theorem holds when $\dim X = n+1$.

As the statement is local around $z \in Z$, we are free to shrink Z. Possibly shrinking Z near z, we may assume that $\bar{z}$ is a Cartier divisor on Z. Set $t := \mathrm{lct}(X/Z \ni z, B; \pi^*\bar{z})$. Possibly shrinking Z near z, we may assume that $(X, B + t\pi^*\bar{z})$ is lc.

Pick a general hyperplane section $H \subset Z$ intersecting $\overline{z}$. Possibly shrinking Z near z, we may assume that $H \cap \overline{z}$ is irreducible. Let z_H be the generic point of $H \cap \overline{z}$ and $G := \pi^* H$; then by Bertini's theorem, the restriction $\pi_G = \pi|_G : G \longrightarrow H$ is a contraction between normal varieties such that $K_G + B|_G \sim_{\mathbb{R},H} 0$. Since H is general, by [27, Lemma 5.17(2)], we may assume that

- the geometric generic fibre of π_G is a rational curve, and
- $(X, B + G + t\pi^*\overline{z})$ is lc.

Let $\phi : Y \longrightarrow X$ be a log resolution of $(X, \operatorname{Supp} B + \pi^*\overline{z})$, we may write

$$K_Y + \phi_*^{-1}B + \sum_i (1 - a_i)E_i = \phi^*(K_X + B),$$

where E_i are ϕ-exceptional prime divisors. Possibly shrinking Z near z, we may further assume that $z \in \pi \circ \phi(E_i)$ for each i. By taking H general enough, we may assume that

- $\phi^* G = \phi_*^{-1} G$, and
- ϕ is a log resolution of $(X, \operatorname{Supp} B + \pi^*\overline{z} + G)$.

Since $\phi_*^{-1}G = \phi^* G = \phi^*\pi^* H$, we have $\pi \circ \phi(E_i \cap \phi_*^{-1}G) = (\pi \circ \phi(E_i)) \cap H$ for each i.

Since

$$K_Y + \phi_*^{-1}B + \phi_*^{-1}G + \sum_i (1 - a_i)E_i = \phi^*(K_X + B + G),$$

by the adjunction formula [27, Proposition 5.73],

$$K_{\phi_*^{-1}G} + \phi_*^{-1}B|_{\phi_*^{-1}G} + \sum_i (1 - a_i)E_i|_{\phi_*^{-1}G} = \phi^*(K_G + B|_G),$$

which implies that the induced morphism $\phi_*^{-1}(G) \longrightarrow G$ is a log resolution of $(G, B|_G + \pi_G^*\overline{z_H})$. Since z and z_H are codimension 1 points of Z and H, respectively, we have

$$\begin{aligned} \operatorname{mld}(G/H \ni z_H, B|_G) &= \min\{a_i \mid \pi \circ \phi(E_i \cap \phi_*^{-1}G) = \overline{z_H}\} \\ &= \min\{a_i \mid \pi \circ \phi(E_i) = \overline{z}\} = \operatorname{mld}(X/Z \ni z, B). \end{aligned}$$

Here, the formula computing minimal log discrepancies is by [11, Lemma 3.5], which says that one can compute the minimal log discrepancy of an lc sub-pair on a log resolution. Similarly, we have

$$K_Y + \phi_*^{-1}B + \phi_*^{-1}G + t\phi_*^{-1}\pi^*\overline{z} + \sum_i (1 - a_i')E_i = \phi^*(K_X + B + G + t\pi^*\overline{z}),$$

$$K_{\phi_*^{-1}G} + \phi_*^{-1}B|_{\phi_*^{-1}G} + t\phi_*^{-1}\pi^*\overline{z}|_{\phi_*^{-1}G} + \sum_i (1 - a_i')E_i|_{\phi_*^{-1}G} = \phi^*(K_G + B|_G + t\pi_G^*\overline{z_H}).$$

As $(X, B + G + t\pi^*\overline{z})$ is lc, so is $(G, B|_G + t\pi_G^*\overline{z_H})$. On the other hand, by the definition of t, there exists an index i such that $a_i' = 0$ and $E_i \subseteq \operatorname{Supp}(\phi^*\pi^*\overline{z})$. In particular, $\pi \circ \phi(E_i) = \overline{z}$. Then by the construction, $E_i \cap \phi_*^{-1}G \neq \emptyset$, which gives a non-klt place of $(G, B|_G + t\pi_G^*\overline{z_H})$ whose image on H is $\overline{z_H}$. Thus $t = \operatorname{lct}(G/H \ni z_H, B|_G; \pi_G^*\overline{z_H})$. As $(G/H \ni z_H, B|_G)$ satisfies the conditions of Theorem 1.10,

$$\begin{aligned} \operatorname{lct}(X/Z \ni z, B; \pi^*\overline{z}) &= \operatorname{lct}(G/H \ni z_H, B|_G; \pi_G^*\overline{z_H}) \\ &\geq \operatorname{mld}(G/H \ni z_H, B|_G) - \frac{1}{2} = \operatorname{mld}(X/Z \ni z, B) - \frac{1}{2} \end{aligned}$$

by the induction hypothesis.

For the last statement, note that $\mathrm{lct}(X/Z \ni z, B; \pi^*\overline{z}) \geq \frac{1}{2}$ implies that the coefficients of $B + \frac{1}{2}\pi^*\overline{z}$ are at most 1 over a neighbourhood of $z \in Z$. So if B is effective, then the multiplicity of each irreducible component of $\pi^* z$ is bounded from above by 2. □

The following example shows that the bounds in Theorems 1.7 and 1.10 are optimal.

Example 4.1. Consider $C \simeq \mathbb{P}^1$. Consider $Y = C \times \mathbb{P}^1$ and the natural projection $\pi : Y \to C$. Let D be a smooth curve on Y of type $(1, 2)$. Note that there exists a closed point $p \in C$ such that D intersects $\pi^{-1}(p)$ at a single closed point with intersection multiplicity 2. Set $F = \pi^{-1}(p)$. Then for any real number $s \geq 0$, we consider the sub-pair $(Y, D - sF)$. We can get a log resolution of $(Y, D - sF)$ by blowing up twice as follows. Let $Y_1 \to Y$ be the blow-up at $F \cap D$. Denote by F_1, D_1 the strict transforms of F, D on Y_1, respectively, and E_1 the exceptional divisor. Then F_1, D_1, E_1 intersect at one point. Let $Y_2 \to Y_1$ be the blow-up at $F_1 \cap D_1 \cap E_1$, denote by F_2, D_2, E_2 the strict transforms of F_1, D_1, E_1 on Y_2, respectively, and G_2 the exceptional divisor on Y_2. Then Y_2 is a log resolution of $(Y, D - sF)$. Let $\pi : Y_2 \to C$ and $f : Y_2 \to Y$ be the induced maps. Then we have

$$K_{Y_2} + D_2 - sF_2 - sE_2 - 2sG_2 = f^*(K_Y + D - sF) \sim_{\mathbb{R},C} 0,$$

and

$$\pi^* p = f^* F = F_2 + E_2 + 2G_2.$$

Set $B_2 = D_2 - sF_2 - sE_2 - 2sG_2$. Then $(Y_2/C \ni p, B_2)$ satisfies the conditions of Theorem 1.10. It is easy to compute that $\mathrm{mld}(Y_2/C \ni p, B_2) = 1 + s$ and $\mathrm{lct}(Y_2/C \ni p, B_2; \pi^* p) = \frac{1}{2} + s$. We also have $\mathrm{mult}_{G_2} \pi^* p = 2$. This shows that Theorem 1.10 is optimal.

In this case, if we consider the canonical bundle formula for (Y_2, B_2) over C, then the discriminant part is $B_C = (\frac{1}{2} - s)p$, and hence for any $M_C \geq 0$ on C,

$$\mathrm{mld}(C \ni p, B_C + M_C) \leq \mathrm{mld}(C \ni p, B_C) = \frac{1}{2} + s.$$

This shows that Theorem 1.7 is optimal.

The next example shows that Theorem 1.10 does not hold when B is not effective on the generic fibre.

Example 4.2. Consider $C \simeq \mathbb{P}^1$. Consider the pair $(C \times \mathbb{P}^1, B := B_1 - B_2)$ and the natural projection $\pi : C \times \mathbb{P}^1 \to C$, where B_1 is a curve on $C \times \mathbb{P}^1$ of type $(2, 3)$ with a cusp $q \in B_1$, and B_2 is the section of π containing q. Set $p = \pi(q)$ and $D = \pi^{-1}(p) = \pi^* p$. We can take B_1, B_2 such that B_1, B_2 and D are locally defined by $(x^2 + y^3 = 0)$, $(y = 0)$ and $(x = 0)$, respectively, for some local coordinates x, y near $q \in C \times \mathbb{P}^1$. Then $\mathrm{lct}(C \times \mathbb{P}^1/C \ni p, B; D) = \frac{1}{3} < \frac{1}{2}$. More generally, if B is not effective on the generic fibre, then there is no uniform lower bound for $\mathrm{lct}(C \times \mathbb{P}^1/C \ni p, B; D)$ as in Theorem 1.10.

4.2. Proofs of Theorems 1.4 and 1.7

We first reduce Theorem 1.7 to the case when B is a $\mathbb{Q}$-divisor.

Lemma 4.3. *Assume that Theorem 1.7 holds when B is a $\mathbb{Q}$-divisor; then Theorem 1.7 holds.*

Proof. Fix the choice of the Weil divisor K_X. We may write

$$K_X + B = \sum_{i=1}^{m} d_i D_i,$$

where D_i are Cartier divisors on X and $d_1, \ldots, d_m$ are $\mathbb{Q}$-linearly independent real numbers. By [17, Lemma 5.3], D_i is $\mathbb{R}$-Cartier and $D_i \sim_{\mathbb{R},Z} 0$ for any $1 \leq i \leq m$.

For a point $\mathbf{t} = (t_1, \dots, t_m) \in \mathbb{R}^m$, we set

$$B(\mathbf{t}) = \sum_{i=1}^{m} t_i D_i - K_X.$$

Then for any $\mathbf{t} \in \mathbb{R}^m$, $K_X + B(\mathbf{t}) \sim_{\mathbb{R},Z} 0$. Set $\mathbf{d} = (d_1, \dots, d_m)$.

Let $f : Y \to X$ be a log resolution of $(X, B + \sum_{i=1}^m D_i)$ such that $\mathrm{Supp}(f^{-1}\pi^{-1}(\bar{z}))$ is a simple normal crossing divisor. Write $K_Y + B_Y(\mathbf{t}) = f^*(K_X + B(\mathbf{t}))$.

Possibly shrinking Z near z, we may assume that (X, B) is lc. Note that $(X, B(\mathbf{t}))$ is lc if and only if the coefficients of $B_Y(\mathbf{t})$ are at most 1. Note that $\mathrm{mld}(X/Z \ni z, B(\mathbf{t})) \geq 1$ if and only if for any prime divisor E on Y with $f(E) = \bar{z}$, $\mathrm{mult}_E\, B_Y(\mathbf{t}) \leq 0$ (compare [11, Lemma 3.3]). So the subset

$$\mathcal{P}_1 := \{\mathbf{t} \in \mathbb{R}^m \mid (X, B(\mathbf{t})) \text{ is lc}, \mathrm{mld}(X/Z \ni z, B(\mathbf{t})) \geq 1\}$$

is determined by finitely many linear functions in $\mathbf{t}$ with coefficients in $\mathbb{Q}$. In other words, $\mathcal{P}_1$ is a non-compact rational polytope containing $\mathbf{d}$. Note that $\mathrm{mld}(X/Z \ni z, B(\mathbf{t}))$ can be computed on Y as the minimum of finitely many linear functions in $\mathbf{t}$ with coefficients in $\mathbb{Q}$. Possibly replacing $\mathcal{P}_1$ with a smaller rational polytope containing $\mathbf{d}$, we may assume that $\mathrm{mld}(X/Z \ni z, B(\mathbf{t}))$ is linear on $\mathcal{P}_1$ and $\mathcal{P}_1$ is bounded.

By Remark 2.13, B is effective on the generic fibre of π. It is easy to see that

$$\mathcal{P}_2 := \{\mathbf{t} \in \mathbb{R}^m \mid B(\mathbf{t}) \text{ is effective on the generic fibre of } \pi\}$$

is a rational polytope.

By the construction, $\mathcal{P} := \mathcal{P}_1 \cap \mathcal{P}_2$ is a bounded rational polytope containing $\mathbf{d}$. If $\mathbf{t} \in \mathcal{P}$, then $\pi : (X, B(\mathbf{t})) \to Z$ is an lc-trivial fibration satisfying the assumptions of Theorem 1.7. So we can consider the canonical bundle formula

$$K_X + B(\mathbf{t}) = \pi^*(K_Z + B(\mathbf{t})_Z + M(\mathbf{t})_Z).$$

By the convexity of log canonical thresholds, the irreducible components of $\mathrm{Supp}(B(\mathbf{t})_Z)$ belong to a finite set $\{P_1, P_2, \dots, P_k\}$ for any $\mathbf{t} \in \mathcal{P}$, here $\{P_1, P_2, \dots, P_k\}$ is the set of prime divisors on Z in $\bigcup_{\mathbf{t}'} \mathrm{Supp}(B(\mathbf{t}')_Z)$, where the union runs over all vertex points $\mathbf{t}' \in \mathcal{P}$. Denote the generic point of P_j by z_j for $1 \leq j \leq k$. Note that for any $1 \leq j \leq k$, $\mathrm{lct}(X/Z \ni z_j, B(\mathbf{t}); \pi^*P_j)$ is computed on a log resolution as the minimum of finitely many linear functions in $\mathbf{t}$ with coefficients in $\mathbb{Q}$. So possibly replacing $\mathcal{P}$ with a smaller rational polytope containing $\mathbf{d}$, we may assume that $\mathrm{lct}(X/Z \ni z_j, B(\mathbf{t}); \pi^*P_j)$ is linear in $\mathbf{t}$ for any $1 \leq j \leq k$.

Now we can take $\mathbf{t}_1, \dots, \mathbf{t}_l \in \mathcal{P} \cap \mathbb{Q}^m$ and positive real numbers $s_1, \dots, s_l$ such that $\sum_{i=1}^l s_i = 1$ and $\sum_{i=1}^l s_i \mathbf{t}_i = \mathbf{d}$. By the construction,

$$\begin{aligned} B_Z &= \sum_{j=1}^{k} (1 - \mathrm{lct}(X/Z \ni z_j, B; \pi^*P_j))P_j \\ &= \sum_{j=1}^{k} \sum_{i=1}^{l} s_i (1 - \mathrm{lct}(X/Z \ni z_j, B(\mathbf{t}_i); \pi^*P_j))P_j = \sum_{i=1}^{l} s_i B(\mathbf{t}_i)_Z. \end{aligned}$$

By assumption, Theorem 1.7 holds for $(X/Z \ni z, B(\mathbf{t}_i))$ for each i: that is, we can choose $M(\mathbf{t}_i)_Z \geq 0$ such that

$$\mathrm{mld}(Z \ni z, B(\mathbf{t}_i)_Z + M(\mathbf{t}_i)_Z) \geq \mathrm{mld}(X/Z \ni z, B(\mathbf{t}_i)) - \frac{1}{2}.$$

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Then setting $M_Z := \sum_{i=1}^{l} s_i M(\mathbf{t}_i)_Z \geq 0$, we have

$$\begin{aligned}\operatorname{mld}(Z \ni z, B_Z + M_Z) &\geq \sum_{i=1}^{l} s_i \operatorname{mld}(Z \ni z, B(\mathbf{t}_i)_Z + M(\mathbf{t}_i)_Z) \\ &\geq \sum_{i=1}^{l} s_i \operatorname{mld}(X/Z \ni z, B(\mathbf{t}_i)) - \frac{1}{2} = \operatorname{mld}(X/Z \ni z, B) - \frac{1}{2}.\end{aligned}$$

Here, for the first inequality, we use the convexity of minimal log discrepancies, and for the last equality, we use the linearity of $\operatorname{mld}(X/Z \ni z, B(\mathbf{t}))$ on $\mathcal{P}$. □

Proof of Theorem 1.7. By Lemma 4.3, we may assume that B is a $\mathbb{Q}$-divisor. As we described in Section 2.4, there are b-divisors $\mathbf{B}$ and $\mathbf{M}$ such that

- $\mathbf{B}_Z = B_Z, \mathbf{M}_Z = M_Z$, and
- for any birational contraction $g : Z' \to Z$, let X' be a resolution of the main component of $X \times_Z Z'$ with induced morphisms $g' : X' \to X$ and $\pi' : X' \to Z'$. Write $K_{X'} + B' = g'^*(K_X + B)$; then $\mathbf{B}_{Z'}$ (respectively, $\mathbf{M}_{Z'}$) is the discriminant part (respectively, the moduli part) of the canonical bundle formula of $K_{X'} + B'$ on Z'.

We may write $\mathbf{B} = \sum d_P P$, where P is the birational component of $\mathbf{B}$ and d_P the corresponding coefficient.

Claim 4.4. *For any birational component P of $\mathbf{B}$ whose center on Z is $\bar{z}$, $d_P \leq \frac{3}{2} - \operatorname{mld}(X/Z \ni z, B)$.*

We will proceed with the proof assuming Claim 4.4. The proof of Claim 4.4 will be given after the proof.

By [35, Theorem 8.1] (see Remark 2.15), $\mathbf{M}$ is b-semi-ample. Then there exists a resolution $g : Z' \to Z$ such that $\mathbf{M}_{Z'}$ is semi-ample, and $\mathbf{B}_{Z'} + \operatorname{Supp}(g^{-1}(\bar{z}))$ is a simple normal crossing divisor. Thus we may take a general $\mathbb{Q}$-divisor $L_{Z'} \geq 0$ on Z' such that $\mathbf{M}_{Z'} \sim_{\mathbb{Q}} L_{Z'}$, $\mathbf{B}_{Z'} + L_{Z'}$ is simple normal crossing, and for each prime divisor P on Z' whose center on Z is $\bar{z}$, the coefficient of P in $\mathbf{B}_{Z'} + L_{Z'}$ is at most $\frac{3}{2} - \operatorname{mld}(X/Z \ni z, B)$. In this case, $\operatorname{mld}(Z'/Z \ni z, \mathbf{B}_{Z'} + L_{Z'}) \geq \operatorname{mld}(X/Z \ni z, B) - \frac{1}{2}$. Note that

$$K_{Z'} + \mathbf{B}_{Z'} + L_{Z'} \sim_{\mathbb{Q}} K_{Z'} + \mathbf{B}_{Z'} + \mathbf{M}_{Z'} = g^*(K_Z + B_Z + \mathbf{M}_Z) \sim_{\mathbb{Q},Z} 0,$$

hence by the negativity lemma [9, Lemma 3.6.2],

$$g^*(K_Z + B_Z + g_* L_{Z'}) = g^* g_*(K_{Z'} + \mathbf{B}_{Z'} + L_{Z'}) = K_{Z'} + \mathbf{B}_{Z'} + L_{Z'}.$$

Thus $M_Z \sim_{\mathbb{Q}} g_* L_{Z'} \geq 0$ and $\operatorname{mld}(Z \ni z, B_Z + g_* L_{Z'}) \geq \operatorname{mld}(X/Z \ni z, B) - \frac{1}{2}$. □

Proof of Claim 4.4. Fix a birational component P_0 of $\mathbf{B}$ whose center on Z is $\bar{z}$:

$$\begin{array}{ccc} (X', B') & \xrightarrow{g'} & (X, B) \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ Z' & \xrightarrow[g]{} & Z. \end{array}$$

Take a resolution $g : Z' \to Z$ such that P_0 is a prime divisor on Z'. Denote the generic point of P_0 on Z' by z' and hence $P_0 = \overline{z'}$. Let X' be a resolution of the main component of $X \times_Z Z'$ with induced maps $g' : X' \to X$ and $\pi' : X' \to Z'$. We may write $K_{X'} + B' = g'^*(K_X + B)$. Then

$$\operatorname{mld}(X'/Z \ni z, B') = \operatorname{mld}(X/Z \ni z, B) \geq 1.$$

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In particular, this implies that

$$\mathrm{mld}(X'/Z' \ni z', B') \geq \mathrm{mld}(X/Z \ni z, B) \geq 1.$$

By the construction, the geometric generic fibre of π' is a rational curve. So $(X'/Z' \ni z', B')$ satisfies the assumptions of Theorem 1.10. By Theorem 1.10,

$$\mathrm{lct}(X'/Z' \ni z', B'; \pi'^*\overline{z'}) \geq \mathrm{mld}(X'/Z' \ni z', B') - \frac{1}{2}.$$

Hence by the definition of $\mathbf{B}$,

$$\begin{aligned} d_{P_0} &= 1 - \mathrm{lct}(X'/Z' \ni z', B'; \pi'^*\overline{z'}) \\ &\leq \frac{3}{2} - \mathrm{mld}(X'/Z' \ni z', B') \leq \frac{3}{2} - \mathrm{mld}(X/Z \ni z, B). \end{aligned}$$ □

Proof of Corollary 1.8. Note that we cannot get Corollary 1.8 by directly applying Theorem 1.7 to all codimension ≥ 1 points on Z, as the choice of M_Z depends on $z \in Z$ in Theorem 1.7. But we can follow the same line as in Theorem 1.7 to prove Corollary 1.8.

By the same reduction in Lemma 4.3, we may assume that B is a $\mathbb{Q}$-divisor. As we described in Section 2.4, there are b-divisors $\mathbf{B}$ and $\mathbf{M}$ such that

- $\mathbf{B}_Z = B_Z$, $\mathbf{M}_Z = M_Z$, and
- for any birational contraction $g : Z' \to Z$, let X' be a resolution of the main component of $X \times_Z Z'$ with induced morphisms $g' : X' \to X$ and $\pi' : X' \to Z'$. Write $K_{X'} + B' = g'^*(K_X + B)$; then $\mathbf{B}_{Z'}$ (respectively, $\mathbf{M}_{Z'}$) is the discriminant part (respectively, the moduli part) of the canonical bundle formula of $K_{X'} + B'$ on Z'.

By [35, Theorem 8.1] (see Remark 2.15), $\mathbf{M}$ is b-semi-ample. Then there exists a resolution $g : Z' \to Z$ such that $\mathbf{M}_{Z'}$ is semi-ample, and $\mathbf{B}_{Z'} + \mathrm{Supp}(g^{-1}(\overline{z}))$ is a simple normal crossing divisor. By applying Claim 4.4 to all codimension ≥ 1 points on Z, we get that the coefficients of $\mathbf{B}_{Z'}$ are at most $\frac{1}{2}$. Thus we may take a general $\mathbb{Q}$-divisor $L_{Z'} \geq 0$ on Z' such that $\mathbf{M}_{Z'} \sim_{\mathbb{Q}} L_{Z'}$, $\mathbf{B}_{Z'} + L_{Z'}$ is simple normal crossing, and the coefficients of $\mathbf{B}_{Z'} + L_{Z'}$ are at most $\frac{1}{2}$. In this case, $(Z', \mathbf{B}_{Z'} + L_{Z'})$ is $\frac{1}{2}$-lc. Note that

$$K_{Z'} + \mathbf{B}_{Z'} + L_{Z'} \sim_{\mathbb{Q}} K_{Z'} + \mathbf{B}_{Z'} + \mathbf{M}_{Z'} = g^*(K_Z + B_Z + \mathbf{M}_Z) \sim_{\mathbb{Q},Z} 0,$$

hence by the negativity lemma [9, Lemma 3.6.2],

$$g^*(K_Z + B_Z + g_*L_{Z'}) = g^*g_*(K_{Z'} + \mathbf{B}_{Z'} + L_{Z'}) = K_{Z'} + \mathbf{B}_{Z'} + L_{Z'}.$$

Thus $M_Z \sim_{\mathbb{Q}} g_*L_{Z'} \geq 0$ and $(Z, B_Z + g_*L_{Z'})$ is $\frac{1}{2}$-lc. □

Proof of Theorem 1.4. As the statement is local, we may assume that Z is affine. Since $-K_X$ is ample over Z, there exists a positive integer N such that $-NK_X$ is very ample over Z. Let H be a general very ample divisor on X such that $H \sim_Z -NK_X$, and take $B = \frac{1}{N}H$. Then $K_X + B \sim_{\mathbb{Q},Z} 0$, B has no vertical irreducible component over Z, and (X, B) is canonical. By Corollary 1.8, we can choose $M_Z \geq 0$ representing the moduli part and B_Z the discriminant part of the canonical bundle formula of $K_X + B$ on Z such that $(Z, B_Z + M_Z)$ is $\frac{1}{2}$-lc. Note that $B \geq 0$ implies that $B_Z \geq 0$. Thus Z is $\frac{1}{2}$-lc. □

Finally, as an application of Corollary 1.8, we show the following weaker version of Iskovskikh's conjecture under a more general setting without using the classification of terminal singularities in dimension 3 as in [30].

Corollary 4.5. *Let $\pi : X \to Z$ be a contraction between normal varieties such that*

1. $\dim X - \dim Z = 1$,
2. *there is no prime divisor D on X such that* $\mathrm{codim}(\pi(D), Z) \geq 2$,

3. *X is terminal,*
4. *K_Z is $\mathbb{Q}$-Cartier, and*
5. *$-K_X$ is ample over Z.*

Then Z is $\frac{1}{2}$-klt.

Here, the assumption in (2) is a natural geometric condition: for example, it holds if all fibres of π are 1-dimensional or if $\rho(X/Z) = 1$.

Proof. As the statement is local, we may assume that Z is affine. By Theorem 1.4, Z is $\frac{1}{2}$-lc. Assume to the contrary that Z is not $\frac{1}{2}$-klt; then there exists an exceptional prime divisor E over Z such that $a(E, Z) = \frac{1}{2}$. Denote by $c_Z(E)$ the center of E on Z.

By [9, Corollary 1.4.3], we can find a proper birational morphism $g : Z' \to Z$ such that E is the only g-exceptional divisor. Let X' be a resolution of the main component of $X \times_Z Z'$ with induced morphisms $g' : X' \to X$ and $\pi' : X' \to Z'$:

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "\pi'"'] & X \arrow[d, "\pi"] \\ Z' \arrow[r, "g"'] & Z. \end{tikzcd}$$

We can write $K_{X'} + G = g'^* K_X$, $K_{Z'} + \frac{1}{2}E = g^* K_Z$.

As $-K_X$ is ample over Z, for any $t \in (0, 1) \cap \mathbb{Q}$, we can take an effective $\mathbb{Q}$-divisor B^t on X such that

- (X, B^t) is canonical,
- B^t has no vertical irreducible component over Z,
- $K_X + B^t \sim_{\mathbb{Q},Z} 0$, and
- $\operatorname{Supp}(B^t) \supset \operatorname{Supp}(\pi^{-1}(c_Z(E)))$, and the multiplicity of each irreducible component of $\operatorname{Supp}(\pi^{-1}(c_Z(E)))$ in B^t is a non-constant linear function in t.

The construction is as follows. Take a sufficiently large N such that $-NK_X \sim_Z H$ is a very ample divisor on X, and $\mathcal{O}_X(H)$ and $\mathcal{O}_X(H) \otimes I_{\operatorname{Supp}(\pi^{-1}(c_Z(E)))}$ are generated by global sections. Now take B_1 to be a general global section of $\mathcal{O}_X(H)$ and B_2 a general global section of $\mathcal{O}_X(H) \otimes I_{\operatorname{Supp}(\pi^{-1}(c_Z(E)))}$. Then $B^t = \frac{(1-st)}{N} B_1 + \frac{st}{N} B_2$ satisfies the requirements for sufficiently small positive rational number s. Here, the assumption in (3) guarantees that (X, B^t) is canonical, and the assumption in (2) guarantees that B^t has no vertical irreducible component over Z as $\operatorname{Supp}(\pi^{-1}(c_Z(E)))$ has codimension at least 2 in X.

Then by Corollary 1.8, for any $t \in (0, 1) \cap \mathbb{Q}$, we can choose $M^t_Z \geq 0$ representing the moduli part of the canonical bundle formula of $K_X + B^t$ on Z such that $(Z, B^t_Z + M^t_Z)$ is $\frac{1}{2}$-lc, where $B^t_Z \geq 0$ is the discriminant part. In particular, $c_Z(E)$ is not contained in $\operatorname{Supp}(B^t_Z + M^t_Z)$, otherwise $a(E, Z, B^t_Z + M^t_Z) < a(E, Z) = \frac{1}{2}$, which is absurd. As we described in Section 2.4, there are b-divisors $\mathbf{B}^t$ and $\mathbf{M}^t$ such that

- $\mathbf{B}^t_Z = B^t_Z$, $\mathbf{M}^t_Z = M^t_Z$,
- $K_{X'} + G + g'^* B^t = \pi'^*(K_{Z'} + \mathbf{B}^t_{Z'} + \mathbf{M}^t_{Z'})$,
- $K_{Z'} + \mathbf{B}^t_{Z'} + \mathbf{M}^t_{Z'} = g^*(K_Z + B^t_Z + M^t_Z) = K_{Z'} + \frac{1}{2}E + g^*(B^t_Z + M^t_Z)$.

Recall that $\mathbf{M}^t$ is b-semi-ample by [35, Theorem 8.1] (see Remark 2.15), so $\mathbf{M}^t_{Z'} \leq g^* M^t_Z$ by the negativity lemma [9, Lemma 3.6.2]. As $c_Z(E)$ is not contained in $\operatorname{Supp}(B^t_Z + M^t_Z)$, we get $\operatorname{mult}_E \mathbf{M}^t_{Z'} = 0$ and then $\operatorname{mult}_E \mathbf{B}^t_{Z'} = \frac{1}{2}$. The latter one implies that $\operatorname{lct}(X'/Z' \ni \eta_E, G + g'^* B^t; \pi'^* E) = \frac{1}{2}$ by definition, where η_E is the generic point of E. This is absurd, as by the construction of B^t, $\operatorname{lct}(X'/Z' \ni \eta_E, G + g'^* B^t; \pi'^* E)$ is a non-constant function in t. $\square$

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Remark 4.6. 1. By Example 1.3, the assumption in (3) of Corollary 4.5 cannot be replaced by 'X is canonical'.
2. We expect that the assumptions in (2) and (5) of Corollary 4.5 are all necessary. In fact, by the terminalisation of Example 1.3, the assumptions in (2) and (5) cannot be removed at the same time.

Prokhorov provided us with the following example, which shows that Corollary 4.5 cannot be improved if $\dim X \geq 4$.

Example 4.7. Consider the following action of $\boldsymbol{\mu}_{2m+1}$ on $\mathbb{P}^1_x \times \mathbb{C}^3_{u,v,w}$:

$$(x; u, v, w) \mapsto (\xi^m x; \xi u, \xi v, \xi^m w),$$

where m is a positive integer and ξ is a primitive $(2m+1)$th root of unity. Let $X = (\mathbb{P}^1 \times \mathbb{C}^3)/\boldsymbol{\mu}_{2m+1}$, $Z = \mathbb{C}^3/\boldsymbol{\mu}_{2m+1}$ and $\pi : X \to Z$ the natural projection. Since $\boldsymbol{\mu}_{2m+1}$ acts freely in codimension 1, $-K_X$ is π-ample and $\rho(X/Z) = 1$. Note that Z has an isolated cyclic quotient singularity of type $\frac{1}{2m+1}(1,1,m)$ at the origin $o \in Z$, and $\operatorname{mld}(Z \ni o) = \frac{m+2}{2m+1}$ (see [4] for the computation of minimal log discrepancies of toric varieties). On the other hand, X has two isolated cyclic quotient singularities of types $\frac{1}{2m+1}(m,1,1,m)$ and $\frac{1}{2m+1}(m+1,1,1,m)$ that are terminal (see [36, (4.11) Theorem]).

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