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GENERIC UNLABELED GLOBAL RIGIDITY

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Abstract

Let $\mathbf{p}$ be a configuration of n points in $\mathbb{R}^d$ for some n and some $d \geqslant 2$. Each pair of points has a Euclidean distance in the configuration. Given some graph G on n vertices, we measure the point-pair distances corresponding to the edges of G. In this paper, we study the question of when a generic $\mathbf{p}$ in d dimensions will be uniquely determined (up to an unknowable Euclidean transformation) from a given set of point-pair distances together with knowledge of d and n. In this setting the distances are given simply as a set of real numbers; they are not labeled with the combinatorial data that describes which point pair gave rise to which distance, nor is data about G given. We show, perhaps surprisingly, that in terms of generic uniqueness, labels have no effect. A generic configuration is determined by an unlabeled set of point-pair distances (together with d and n) if and only if it is determined by the labeled distances.

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1. Introduction

Let d be some fixed dimension.

Definition 1.1. An *ordered graph* $G = (V, E)$ on n vertices $V = \{1, \ldots, n\}$ is an ordered sequence of edges (unordered vertex pairs). We do not allow self-loops or duplicate edges. (The ordering is just a notational convenience.)

Let G be an ordered graph (with $n \geqslant d + 2$ vertices and m edges) and $\mathbf{p} = (\mathbf{p}_1, \ldots, \mathbf{p}_n)$ be a configuration of n points in $\mathbb{R}^d$, which we associate with the vertices of G in the natural way. One can measure the squared Euclidean distances in $\mathbb{R}^d$ between vertex pairs corresponding to the edges of G. This gives us an ordered sequence, $\mathbf{v}$, of m squared-distance real values. We write this as $\mathbf{v} = m_G^{\mathbb{E}}(\mathbf{p})$, where $m_G^{\mathbb{E}}(\cdot)$ maps from configurations to squared edge lengths along the edges of G (the $\mathbb{E}$ superscript denotes Euclidean). Importantly, $\mathbf{v}$ does not contain any labeling information describing which squared-length value is associated to which vertex pair; it is simply a sequence of real numbers.

A natural question is:

When does $\mathbf{v}$ *(together with d and n) determine G and* $\mathbf{p}$*?*

We can only hope for G to be unique up to a relabeling of its vertices. A relabeling is simply a permutation on the vertices, $\{1, \ldots, n\}$. Moreover, under this relabeling, we can only hope that $\mathbf{p}$ is unique up to a congruence (affine isometry) of $\mathbb{R}^d$. Thus, given some other configuration $\mathbf{q}$ and ordered graph H, also with n vertices and m edges, such that $\mathbf{v} = m_H^{\mathbb{E}}(\mathbf{q})$, under what conditions will we know that $G = H$ up to a vertex relabeling and $\mathbf{p} = \mathbf{q}$ up to congruence? The restriction that H has exactly n vertices is natural; if H were, say, a tree over $m + 1$ vertices, it would be able to produce any m-tuple of real numbers including $\mathbf{v}$ as the squared-distance measurement of some configuration.

We will be interested in studying this problem under the nondegeneracy assumption that $\mathbf{p}$ is generic.

DEFINITION 1.2. A configuration $\mathbf{p}$ in $\mathbb{R}^d$ is *generic* if there is no nonzero polynomial relation, with coefficients in $\mathbb{Q}$, among the coordinates of $\mathbf{p}$.

Boutin and Kemper [4] proved that if G consists of an ordering of the edges of the complete graph, K_n, and $\mathbf{p}$ is generic, then uniqueness is guaranteed. There is only one $\mathbf{p}$, up to a congruence, consistent with its unlabeled $\mathbf{v}$. With this result in hand, one can immediately weaken the completeness requirement for G, and only require that it 'allows for trilateration' in d dimensions. Loosely speaking, this means that G can be built by gluing together overlapping K_{d+2} graphs (see [10] for formal definitions). This unlabeled trilateration concept was first explored in [20], and a formal proof of uniqueness is given in [10].

Our goal in this paper is to weaken the conditions on G as much as possible.

DEFINITION 1.3. Let G be an ordered graph and $\mathbf{p}$ a configuration in $\mathbb{R}^d$. We say that the pair $(G, \mathbf{p})$ is *globally rigid* in $\mathbb{R}^d$ if for all configurations $\mathbf{q}$ in $\mathbb{R}^d$, $m_G^{\mathbb{E}}(\mathbf{p}) = m_G^{\mathbb{E}}(\mathbf{q})$ implies $\mathbf{p} = \mathbf{q}$ (up to congruence).

We say that G is *generically globally rigid* in $\mathbb{R}^d$ if $(G, \mathbf{p})$ is globally rigid for all generic $\mathbf{p}$ in $\mathbb{R}^d$.

Gortler *et al.* [13] proved:

THEOREM 1.4 [13]. *If an ordered graph G is not generically globally rigid in $\mathbb{R}^d$, then for any generic $\mathbf{p}$, there is a noncongruent $\mathbf{q}$ so that $m_G^{\mathbb{E}}(\mathbf{p}) = m_G^{\mathbb{E}}(\mathbf{q})$.*

This means, in particular, that every graph is either generically globally rigid or generically not globally rigid.

Ordered graphs that allow for d-dimensional trilateration are generically globally rigid in $\mathbb{R}^d$ (see, for example, [12]), but there are many graphs that are generically globally rigid but do not allow for trilateration. A small example in two dimensions is when G comprises the edges of the complete bipartite graph $K_{4,3}$ (generic global rigidity follows from the combinatorial considerations of [5, 18] and can be directly confirmed using the algorithm from [5, 13]). This graph does not even contain a single triangle! (For $d = 2$ and G with $m = O(n \log n)$ edges, results from [19, 21] imply that almost all globally rigid graphs do not allow for trilateration.)

If an ordered graph G is not generically globally rigid, then one generally cannot recover $\mathbf{p}$ when given *both* $\mathbf{v}$ and G (that is, labeled data). The recovery problem is simply not well posed. When an ordered graph is generically globally rigid, then generally this labeled recovery problem will be well posed, though it still might be intractable to perform [27]. We note that testing whether an ordered graph is generically globally rigid can be done with an efficient randomized algorithm [13].

From the above, it is clear that generic global rigidity is necessary for generic unlabeled uniqueness. In this paper we prove the following theorem which states that the property of generic global rigidity of a graph is also sufficient for generic unlabeled uniqueness. This result answers a question posed in [10].

THEOREM 1.5. *In any fixed dimension $d \geqslant 2$, let $\mathbf{p}$ be a generic configuration of $n \geqslant d+2$ points. Let $\mathbf{v} = m_G^{\mathbb{E}}(\mathbf{p})$, where G is an ordered graph (with n vertices and m edges) that is generically globally rigid in $\mathbb{R}^d$.*

Suppose there is a configuration $\mathbf{q}$, also of n points, along with an ordered graph H (with n vertices and m edges) such that $\mathbf{v} = m_H^{\mathbb{E}}(\mathbf{q})$.

Then there is a vertex relabeling of H such that $G = H$. Moreover, under this vertex relabeling, up to congruence, $\mathbf{q} = \mathbf{p}$.

REMARK 1.6. This theorem is true in one dimension as well, if we add the assumption that G is 3-connected. (This assumption will come for free in higher dimension.) We will, in fact, use 3-connectivity in the proof of the more technical Theorem 3.4 that underlies our main result.

REMARK 1.7. We can state Theorem 1.5 without ordered graphs as follows. Let G be an unordered generically globally rigid graph in dimension $d \geqslant 2$, and $\mathbf{p}$ be a generic configuration in dimension d. If H is some other unordered graph with the same number of vertices as G and $\mathbf{q}$ any configuration so that $(H, \mathbf{q})$ has the same unordered set of edge lengths as $(G, \mathbf{p})$, Theorem 1.5 implies that there is an isomorphism between G and H consistent with the bijection on edges induced by the distinct edge lengths of a generic measurement. Furthermore, under this isomorphism, $\mathbf{p}$ is congruent to $\mathbf{q}$.

Ordered graphs are a convenience to avoid referring to an implicit isomorphism throughout.

REMARK 1.8. Theorem 1.4 implies that a generic configuration $\mathbf{p}$ is determined by its labeled edge lengths if and only if these edges form a generically globally rigid graph. Hence, Theorem 1.5 says that a generic configuration $\mathbf{p}$, with known d and n, is uniquely determined (up to relabeling and congruence) from its (unordered) unlabeled edge lengths if and only if it is uniquely determined (up to congruence) by its labeled edge lengths.

Note that for a generically globally rigid graph G, there can be a nongeneric $(G, \mathbf{p})$ which is still globally rigid, but for which $m_G(\mathbf{p})$ (and n) does not uniquely determine $\mathbf{p}$ in the unlabeled setting. (See [4, Figure 4] for an example in the plane where G is K_4.)

Since the nongeneric failures of this theorem are due to a finite collection of algebraically expressible exceptions, the uniqueness promised by this theorem holds over a Zariski open set of configurations.

Our result is information theoretic; it does not give an efficient algorithm for determining $\mathbf{p}$ from $\mathbf{v}$. Indeed, determining $\mathbf{p}$ is NP-hard, even when given $\mathbf{v}$ *and* G [27]. We will discuss some practical implications and related questions in Section 7.2.

The body of this paper will be concerned with the proof of Theorem 1.5. Our approach is to reduce the question to one about the so-called 'measurement variety' (defined in Section 3) of G, which represents all possible $\mathbf{v}$, as $\mathbf{p}$ varies over all d-dimensional configurations. We will want to understand when two distinct ordered graphs, G and H, can give rise to the same measurement variety. We will find (see Theorem 3.4) that when G is generically globally rigid in d dimensions, then this cannot happen. Theorem 1.5 then follows quickly.

2. Rigidity background

In this section we will recall the needed definitions and results from graph rigidity theory.

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2.1. Local rigidity.

DEFINITION 2.1. A *framework* $(G, \mathbf{p})$ is a pair of an ordered graph and a configuration. Two frameworks $(G, \mathbf{p})$ and $(G, \mathbf{q})$ are *equivalent* if $m_G^{\mathbb{E}}(\mathbf{p}) = m_G^{\mathbb{E}}(\mathbf{q})$; they are *congruent* if $\mathbf{p}$ and $\mathbf{q}$ are congruent.

DEFINITION 2.2. Let G be an ordered graph. We say that $(G, \mathbf{p})$ is *locally rigid* in $\mathbb{R}^d$ if, a sufficiently small enough neighborhood of $\mathbf{p}$ in the fiber $(m_G^{\mathbb{E}})^{-1}(m_G^{\mathbb{E}}(\mathbf{p}))$ consists only of $\mathbf{q}$ that are congruent to $\mathbf{p}$. Otherwise we say that $(G, \mathbf{p})$ is *locally flexible* in $\mathbb{R}^d$.

The fiber of $m_G^{\mathbb{E}}$ consists of the configurations $\mathbf{q}$ such that $(G, \mathbf{q})$ is equivalent to $(G, \mathbf{p})$. So local rigidity means that there is a neighborhood of $\mathbf{p}$ in which any $\mathbf{q}$ with $(G, \mathbf{q})$ equivalent to $(G, \mathbf{p})$ must be congruent to $\mathbf{p}$, in parallel to Definition 1.3.

DEFINITION 2.3. A *first-order flex* or *infinitesimal flex* $\mathbf{p}'$ in $\mathbb{R}^d$ of $(G, \mathbf{p})$ is a corresponding assignment of vectors $\mathbf{p}' = (\mathbf{p}'_1, \ldots, \mathbf{p}'_n)$, $\mathbf{p}'_i \in \mathbb{R}^d$ such that for each $\{i, j\}$, an edge of G, the following holds:

$$(\mathbf{p}_i - \mathbf{p}_j) \cdot (\mathbf{p}'_i - \mathbf{p}'_j) = 0. \tag{2.1}$$

A first-order flex $\mathbf{p}'$ in $\mathbb{R}^d$ is *trivial* if it is the restriction to the vertices of the time-zero derivative of a smooth motion of isometries of $\mathbb{R}^d$.

The property of being trivial is independent of the graph G.

DEFINITION 2.4. A framework $(G, \mathbf{p})$ in $\mathbb{R}^d$ is called *infinitesimally rigid* in $\mathbb{R}^d$ if it has no infinitesimal flexes in $\mathbb{R}^d$ except for trivial ones. When $n \geqslant (d+1)$ this is the same as saying that the rank of the differential of $m_G^{\mathbb{E}}(\cdot)$ at $\mathbf{p}$ is $nd - \binom{d+1}{2}$. If a framework is not infinitesimally rigid in $\mathbb{R}^d$, it is called *infinitesimally flexible* in $\mathbb{R}^d$.

We need some standard facts about infinitesimal rigidity.

THEOREM 2.5 (See for example, [11]). *If $(G, \mathbf{p})$ is infinitesimally rigid in $\mathbb{R}^d$, then $(G, \mathbf{p})$ is locally rigid in $\mathbb{R}^d$.*

Affine transformations A on $\mathbb{R}^d$ act on configurations pointwise to produce another configuration, that is, $A(\mathbf{p})_i := A(\mathbf{p}_i)$.

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LEMMA 2.6 [7]. *Let* $(G, \mathbf{p})$ *be a framework in* $\mathbb{R}^d$ *and let* A *be a nonsingular affine transformation. Then* $(G, \mathbf{p})$ *is infinitesimally rigid if and only if* $(G, A(\mathbf{p}))$ *is.*

In other words, infinitesimal rigidity is invariant under affine transformations. The following two statements are folklore, but we give proofs for completeness.

LEMMA 2.7. *Let* G *be a graph with* $n \geqslant d + 1$ *vertices and let* $(G, \mathbf{p})$ *be an infinitesimally rigid framework in* $\mathbb{R}^d$. *Then* $\mathbf{p}$ *has* d*-dimensional affine span.*

Proof. Any assignment of vectors orthogonal to the affine span of $\mathbf{p}$ is an infinitesimal flex of $(G, \mathbf{p})$. Hence, if $\mathbf{p}$ has defective affine span, there is, at least, an n-dimensional space of infinitesimal flexes of $(G, \mathbf{p})$ orthogonal to the affine span of $\mathbf{p}$. There is also, at least, a $\binom{d}{2}$-dimensional space (from rigid motions in dimension $d - 1$) of infinitesimal flexes within the affine span of $(G, \mathbf{p})$. Thus $(G, \mathbf{p})$ has infinitesimal flex space of dimension at least $\binom{d}{2} + d + 1 > \binom{d+1}{2}$. □

LEMMA 2.8. *Let* $(G, \mathbf{p})$ *be a framework. Then, up to congruence, there are only a finite number of configurations* $\mathbf{q}$ *so that* $(G, \mathbf{q})$ *is locally rigid and equivalent to* $(G, \mathbf{p})$.

Proof. The set of frameworks that are equivalent to $\mathbf{p}$ form an algebraic variety V. From the definition of local rigidity, if $\mathbf{q}$ is in V and locally rigid, then it is only connected in V to other frameworks in its congruence class (in fact only ones that do not involve reflection). Thus an infinite number of such $\mathbf{q}$ would imply an infinite number of connected components in V. But as a variety, V must have a finite number of connected components. □

DEFINITION 2.9. If $(G, \mathbf{p})$ is locally rigid for all generic configurations $\mathbf{p}$ in $\mathbb{R}^d$, then we say that G is *generically locally rigid* in $\mathbb{R}^d$. If $(G, \mathbf{p})$ is locally flexible for all generic configurations $\mathbf{p}$ in $\mathbb{R}^d$, then we say that G is *generically locally flexible* in $\mathbb{R}^d$.

If $(G, \mathbf{p})$ is infinitesimally rigid for all generic configurations $\mathbf{p}$ in $\mathbb{R}^d$, then we say that G is *generically infinitesimally rigid* in $\mathbb{R}^d$. If $(G, \mathbf{p})$ is infinitesimally flexible for all generic configurations $\mathbf{p}$ in $\mathbb{R}^d$, then we say that G is *generically infinitesimally flexible* in $\mathbb{R}^d$.

As described in [1], generic local rigidity is determined by generic infinitesimal rigidity.

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THEOREM 2.10 [1]. *If some framework* $(G, \mathbf{p})$ *in* $\mathbb{R}^d$ *is infinitesimally rigid in* $\mathbb{R}^d$*, then* G *is generically infinitesimally rigid in* $\mathbb{R}^d$ *and thus generically locally rigid in* $\mathbb{R}^d$*. If* G *is not generically infinitesimally rigid in* $\mathbb{R}^d$ *then it is generically locally flexible in* $\mathbb{R}^d$*. Thus, if* G *is not generically locally rigid in* $\mathbb{R}^d$ *then it is generically locally flexible in* $\mathbb{R}^d$*.*

2.2. Global rigidity. The following two results about generic global rigidity will be useful.

LEMMA 2.11. *Let* G *be generically globally rigid in* $\mathbb{R}^d$*. Then* G *is generically globally rigid in* $\mathbb{R}^{d-1}$*.*

Proof. If G is generically globally rigid in dimension d, then it remains so under *coning*, the process of adding one vertex and attaching it to all vertices in G. A result of Connelly and Whiteley, [6, Corollary 10], then implies that G is generically globally rigid in $\mathbb{R}^{d-1}$. □

A theorem of Hendrickson relates generic global rigidity and connectivity:

THEOREM 2.12 [16]. *Let* G *be generically globally rigid in* $\mathbb{R}^d$*. Then* G *is* $d+1$*-connected.*

Now we review idea of equilibrium stresses and how they relate to global rigidity.

DEFINITION 2.13. Given an ordered graph G, a *stress vector* $\omega = (\ldots, \omega_{ij}, \ldots)$, is an assignment of a real scalar $\omega_{ij} = \omega_{ji}$ to each edge, $\{i, j\}$ in G. (We have $\omega_{ij} = 0$ when $\{i, j\}$ is not an edge of G.)

We say that ω is an *equilibrium stress vector* for $(G, \mathbf{p})$ if the vector equation

$$\sum_j \omega_{ij}(\mathbf{p}_i - \mathbf{p}_j) = 0 \tag{2.2}$$

holds for all vertices i of G.

We associate an n-by-n *stress matrix* Ω to a stress vector ω, by setting the i, jth entry of Ω to $-\omega_{ij}$, for $i \neq j$, and the diagonal entries of Ω are set such that the row and column sums of Ω are zero.

If ω is an equilibrium stress vector for $(G, \mathbf{p})$ then we say that the associated Ω is an *equilibrium stress matrix* for $(G, \mathbf{p})$. For each of the d spatial dimensions, if we define a vector $\mathbf{v}$ in $\mathbb{R}^n$ by collecting the associated coordinate over all of the points in $\mathbf{p}$, we have $\Omega\mathbf{v} = 0$. The all-ones vector is also in the kernel of Ω. Thus

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if the dimension of the affine span of the vertices $\mathbf{p}$ is d, then the rank of Ω is at most $n-d-1$, but it could be less.

DEFINITION 2.14. Let S be a linear space of stress matrices. We define the *shared stress kernel* of S to be the subspace of $\mathbb{R}^n$ consisting of vectors in the kernel of every $\Omega \in S$.

The shared stress kernel of a framework $(G, \mathbf{p})$ is the shared stress kernel of the linear space of equilibrium stress matrices for $(G, \mathbf{p})$.

From the equilibrium condition, we see that the shared stress kernel of $(G, \mathbf{p})$ contains the d coordinates of $\mathbf{p}$ along with the all-ones vector. Thus, if the dimension of the affine span of the vertices $\mathbf{p}$ is d, then the dimension of the shared stress kernel is at least $d+1$, but it could be more.

Below is the central theorem we shall use that connects generic global rigidity with the dimension of the shared stress kernel at generic $\mathbf{p}$.

THEOREM 2.15 [13, Theorems 1.14 and 4.4]. *Let G be an ordered graph with $n \geqslant d+2$ vertices. If G is generically globally rigid in $\mathbb{R}^d$, then there is a generic $\mathbf{p}$ with an equilibrium stress matrix of rank $n-d-1$. Thus there is a generic $\mathbf{p}$ with a shared stress kernel of dimension $d+1$.*

If G is not generically globally rigid in $\mathbb{R}^d$, then every generic $\mathbf{p}$ has shared stress kernel of dimension $> d+1$. (This direction is essentially Connelly's sufficient condition [5] *as strengthened slightly in* [13, Section 4.2].*)*

REMARK 2.16. From general principles about matrices and rank, if one generic framework has an equilibrium stress matrix of rank $n-d-1$, then so too must all generic frameworks (see [17, Theorem 2.5] and [13, Lemma 5.8]). This also implies that every complex generic framework also must have an equilibrium stress matrix of rank $n-d-1$.

3. Measurement variety

In this section, we define the measurement variety and reduce Theorem 1.5 to a statement about measurement varieties.

From here on out, (unless where explicitly stated) we move the complex setting, where $\mathbf{p}$ is a configuration of n points in $\mathbb{C}^d$. This will allow us to apply basic machinery from algebraic geometry to our problem. Unless stated otherwise, we will always be dealing with the Zariski topology, where the closed sets are the algebraic subsets, and Zariski open subsets are obtained by removing a subvariety from a variety.

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DEFINITION 3.1. Let d be some fixed dimension and n a number of vertices. Let $G := \{E_1, \ldots, E_m\}$ be an ordered graph. The ordering on the edges of G fixes an association between each edge in G and a coordinate axis of $\mathbb{C}^m$. Let $m_G(\mathbf{p})$ be the map from d-dimensional configuration space to $\mathbb{C}^m$ measuring the squared lengths of the edges of G.

The *complex squared length* of the edge ij is

$$m_{ij}(\mathbf{p}) := \sum_{k=1}^{d} (\mathbf{p}_i^k - \mathbf{p}_j^k)^2$$

where k indexes over the d coordinates of $\mathbb{C}^d$. Here, we measure complex squared length using the complex square operation with no conjugation.

We denote by $M_{d,G}$ the closure of the image of $m_G(\cdot)$ over all d-dimensional configurations. This is an algebraic set, defined over $\mathbb{Q}$. We call this the (squared) *measurement variety* of G (in d dimensions).

As the closure of the image of an irreducible set (configuration space), under a polynomial map, the variety $M_{d,G}$ is irreducible. As $M_{d,G}$ contains all scales of all of its points, the variety is homogeneous.

In the complex setting, using the above definition for complex squared length, we can also define the concepts of congruence and infinitesimal/local/global rigidity in $\mathbb{C}^d$. Importantly, as described in the following result, moving to the complex setting will maintain the rigidity properties relevant to us. Thus, we may simply talk about ‘rigidity in d dimensions’, without specifying $\mathbb{R}^d$ or $\mathbb{C}^d$.

THEOREM 3.2. *A graph G is generically infinitesimally/locally/globally rigid in $\mathbb{R}^d$ if and only if it is so in $\mathbb{C}^d$.*

The case of generic global rigidity is proven in [**15**]. One direction of generic local rigidity is in [**29**]. For completeness, here we sketch a proof of the equivalence for generic infinitesimal and generic local rigidity.

Proof. First, we note that a generic real configuration in $\mathbb{R}^d$ is also generic as a complex configuration.

Second, the proof of Theorem 2.10 in [**1**], which equates generic infinitesimal rigidity to generic local rigidity, equally applies to the complex setting.

Finally, the rank of the rigidity matrix does not change when enlarging the field from $\mathbb{R}$ to $\mathbb{C}$ (because the determinant is defined over $\mathbb{Z}$), and so infinitesimal rigidity of a real generic $(G, \mathbf{p})$ will be the same in both fields. By the complex version of Theorem 2.10, generic local rigidity is proved as well. $\square$

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LEMMA 3.3. *If G is generically locally rigid in $\mathbb{C}^d$, then the image of $m_G(\cdot)$ acting on all configurations is $dn - \binom{d+1}{2}$-dimensional. Otherwise, the dimension of the image is smaller.*

Proof sketch. From Theorem 2.10, if G is generically locally rigid in $\mathbb{C}^d$ then it is generically infinitesimally rigid in $\mathbb{C}^d$. Thus the generic rank of the differential of $m_G(\cdot)$ is $dn - \binom{d+1}{2}$. From the constant rank theorem (as used in [1, Proposition 2]), the dimension of the image of $m_G(\cdot)$ is at least as big as the rank r of the differential at a generic $\mathbf{p}$. This is the largest differential rank of $m_G(\cdot)$ over the domain. Applying Sard's Theorem to $m_G(\cdot)$ (once the nonsmooth points of the image are removed, and then the preimages of these nonsmooth points are removed from the domain) tells us that inverse image of some (in fact, almost every) point in the image consists entirely of configurations $\mathbf{p}$, where the differential has rank at least as big as the dimension of the image of $m_G(\cdot)$. □

The main theorem about measurement varieties we will prove in this paper is the following:

THEOREM 3.4. *Suppose that $d \geqslant 2$ (or suppose that $d = 1$ and G is 3-connected). Let G and H be ordered graphs, both with $n \geqslant d + 2$ vertices and m edges. Suppose G is generically globally rigid in d dimensions. Suppose $M_{d,G} = M_{d,H}$. Then there is a vertex relabeling under which $G = H$.*

Assuming Theorem 3.4, we are ready to prove our main result.

Proof of Theorem 1.5. Lemma 3.3 implies that $M_{d,G}$ is an irreducible variety of dimension $dn - \binom{d+1}{2}$. Meanwhile, using Lemma 3.3 again, $M_{d,H}$ is an irreducible variety of dimension $\leqslant dn - \binom{d+1}{2}$, with equality if H is generically locally rigid in $\mathbb{C}^d$. (It is here where we need that H does not have more vertices than G.) The generic real configuration $\mathbf{p}$ is also generic as a point in $\mathbb{C}^{dn}$. The point $\mathbf{v} \in \mathbb{C}^m$ is, by assumption, in both $M_{d,G}$ and $M_{d,H}$ and by Lemma A.7, $\mathbf{v}$ is generic in $M_{d,G}$. This implies that we must have $M_{d,G} \subseteq M_{d,H}$, otherwise $\mathbf{v}$ would be cut out from $M_{d,G}$ by the one of the equations defining $M_{d,H}$, and thus rendering $\mathbf{v}$ nongeneric in $M_{d,G}$. So $M_{d,H}$ must be of dimension at least $dn - \binom{d+1}{2}$, and thus exactly $dn - \binom{d+1}{2}$.

Since $M_{d,G}$ and $M_{d,H}$ have the same dimension and $M_{d,H}$ is irreducible, $M_{d,G} \subseteq M_{d,H}$ implies that $M_{d,G} = M_{d,H}$.

Now we may apply Theorem 3.4 to conclude that there is a vertex relabeling such that $G = H$. Finally, from the assumption that G is generically globally rigid, we must have $\mathbf{p}$ congruent to $\mathbf{q}$. □

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With this settled, the next two sections develop the proof of Theorem 3.4. Briefly, the approach is by induction on dimension. This kind of induction was used in [10] to obtain a new proof of the result of Boutin and Kemper on complete graphs. The base case, $d = 1$, follows from a graph-theoretic result of Whitney via a connection between cycle spaces of graphs and projections of 1-dimensional measurement sets. This is done in Section 5. The connection between measurement varieties and Whitney's theorem was first explored in the unpublished manuscript [14]. The main results from [14] are included in Section 6. The more difficult step is the inductive one, which requires understanding the geometry of the measurement set $M_{d,G}$ well enough to identify the subvariety corresponding to $M_{d-1,G}$ intrinsically. That is the topic of the next section.

4. Getting down to $d = 1$

In this section we will prove the following proposition:

PROPOSITION 4.1. *Let G, an ordered graph on $n \geqslant d+2$ vertices with m edges, be generically globally rigid in $\mathbb{C}^d$ and let H be some ordered graph on n vertices with m edges. Suppose $M_{d,G} = M_{d,H}$. Then $M_{d-1,G} = M_{d-1,H}$ and, by induction, $M_{1,G} = M_{1,H}$.*

The basic strategy is to show that points $\mathbf{x}$ in $M_{d,G} \setminus M_{d-1,G}$ look intrinsically different in $M_{d,G}$ than points $\mathbf{y}$ of $M_{d,G}$ that are also in $M_{d-1,G}$. This means that these cases can be distinguished from the variety alone, without knowing the generating graph G. We will not simply be able to use smoothness as the distinguishing factor as there can be points in $M_{d,G} \setminus M_{d-1,G}$ that are not smooth. Our characterization will involve looking at Gauss fibers in $M_{d,G}$ in the neighborhood around such points. Luckily, from results in [5, 13], we have a reasonable understanding of these Gauss fibers (at least generically) and how they relate to equilibrium stresses of $(G, \mathbf{p})$ and affine transformations of $\mathbf{p}$. The key distinguishing features of these points are described in Propositions 4.20 and 4.21. The geometry that distinguishes points in $M_{d,G}$ that are also in $M_{d-1,G}$ is illustrated schematically in Figure 1.

In what follows, we will make the formal argument as weak as possible, only focusing on generic points, but we will also add remarks as we go along, with stronger statements for geometric intuition.

LEMMA 4.2. *Let G be generically locally rigid in $\mathbb{C}^d$, with $n \geqslant d+1$ vertices. Suppose $(G, \mathbf{p})$ is an infinitesimally flexible framework. Then the point $\mathbf{x} := m_G(\mathbf{p})$ is not generic in $M_{d,G}$.*

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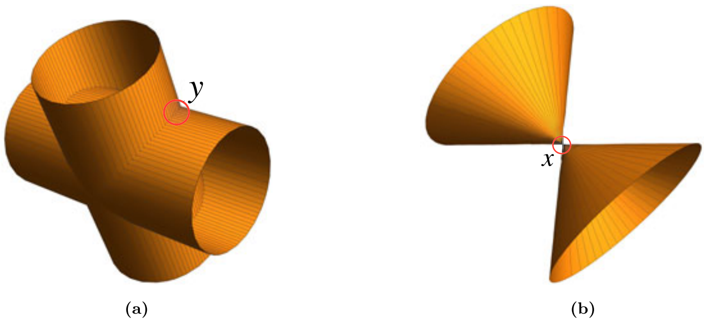

Figure 1. Two types of singular points on ruled varieties. (a) The Gauss fibers on a variety consisting of two intersecting cylinders consist of the ruling lines indicated. Points in the intersection of the two cylinders, such as the one marked y are in the singular locus, but still lie in the closure of a finite number of generic Gauss fibers (in this case, one ruling line from each cylinder). Proposition 4.20 says that measurements y that arise from configurations with full spans are either smooth points (in a single fiber closure) or lie in the closure of a finite number of generic Gauss fibers as in this figure. (b) The Gauss fibers on the elliptic cone also consist of ruling lines, as indicated. The cone point, marked as x, lies in the closure of an infinite number of ruling lines. This is a different situation than we saw (for y) in (a). Proposition 4.21 says that measurements x that arise from configurations with deficient spans lie in the closure of an infinite number of generic Gauss fibers as in this figure.

In particular, if $\mathbf{p}$ *has deficient affine span, then* $m_G(\mathbf{p})$ *is nongeneric.*

Proof sketch. From Theorem 2.10, G is generically locally rigid if and only if it is generically infinitesimally rigid if and only if the generic dimension of the differential of $m_G(\cdot)$ is $dn - \binom{d+1}{2}$.

If $\mathbf{x}$ is not a smooth point of $M_{d,G}$ then it cannot be generic and we are done.

Next we restrict the map $m_G(\cdot)$ by removing the nonsmooth points from $M_{d,G}$ and then removing the preimages of these nonsmooth points from the domain. By assumption, the configuration $\mathbf{p}$ is not a regular point of $m_G(\cdot)$, making $\mathbf{x}$ not a regular value of its image.

But from Sard's theorem applied to $m_G(\cdot)$, the set of critical values is of lower dimension. This set is also constructible and defined over $\mathbb{Q}$. This set remains of lower dimension under closure, thus the critical values must satisfy some extra equation, making them nongeneric.

By Lemma 2.7, any $(G, \mathbf{p})$ with deficient affine span is infinitesimally flexible when G has at least $d + 1$ vertices, giving the second part of the lemma. □

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DEFINITION 4.3. Fix d and G. We say that $\mathbf{x}$ is an *unhit* point of $M_{d,G}$ if there is no configuration $\mathbf{p}$ such that $\mathbf{x} = m_G(\mathbf{p})$. Otherwise it is *hit*.

LEMMA 4.4. *Let G be an ordered graph on $n \geqslant d+1$ vertices that is generically locally rigid in $\mathbb{C}^d$. Let $\mathbf{x}$ be generic in $M_{d,G}$. Then $\mathbf{x}$ is hit. Moreover, any configuration $\mathbf{p}$ hitting $\mathbf{x}$ is infinitesimally rigid and has full affine span.*

Proof. The hit set is an irreducible constructible set with $M_{d,G}$ as its closure. By Lemma A.4, it must then contain a nonempty (Zariski) open subset of $M_{d,G}$. Thus the unhit set must be contained in a closed subset (that is, a subvariety). This renders all unhit points nongeneric.

Infinitesimal rigidity follows from Lemma 4.2, which also gives us the stated span. □

DEFINITION 4.5. Let $\mathbf{p}$ be a configuration in $\mathbb{C}^d$ with a full affine span. Then the *open affine class* $\mathcal{A}(\mathbf{p})$ is the set of configurations that are affine images of $\mathbf{p}$, and are nondegenerate (have full span). An affine class is *generic* if it contains a generic configuration. (Generic affine classes exist, since $\mathcal{A}(\mathbf{p})$ is defined for every $\mathbf{p}$ with full span.)

Given a generic affine class $\mathcal{A}$, we define the generic locus $\mathcal{A}^g$ to be the subset of configurations in $\mathcal{A}$ that are also generic as configurations.

Let $\overline{\mathcal{A}(\mathbf{p})}$ be the closure of an affine class. This includes the degenerate affine images. $\overline{\mathcal{A}(\mathbf{p})}$ is a linear space.

LEMMA 4.6. *Let G be an ordered graph on n vertices and $\mathbf{p}$ a configuration of n points in $\mathbb{C}^d$. Then $m_G(\overline{\mathcal{A}(\mathbf{p})})$ is a linear space, and in particular, it is closed.*

Proof. For each edge ij of G, define its edge vector as $\mathbf{e} := \mathbf{p}_i - \mathbf{p}_j$ in $\mathbb{C}^d$. Then the complex squared length on that edge is the vector product $\mathbf{e}^t\mathbf{e}$.

An affine transform, A, applied to $\mathbf{p}$ can be expressed as $\mathbf{p}_i \to \mathbf{M}\mathbf{p}_i + \mathbf{t}$, where $\mathbf{M}$ is some d-by-d complex matrix and $\mathbf{t}$ is some (translation) vector. The effect on each edge vector is of the form $\mathbf{e}_{ij} \to \mathbf{M}\mathbf{e}_{ij}$. The effect on its squared length is $\mathbf{e}^t\mathbf{e} \to \mathbf{e}^t\mathbf{M}^t\mathbf{M}\mathbf{e} =: \mathbf{e}^t\mathbf{Q}\mathbf{e} = \mathrm{tr}(\mathbf{Q}\mathbf{e}\mathbf{e}^t)$, where $\mathbf{Q}$ is a symmetric matrix. Note that the rightmost expression is linear in $\mathbf{Q}$.

Since we are in the complex setting, using a Takagi factorization, every symmetric matrix $\mathbf{Q}$ arises in this form from some $\mathbf{M}$.

Thus, we can model the action of $m_G(\cdot)$ on $\overline{\mathcal{A}(\mathbf{p})}$ by defining a map $n_{G,\mathbf{p}}(\mathbf{Q})$ from symmetric $d \times d$ matrices $\mathbf{Q}$ to $\mathbb{C}^m$ that acts coordinate-wise as $n_{G,\mathbf{p}}(\mathbf{Q})_{ij} := \mathrm{tr}(\mathbf{Q}\mathbf{e}_{ij}\mathbf{e}_{ij}^t)$. Since $n_{G,\mathbf{p}}(\cdot)$ is a linear map acting on the linear space of symmetric matrices, its image, which is $n_G(\overline{\mathcal{A}(\mathbf{p})})$, is a linear subspace of $\mathbb{C}^m$ as claimed. □

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DEFINITION 4.7. Let V be an irreducible homogeneous variety. We define an (open) *Gauss fiber* F of V to be a maximal set of smooth points of V with a common tangent space. (For an inhomogeneous variety, we would instead have to work with affine tangent planes.) We say that F is a *generic Gauss fiber* if it contains a point that is generic in V. Given a generic Gauss fiber F of V, we define the generic locus F^g to be the subset of points in F that are also generic in V.

The term 'Gauss fiber' is used as it is the fiber above a point in the image of the (rational) Gauss map $\mathbf{x} \mapsto T_{\mathbf{x}}V$, taking each smooth point of V to the appropriate Grassmanian. Importantly, the definitions of F and F^g only depend on the geometry of the variety V, and not on any other information (such as how V may have been generated from some graph).

REMARK 4.8. A deeper result about ruled varieties states that if F is a generic Gauss fiber of any irreducible homogeneous variety, then its closure, $\overline{F}$, is always a linear space [8, Section 2.3.2], and in particular, irreducible. This also tells us that F^g is dense in F (Lemma A.6) and so $\overline{F^g} = \overline{F}$.

The next set of lemmas will establish a correspondence between generic Gauss fibers of $M_{d,G}$ and affine classes of configurations.

DEFINITION 4.9. Let V be a homogeneous variety in $\mathbb{C}^m$. Let $\mathbf{x}$ be a smooth point in V. Let ϕ be a nonzero element of $(\mathbb{C}^m)^*$. We say that ϕ is *tangent* to V at $\mathbf{x}$ if $T_{\mathbf{x}}V \subseteq \ker(\phi)$. We will call (with slight abuse of duality) such a ϕ a *tangential hyperplane*.

The following lemma relates an equilibrium stress vector for $(G, \mathbf{p})$ to the geometry of $M_{d,G}$ around $m_G(\mathbf{p})$.

LEMMA 4.10 [13, Lemma 2.21]. *Let G be an ordered graph with $n \geqslant d + 2$ vertices. Let $(G, \mathbf{p})$ be an infinitesimally rigid framework with $m_G(\mathbf{p})$ smooth in $M_{d,G}$ (such as when $\mathbf{p}$ is generic). A nonzero $\omega \in (\mathbb{C}^m)^*$ is tangent to $M_{d,G}$ at $m_G(\mathbf{p})$ if and only if ω is an equilibrium stress for $(G, \mathbf{p})$.*

REMARK 4.11. If $m_G(\mathbf{p})$ is smooth, but $(G, \mathbf{p})$ is infinitesimally flexible, then every tangential hyperplane ω is still an equilibrium stress for $(G, \mathbf{p})$, but the framework will also satisfy extra equilibrium stresses.

LEMMA 4.12. *Let G, an ordered graph on $n \geqslant d+2$ vertices with m edges, be generically globally rigid in $\mathbb{C}^d$. Let F be a generic Gauss fiber of $M_{d,G}$. Then there exists a single affine class $\mathcal{A}$ such that all $\mathbf{p}$ with $m_G(\mathbf{p}) \in F^g$ are in $\mathcal{A}$; that is, $m_G^{-1}(F^g) \subseteq \mathcal{A}$. This class $\mathcal{A}$ is generic.*

Proof. From Lemma 4.4, each $\mathbf{x} \in F^g$ is hit, giving us at least one $\mathbf{p}$ with $m_G(\mathbf{p}) = \mathbf{x}$. Also from Lemma 4.4, each such $\mathbf{p}$ is infinitesimally rigid and thus has a full span.

Lemma 4.10 then tells us that the equilibrium stresses for $(G, \mathbf{p})$ with $m_G(\mathbf{p}) \in F^g$ correspond to the tangential hyperplanes at $m_G(\mathbf{p})$. Since the tangents, and thus tangential hyperplanes, agree for all $\mathbf{x} \in F^g$, all such $\mathbf{p}$ share the same space S of equilibrium stresses.

From Lemma A.8, above any $\mathbf{x} \in F^g$ there is a generic configuration $\mathbf{q}$ and from Theorem 2.15 (see also Remark 2.16) $\mathbf{q}$ has a shared stress kernel of dimension $d+1$. Thus S must have a shared stress kernel of dimension $d+1$. This makes the dimension of the set of d-dimensional configurations having this stress space S equal to $d(d+1)$. In particular, this places all such $\mathbf{p}$ in some unique closed affine class $\bar{\mathcal{A}}$. This, along with the established affine span of $\mathbf{p}$ places it in $\mathcal{A}$. Genericity of $\mathcal{A}$ comes from the genericity of $\mathbf{q}$. □

In light of Lemma 4.12, the following is well defined.

DEFINITION 4.13. Let G, an ordered graph on $n \geqslant d+2$ vertices with m edges, be generically globally rigid in $\mathbb{C}^d$. Let F be a generic Gauss fiber of $M_{d,G}$. Define, by an overloading of notation, $\mathcal{A}(F)$ to be the generic affine class $\mathcal{A}(\mathbf{p})$ for any/every $\mathbf{p}$ above any $\mathbf{x} \in F^g$. We also denote by $\mathcal{A}(\cdot)$ the map $F \mapsto \mathcal{A}(F)$, which is defined for generic Gauss fibers of $M_{d,G}$.

REMARK 4.14. From Remark 4.11, when G is generically globally rigid and $\mathbf{q}$ is any configuration so that $m_G(\mathbf{q})$ is smooth and in a generic Gauss fiber F (even if $m_G(\mathbf{q})$ is not in F^g), then $\mathbf{q} \in \overline{\mathcal{A}(F)}$. Additionally, any such $\mathbf{q}$ must have an equilibrium stress matrix of rank $n-d-1$. If additionally, $\mathbf{q}$ has a full affine span, then $\mathbf{q} \in \mathcal{A}(F)$. (Later we will see that such $\mathbf{q}$, with $m_G(\mathbf{q})$ smooth, must in fact always have full affine span.)

LEMMA 4.15. *Let G, an ordered graph on $n \geqslant d+2$ vertices with m edges, be generically globally rigid in $\mathbb{C}^d$. If $\mathbf{p}^1$ and $\mathbf{p}^2$ are generic configurations and $\mathcal{A}(\mathbf{p}^2) = \mathcal{A}(\mathbf{p}^1)$, then $m_G(\mathbf{p}^1)$ and $m_G(\mathbf{p}^2)$ are both generic and in the same generic Gauss fiber of $M_{d,G}$. Thus, if F^1 and F^2 are two different generic Gauss fibers, then $\mathcal{A}(F^1) \neq \mathcal{A}(F^2)$.*

Proof. We proceed as in the proof of Lemma 4.12. Since $\mathbf{p}^1$ and $\mathbf{p}^2$ are nonsingular affine images of each other, they must satisfy all of the same equilibrium stress matrices. Thus $m_G(\mathbf{p}^1)$ and $m_G(\mathbf{p}^2)$ must have the same tangential hyperplanes, and be in the same Gauss fiber F of $M_{d,G}$. From Lemma A.7, the images $m_G(\mathbf{p}^i)$, $i = 1, 2$ are generic, so F is a generic Gauss fiber. □

We get the following corollary, which is also interesting in its own right.

PROPOSITION 4.16. *Let G, an ordered graph on $n \geqslant d + 2$ vertices with m edges, be generically globally rigid in $\mathbb{C}^d$. The map $\mathcal{A}(\cdot)$ gives a bijection between generic Gauss fibers of $M_{d,G}$ and generic affine classes. Finally, we have $F^g = m_G(\mathcal{A}(F)^g)$.*

Proof. Lemma 4.15 implies that the map $\mathcal{A}(\cdot)$ from generic Gauss fibers of $M_{d,G}$ to affine classes is injective. Lemma A.8 also implies that if F is a generic Gauss fiber of $M_{d,G}$ that $\mathcal{A}(F)$ is a generic affine class.

The map $\mathcal{A}(\cdot)$ is also surjective. By definition, a generic affine class arises as $\mathcal{A}(\mathbf{p})$ for a generic configuration $\mathbf{p}$. By Lemma A.7, the image $m_G(\mathbf{p})$ is generic in $M_{d,G}$. Hence the Gauss fiber containing $m_G(\mathbf{p})$ is generic. Since $\mathcal{A}(\mathbf{p})$ was an arbitrary generic affine class, we have surjectivity.

From Lemma 4.12 we have $m_G^{-1}(F^g) \subseteq \mathcal{A}(F)$. Since, from Lemma 4.4, each point in F^g is hit, this gives us $F^g \subseteq m_G(\mathcal{A}(F))$. From Lemma A.8, this means $F^g \subseteq m_G(\mathcal{A}(F)^g)$.

In the other direction, Lemma 4.15 gives us $F \supseteq m_G(\mathcal{A}(F)^g)$. From Lemma A.7, this means $F^g \supseteq m_G(\mathcal{A}(F)^g)$. □

The following is the main structural lemma that we will need going forward.

LEMMA 4.17. *Let G, an ordered graph on $n \geqslant d + 2$ vertices with m edges, be generically globally rigid in $\mathbb{C}^d$. Let F be a generic Gauss fiber of $M_{d,G}$. Then $\overline{F^g} = m_G(\overline{\mathcal{A}(F)})$.*

Proof. From Proposition 4.16 we have $F^g = m_G(\mathcal{A}(F)^g) \subseteq m_G(\mathcal{A}(F))$.

From Lemma A.6, $\mathcal{A}(F)^g$ is dense in $\overline{\mathcal{A}(F)}$ and so $\overline{\mathcal{A}(F)^g} = \overline{\mathcal{A}(F)}$. From continuity, we have $\overline{m_G(\mathcal{A}(F)^g)} \supseteq m_G(\overline{\mathcal{A}(F)^g})$. Thus $\overline{F^g} = \overline{m_G(\mathcal{A}(F)^g)} \supseteq m_G(\overline{\mathcal{A}(F)^g}) = m_G(\overline{\mathcal{A}(F)})$.

For the other direction, we have established above that $F^g \subseteq m_G(\mathcal{A}(F))$. Meanwhile, from Lemma 4.6, the image $m_G(\overline{\mathcal{A}(F)})$ is closed, and thus, from continuity, $\overline{m_G(\mathcal{A}(F))} = m_G(\overline{\mathcal{A}(F)})$. Thus $\overline{F^g} \subseteq \overline{m_G(\mathcal{A}(F))} = m_G(\overline{\mathcal{A}(F)})$. □

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REMARK 4.18. In light of Remark 4.8, we see that for a generically globally rigid graph G and generic Gauss fiber F of $M_{d,G}$, we actually have $\overline{F} = m_G(\overline{\mathcal{A}(F)})$. This also means that all points of $\overline{F}$ are hit.

LEMMA 4.19. *Let G, an ordered graph on $n \geqslant d+2$ vertices with m edges, be generically globally rigid in $\mathbb{C}^d$. Let $\mathbf{x}$ be any point (not necessarily generic) in $M_{d,G} \setminus M_{d-1,G}$ and in $\overline{F^g}$, for some generic Gauss fiber F of $M_{d,g}$. There must be a configuration $\mathbf{q}$ that has full span and is in $\mathcal{A}(F)$ such that $m_G(\mathbf{q}) = \mathbf{x}$. For such a $\mathbf{q}$, the framework $(G, \mathbf{q})$ must be infinitesimally rigid, and hence also be locally rigid.*

Proof. From Lemma 4.17 there must be some $\mathbf{q}$ in $\overline{\mathcal{A}(F)}$ such that $m_G(\mathbf{q}) = \mathbf{x}$. From the assumption that $\mathbf{x}$ is not in $M_{d-1,G}$, we know that $\mathbf{q}$ must have an affine span of dimension d. Thus $\mathcal{A}(\mathbf{q})$ is well defined and is equal to $\mathcal{A}(F)$.

If $(G, \mathbf{q})$ were infinitesimally flexible, then from Lemma 2.6 so too would all of the points in $\mathcal{A}(\mathbf{q})$, which equals $\mathcal{A}(F)$. But from Lemma 4.2, this would contradict the assumed genericity of F. Local rigidity follows from infinitesimal rigidity and Theorem 2.5. $\square$

The next two propositions form the central part of our argument. Informally, they say that we can distinguish between points in the measurement set that arise from lower-dimensional configurations from those that are merely singular by looking at the generic Gauss fibers going through them. Figure 1 gives a schematic of the two situations.

PROPOSITION 4.20. *Let G be an ordered graph with $n \geqslant d+2$ vertices that is generically globally rigid in $\mathbb{C}^d$. Let $\mathbf{x}$ be any point (not necessarily generic) in $M_{d,G} \setminus M_{d-1,G}$. Then there are at most a finite number of generic Gauss fibers F of $M_{d,G}$ with $\mathbf{x}$ in $\overline{F^g}$.*

Note that if $\mathbf{x}$ is a smooth point of $M_{d,G}$ then it is in a single generic Gauss fiber closure. But here, we are not making such assumptions on $\mathbf{x}$; for example, we will allow for $\mathbf{x}$ that are measurements of frameworks that are (due to nongenericity) not globally rigid. This can occur even in a generic affine class [6, Example 8.3].

Informally, the key idea is that if $\mathbf{x}$ is in an infinite number of $\overline{F^g}$, then it has preimage configurations from an infinite number of affine classes. From the full span assumption, this gives us an infinite number of preimage configurations, unrelated by congruence. Each of these preimage configurations will have to be locally rigid from the assumed genericity of each Gauss fiber. This would contradict the fact that, up to congruence, there can only be a finite number of locally rigid configurations with the same edge lengths.

Proof of Proposition 4.20. Let $\{F^i | i \in I\}$ be the collection of generic Gauss fibers of $M_{d,G}$ containing $\mathbf{x}$ in their closures. A priori, the index set I might be infinite. For every such F^i, from Lemma 4.19, there must be a configuration $\mathbf{q}^i$ in $\mathcal{A}(F^i)$ that has full span, is locally rigid and such that $m_G(\mathbf{q}^i) = \mathbf{x}$. Since $\mathbf{q}$ has full affine span, $\mathcal{A}(\mathbf{q}^i)$ is well defined and is equal to $\mathcal{A}(F^i)$. From Lemma 4.15, for any two such distinct F^i and F^j, we have $\mathcal{A}(\mathbf{q}^i) \neq \mathcal{A}(\mathbf{q}^j)$. Thus $\mathbf{q}^i$ cannot be congruent to $\mathbf{q}^j$.

Suppose there were an infinite number of F^i. Then there would be an infinite number of locally rigid congruence classes $[\mathbf{q}^i]$ that map to $\mathbf{x}$. But this contradicts Lemma 2.8. □

PROPOSITION 4.21. *Let G, an ordered graph on $n \geqslant d+2$ vertices with m edges, be generically globally rigid in $\mathbb{C}^d$. There is an $\mathbf{x}$, generic in $M_{d-1,G}$, such that there are an infinite number of Gauss fibers F of $M_{d,G}$ with $\mathbf{x}$ in $\overline{F^g}$.*

The key idea is that if $\mathbf{q}$ is a configuration with a deficient affine span, then there are an infinite number of configurations $\mathbf{p}$ with full affine spans such that $\mathbf{q} \in \overline{\mathcal{A}(\mathbf{p})}$. This will give us an infinite number of generic Gauss fibers with $\mathbf{x} = m_G(\mathbf{q})$ in their closures.

Proof. We start with a generic configuration $\mathbf{p}$. Let π be the projection from d-dimensional configurations to $d-1$ dimensional configurations that simply ignores the last spatial coordinate. Let $\mathbf{q} := \pi(\mathbf{p})$ and $\mathbf{x} := m_G(\mathbf{q})$. Since $\mathbf{p}$ is generic as a d-dimensional configuration, $\mathbf{q}$ is generic as a $(d-1)$-dimensional configuration, and $\mathbf{x}$ is generic in $M_{d-1,G}$.

Let F be the Gauss fiber of $M_{d,G}$ that contains $m_G(\mathbf{p})$. Since $\mathbf{q} \in \overline{\mathcal{A}(F)}$, from Lemma 4.17, $\mathbf{x}$ is in $\overline{F^g}$. This gives us one fiber F for the proposition. Now we show how to get more.

Define $\mathcal{L}(\mathbf{p}) := \pi^{-1}(\mathbf{q})$ to be the space of lifts of $\mathbf{q}$. The space of lifts is an affine space that contains $\mathbf{p}$, and so, by Lemma A.6, $\mathcal{L}(\mathbf{p})$ contains a dense set of generic configurations. Since $n \geqslant d+2$, we can find an infinite number of $\mathbf{p}'$ that are generic configurations, are in $\mathcal{L}(\mathbf{p})$ and with each in a different affine class. (Any finite number of affine classes are contained in a finite number of strict subvarieties of the linear lifting space, and thus cannot cover all of the generic configurations.)

For any such configuration, say $\mathbf{p}'$, that is not in $\mathcal{A}(\mathbf{p})$, we can apply the same argument to get another Gauss fiber F' with $\mathbf{x}$ in $\overline{F'^g}$. From Lemma 4.12, we have

$F \neq F'$. Since we can do this endlessly, we obtain our infinite number of fibers for $\mathbf{x}$. $\square$

REMARK 4.22. Any $\mathbf{x}$ meeting the hypotheses of Proposition 4.21 cannot be a smooth point in $M_{d,G}$ (as it is in the closure of multiple Gauss fibers, and the Gauss map, where defined, is continuous). Since this $\mathbf{x}$ is also generic in $M_{d-1,G}$, we can conclude that all of $M_{d-1,G}$ lies in the singular locus of $M_{d,G}$.

REMARK 4.23. To recap, every $\mathbf{q}$ with $m_G(\mathbf{q})$ smooth and in a generic Gauss fiber F of a generically globally rigid graph G, has a full affine span, is infinitesimally rigid, and is in $\mathcal{A}(F)$. Such a framework $(G, \mathbf{q})$ must be globally rigid [5].

The closure of F is the linear space $m_G(\overline{\mathcal{A}(F)})$. This means that all points in $\overline{F}$ are hit. It also means that, for each point $\mathbf{x} \in \overline{F}$, there must be some point $\mathbf{q}$ in $\overline{\mathcal{A}(F)}$ with $m_G(\mathbf{q}) = \mathbf{x}$.

Points that are smooth in $M_{d,G}$ are in only one Gauss fiber and one Gauss fiber closure.

Let F be a generic Gauss fiber. Let the 'bad' points be $B := \overline{F} \setminus F$. These are nonsmooth in $M_{d,G}$. Points that are in B due to deficient span will, generically, be in the closure of infinitely many distinct generic Gauss fibers. All 'other' points in B (no deficient span in the preimage) can be in only a finite number of generic $\overline{F}$. (Any of the preimages $\mathbf{q}$ of these other bad points is also infinitesimally rigid.) These other bad points can occur, say, when $(G, \mathbf{q})$ is not globally rigid. (Note that global rigidity of frameworks is not an affine invariant property [6, Example 8.3].)

At nongeneric Gauss fibers F of $M_{d,G}$, most bets are off. F can be of some larger dimension, and conceivably be reducible. It is even conceivable that there are $\mathbf{q}$ that have full spans and are infinitesimally flexible but such that $m_G(\mathbf{q})$ is still a smooth point of $M_{d,G}$ (in such a nongeneric F).

The next lemmas will let us treat G and H symmetrically. We start with a technical result about Gauss fiber dimension that allows us to identify generic global rigidity intrinsically in $M_{d,G}$.

LEMMA 4.24. *Let G be a generically locally rigid graph with $n \geqslant d+2$ vertices and m edges. Suppose that, at some generic $\mathbf{p}$ the shared stress kernel of $(G, \mathbf{p})$ has dimension $k \geqslant d+1$. Let F be the Gauss fiber of $M_{d,G}$ containing $m_G(\mathbf{p})$. Then the dimension of $\overline{F^g}$ is $dk - \binom{d+1}{2}$.*

Proof. Let $\mathbf{p}$ be a generic configuration. Define K to be the space of configurations $\mathbf{q}$ such that $(G, \mathbf{q})$ satisfies all the equilibrium stresses $(G, \mathbf{p})$ does. This K is a linear space of dimension dk.

As discussed in Definition 2.14, k, the dimension of the shared stress kernel, is at least $d+1$. Hence, K is of dimension at least $d(d+1)$. Moreover (see Definition 2.13) K must include all affine images of $\mathbf{p}$, including all $\mathbf{q}$ that are congruent to $\mathbf{p}$.

Let K^g be the configurations in K that are generic in configuration space. From Lemma A.6, K^g is a dense subset of K, which is closed, and so $\overline{K^g} = K$.

Let F be the Gauss fiber of $M_{d,G}$ containing $m_G(\mathbf{p})$; F is generic because $\mathbf{p}$ is. Now that we know F is a generic affine class, we can show that $F^g = m_G(K^g)$ by following the proof of the same statement in Proposition 4.16. This gives us $\overline{F^g} = \overline{m_G(K^g)} = \overline{m_G(\overline{K^g})} = \overline{m_G(K)}$. The second equality is due to continuity.

From the above, we get $\dim(\overline{F^g}) = \dim(\overline{m_G(K)}) = \dim(m_G(K))$. The second equality is due to the fact that a constructible set and its Zariski closure have the same dimension. (In the case that G is generically globally rigid, then $K = \overline{\mathcal{A}(F)}$ and due to Lemmas 4.6 and 4.17 we actually have $\overline{F^g} = m_G(K)$.)

By generic local rigidity of G, we know that the maximum rank of dm_G, restricted to its action on K, is $dk - \binom{d+1}{2}$. (Recall that K includes all configurations that are congruent to $\mathbf{p}$. It follows that, at $\mathbf{p}$, the differential of m_G, acting as a map restricted to K, has a kernel of dimension at least $\binom{d+1}{2}$, corresponding to the trivial infinitesimal flexes. If this kernel at our generic $\mathbf{p}$ were any larger, $(G, \mathbf{p})$ could not be infinitesimally rigid.)

Using the constant rank theorem and Sard's Theorem as in the proof of Lemma 3.3, we conclude that the dimension of $\overline{F^g}$ is exactly this size. □

REMARK 4.25. Using the fact that, in fact, F is irreducible (see Remark 4.8), we can improve Lemma 4.24 to give the dimension of $\overline{F}$ instead of $\overline{F^g}$.

We now interpret Lemma 4.24 in terms of generic global rigidity.

LEMMA 4.26. *Let G, an ordered graph on $n \geqslant d+2$ vertices with m edges, be generically globally rigid in $\mathbb{C}^d$. Let H be some ordered graph also with n vertices with m edges. Suppose $M_{d,G} = M_{d,H}$. Then H is also generically globally rigid in $\mathbb{C}^d$.*

Proof. Since $M_{d,G} = M_{d,H}$, they, in particular have the same dimension. If G is generically globally rigid, then it is also generically locally rigid. By Lemma 3.3 this implies that H (having the same number of vertices as G) is also generically locally rigid.

Because $M_{d,G}$ and $M_{d,H}$ are the same variety, the generic Gauss fiber F of $M_{d,G}$ containing $\mathbf{x} := m_G(\mathbf{p})$ is also the generic Gauss fiber of $M_{x,H}$ containing $\mathbf{x}$ (and $\mathbf{x}$ will have some other generic preimage $\mathbf{q}$ under $m_H(\cdot)$ by Lemma A.8).

Lemma 4.24 then lets us compute the dimension of $\overline{F^g}$ in two different ways, using the shared stress kernel of $(G, \mathbf{p})$ and $(H, \mathbf{p})$, respectively. Since G is generically globally rigid, Theorem 2.15 implies its shared stress kernel is $(d+1)$-dimensional. To avoid a contradiction from Lemma 4.24, $(H, \mathbf{p})$ must also have a shared stress kernel of dimension $d+1$, in which case Theorem 2.15 implies that H is also generically globally rigid. □

With all this in hand, we can complete the proof of the main proposition of this section.

Proof of Proposition 4.1. From Proposition 4.21, there is a point $\mathbf{x}$ generic in $M_{d-1,G}$ in an infinite number of $\overline{F^g}$. From Lemma 4.26 and Proposition 4.20, $\mathbf{x}$ must also be in $M_{d-1,H}$. Since $\mathbf{x}$ is generic, this implies that $M_{d-1,G} \subseteq M_{d-1,H}$ (otherwise the equations of $M_{d-1,H}$ would certify $\mathbf{x}$ as nongeneric). From Lemma 4.26 we can apply the same argument in the other direction to conclude $M_{d-1,H} \subseteq M_{d-1,G}$.

For the induction, we use Lemma 2.11. □

5. $d = 1$

In this section, we prove the following proposition:

PROPOSITION 5.1. *Let G and H be ordered graphs, with $\geqslant 3$ vertices. Suppose that G is 3-connected, H has no isolated vertices, and $M_{1,G} = M_{1,H}$. Then there is a vertex relabeling under which $G = H$.*

This establishes the base case for an inductive proof of Theorem 3.4, which we prove at the end of the section.

DEFINITION 5.2. Let $V \subseteq \mathbb{C}^N$ be an irreducible affine variety. Let L be a linear subspace. Let π_L denote the quotient map taking $\mathbb{C}^N$ to $\mathbb{C}^N/L$.

Let $[N] = \{1, \ldots, n\}$. For each $I \subseteq [N]$ the *coordinate subspace* S_I is the linear span of the coordinate vectors indexed by I; that is, $S_I = \operatorname{lin}\{e_i : i \in I\}$. Define $\overline{I} := [N] \setminus I$ for $I \subseteq [N]$. A coordinate subspace S_I is *independent* in V if the dimension of $\pi_{S_{\overline{I}}}(V)$ is $|I|$. Otherwise S_I is *dependent* in V.

LEMMA 5.3. *Let E' be the subset of the edges of G. $S_{E'}$ is independent in $M_{1,G}$ if and only if the edges of E' form a forest over the vertices of G.*

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Proof. If E' is a forest, then given any target measurement values in $\mathbb{C}^{|E'|}$, we can traverse the forest and sequentially place the vertices in $\mathbb{C}^1$ to achieve this measurement.

Conversely, if E' is not a forest, then it contains a cycle C. The sum of the vectors connecting the points of C in $\mathbb{C}^1$ must sum to zero, giving us a nontrivial equation that must be satisfied. □

DEFINITION 5.4. A subset of edges E' of a graph G is *cycle supported* if the edges of E', in some order, form a simple cycle in G. An edge bijection σ between two graphs G and H, is a *cycle isomorphism* if it maps cycle supported sets, and only cycle supported sets, to cycle supported sets.

LEMMA 5.5. *Let G and H be ordered graphs with m edges. Suppose $M_{1,G} = M_{1,H}$. Then the mapping taking the ordered edges of G to the ordered edges of H is a cycle isomorphism.*

Proof. Suppose the mapping is not a cycle isomorphism. Then without loss of generality, there is a set of edges C that form a simple cycle in G and not H.

Suppose that this C forms a forest in H. Then from Lemma 5.3, we have $\pi_{S_{\overline{C}}}(M_{1,G}) \neq \pi_{S_{\overline{C}}}(M_{1,H})$ and thus $M_{1,G} \neq M_{1,H}$.

Suppose instead that this C is neither a simple cycle nor a forest in H, then there must be an edge e such that $C' := C - e$ is not a forest in H, while C' is a forest in G (a simple cycle minus one edge is a path). Then from Lemma 5.3, we have $\pi_{S_{\overline{C'}}}(M_{1,G}) \neq \pi_{S_{\overline{C'}}}(M_{1,H})$ and thus $M_{1,G} \neq M_{1,H}$. □

A theorem of Whitney [30] (see also [26]) allows us to upgrade cycle isomorphisms to graph isomorphisms.

THEOREM 5.6. *Let G and H be two graphs, with G being 3-connected, and H with no isolated vertices. An edge bijection that is a cycle isomorphism must arise from a graph isomorphism.*

REMARK 5.7. Another way to state Whitney's theorem is that if G and H have isomorphic graphic matroids and no isolated vertices, and G is 3-connected, then G and H are isomorphic as graphs. In particular, topological information contained in the ordering of the edges on a cycle is not part of the hypothesis, nor did we consider it in Lemma 5.5.

The notion of cycle isomorphism is equivalent to having isomorphic graphic matroids, so it could also be formulated in terms of 'forest isomorphisms'.

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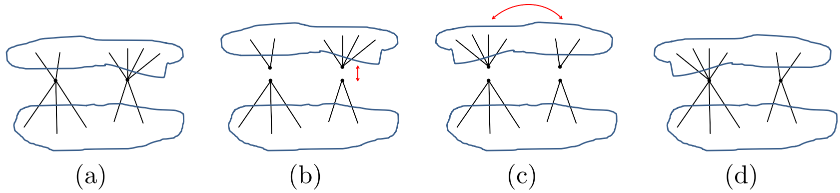

Figure 2. The reversal operation. The graphs, (a) and (d) are 2-isomorphic but not isomorphic. Note that the edge lengths of the frameworks are unchanged under a 2-isomorphism.

REMARK 5.8. If G is not 3-connected, then there are cycle isomorphisms between G and nonisomorphic H. Whitney [31] showed that these belong to a restricted class of '1-isomorphisms' and '2-isomorphisms.' See Figure 2 for an example of a pair of 2-isomorphic graphs.

With this in hand, we can prove the main proposition of this section.

Proof of Proposition 5.1. From Lemma 5.5 the mapping taking the edges of G to H must be a cycle isomorphism. Then from the assumed 3-connectivity and Theorem 5.6 this mapping must arise from a vertex relabeling. □

The main structural theorem of this paper now follows.

Proof of Theorem 3.4. From Proposition 4.1 we can reduce the problem from d dimensions down to 1. Since G is generically globally rigid in $d \geqslant 2$ dimensions, from Theorem 2.12 it must be 3-connected. Since the number of vertices in both graphs is the same and $M_{1,G} = M_{1,H}$, from Lemma 3.3, H cannot have isolated vertices. The result then follows from Proposition 5.1. □

As proven in the end of Section 3, this immediately proves the main result of this paper, Theorem 1.5. □

6. Bonus result

There is an interesting variant of Theorem 3.4 that was originally reported in the unpublished manuscript [14]. For this theorem we will replace the hypothesis that G is generically globally rigid in d dimensions with the far weaker one that G is 3-connected. However, we require not only equality of measurement varieties, but also equality of Euclidean measurement sets.

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DEFINITION 6.1. Let d be some fixed dimension and n a number of vertices. Let $G := \{E_1, \ldots, E_m\}$ be an ordered graph. The ordering on the edges of G fixes an association between each edge in G and a coordinate axis of $\mathbb{R}^m$.

We denote by $M^{\mathbb{E}}_{d,G}$ the image of $m^{\mathbb{E}}_G(\cdot)$ over all real d-dimensional configurations. We call this the (squared) *Euclidean measurement set* of G (in d dimensions). This is a real semialgebraic set, defined over $\mathbb{Q}$.

THEOREM 6.2. *Let d be fixed. Let G be an ordered 3-connected graph on n vertices with m edges, and H some ordered graph with no isolated vertices and with m edges. Suppose $M^{\mathbb{E}}_{d,G} = M^{\mathbb{E}}_{d,H}$. Then there is a vertex relabeling of H such that $G = H$.*

REMARK 6.3. This theorem does not rule out the possibility that there are two nonisomorphic graphs G and H such that $M^{\mathbb{E}}_{d,G} \cap M^{\mathbb{E}}_{d,H}$ contains a standard-topology open set. This would imply that $M_{d,G}$ is equal to $M_{d,H}$ even though $M^{\mathbb{E}}_{d,G} \neq M^{\mathbb{E}}_{d,H}$. In this case, there could be some generic Euclidean measurements that are achievable from both graphs and some generic Euclidean measurements that are achievable only in one graph.

As a result, Theorem 6.2 does not help us to prove Theorem 1.5.

The rest of this section proves Theorem 6.2. The steps are similar to the ones we used in Section 5, but in the present setting, they work immediately in dimensions greater than one.

LEMMA 6.4. *Let E' be the subset of the edges of G. $\pi_{S_{\overline{E'}}}(M^{\mathbb{E}}_{d,G})$ equals the entire first octant if and only if the edges of E' form a forest over the vertices of G.*

Proof. If E' is a forest, then given any target measurements point in the first octant of $\mathbb{R}^{|E'|}$, we can traverse the forest and sequentially place the vertices in $\mathbb{R}^d$ to achieve this measurement.

Conversely, if E' is not a forest, then it contains a cycle C on k edges, for some k. In this case, $\pi_{S_{\overline{E'}}}(M^{\mathbb{E}}_{d,G})$ cannot be the entire first octant of $\mathbb{R}^k$ since there is no real framework (in any dimension) where all but one of the edges of the cycle has zero length. □

LEMMA 6.5. *Let G and H be ordered graphs with m edges. Suppose $M^{\mathbb{E}}_{d,G} = M^{\mathbb{E}}_{d,H}$. Then the mapping taking the ordered edges of G to the ordered edges of H is a cycle isomorphism.*

Proof. Suppose the mapping is not a cycle isomorphism. Then without loss of generality there is a set of edges C that from a simple cycle in G and not H.

Suppose that C is a forest in H. Then from Lemma 6.4, we have $\pi_{S_{\overline{C}}}(M^{\mathbb{E}}_{d,G}) \neq \pi_{S_{\overline{C}}}(M^{\mathbb{E}}_{d,H})$ and thus $M^{\mathbb{E}}_{d,G} \neq M^{\mathbb{E}}_{d,H}$.

Suppose that C is neither a simple cycle nor a forest in H, then there must be an edge e such that $C' := C - e$ is not a forest in H, while C' is a forest in G. Then from Lemma 6.4, we have $\pi_{S_{\overline{C'}}}(M^{\mathbb{E}}_{d,G}) \neq \pi_{S_{\overline{C'}}}(M^{\mathbb{E}}_{d,H})$ and thus $M^{\mathbb{E}}_{d,G} \neq M^{\mathbb{E}}_{d,H}$. □

Proof of Theorem 6.2. The theorem now follows directly from Lemma 6.5, the assumed 3-connectivity, and Theorem 5.6. □

7. Remaining issues

This paper answers some central questions about the relationships between graphs and their measurement varieties/sets. There are a few natural remaining questions.

7.1. Redundant rigidity.

Theorem 1.5 is tight in the sense that if G is not generically globally rigid in d dimensions, then clearly we cannot determine $\mathbf{p}$ from $\mathbf{v}$. But it is still possible that one might be able to determine G from $\mathbf{v}$. Here we discuss a possible strengthening of Theorem 3.4.

DEFINITION 7.1. We say that a graph G is *generically redundantly rigid* in $\mathbb{R}^d$ if G is generically locally rigid in $\mathbb{R}^d$ and remains so after the removal of any single edge.

QUESTION 7.2. *Is the following claim true:*

Let G, an ordered graph with $n \geqslant d+2$ vertices and m edges, be 3-connected and generically redundantly rigid in d dimensions. Let H be some ordered graph on n vertices with m edges. Suppose $M_{d,G} = M_{d,H}$. Then there is a vertex relabeling under which $G = H$.

The claim is true for $d = 2$, since in two dimensions, redundant rigidity and 3-connectivity imply generic global rigidity [**5**, **18**].

In terms of the ingredients used for proving Theorem 3.4, we note that the conclusion of Proposition 4.20 is false when G is merely generically redundantly rigid. For example the complete bipartite graph, $K_{5,5}$, is redundantly rigid in three dimensions and is 4-connected. But for any configuration $\mathbf{q}$ where even one of its 'parts' has a deficient span, there will be an infinite number of generic Gauss

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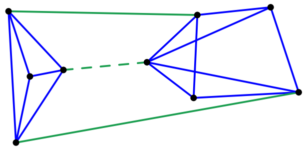

Figure 3. A pair of nonisomorphic graphs with the same measurement variety. Any of the green edges in the graphs (a) and (b), if removed, result in a graph that is generically flexible. As described in Remark 7.4, these graphs have the same measurement variety (it is the product of the measurement varieties of the complete graph K_4, the wheel W_4 and $\mathbb{C}^3$). However, the graphs (a) and (b) are not isomorphic, because the dashed green edges can be distinguished from each other by the degree of the endpoint on the right.

fibers with $m_G(\mathbf{q})$ in their closure. This is because equilibrium stresses for generic $\mathbf{p}$, which are all rank 2, only enforce affine relations within each of the parts [3].

A positive answer to Question 7.2 would directly imply Theorem 3.4 due to Theorem 2.12 and the following theorem of Hendrickson [16].

THEOREM 7.3. *If G is generically globally rigid in $\mathbb{R}^d$, with $n \geqslant d+2$ then it is redundantly rigid in $\mathbb{R}^d$.*

REMARK 7.4. A positive answer to Question 7.2 would give us a reasonably tight characterization of measurement variety agreement in light of the following.

Suppose that G is not generically redundantly rigid, and let e be an edge of G so that $G' := G - e$ is generically locally flexible. From the size of G, there must be a nonedge e' of G different from e whose lengths can be changed under a continuous flex of G'. Let G'' be the graph obtained from G by replacing e with e'. Then $M_{d,G} = M_{d,G''}$ as both are equal to $M_{d,G'} \oplus \mathbb{C}^1$. (See an example in Figure 3.) But the mapping from G to G'' will not be an isomorphism for graphs unless e and e' are in the same orbit of $\mathrm{Aut}(G + e')$. This is a very restrictive condition on G that any extension of our results to graphs that are not generically redundantly rigid will have to include. (Garamvölgyi and Jordán [9] explore the question of when a nonredundantly rigid graph can be reconstructed from edge-length measurements.)

There are also some more unresolved issues about measurement sets.

QUESTION 7.5. *Can the assumption that G and H have the same number of vertices be dropped from Theorem 3.4? (This open up the possibility that H has* *more vertices, but is generically locally flexible. Garamvölgyi and Jordán* [9] *give an affirmative answer in dimensions one and two.)*

More generally, we know very little about what assumptions other than dimension can let us conclude for general d that $M_{d,G} \subseteq M_{d,H}$ implies $M_{d,G} = M_{d,H}$.

7.2. Unlabeled graph realization. The *graph realization* (or *distance geometry*) problem asks to reconstruct an unknown configuration $\mathbf{p}$ given a graph G, dimension d, and labeled edge-length measurements $\mathbf{v} = m_G^{\mathbb{E}}(\mathbf{p})$. From Theorem 1.4, if we assume that $\mathbf{p}$ is generic and know that G is generically globally rigid, and we can find any $\mathbf{q}$ at all (not necessarily generic) so that $\mathbf{v} = m_G^{\mathbb{E}}(\mathbf{q})$, then we know that $\mathbf{q} = \mathbf{p}$ (up to congruence). As a practical matter, it is important that $\mathbf{q}$ need not be generic, since this is a very strong restriction on (or assumption about) any specific algorithm (as opposed to the process generating the input $\mathbf{p}$).

Theorem 1.5 does not immediately give us the analogous result for *unlabeled* distance geometry. The subtlety is that the hypotheses of Theorem 1.5 include knowledge about G, which we will not have access to in an unlabeled distance geometry instance. Unlike in the labeled case, $\mathbf{v}$ by itself does not immediately tell us whether our problem is generically well posed.

Examining our proofs, we get a partial result in this direction:

THEOREM 7.6. *In any fixed dimension $d \geqslant 2$, let $\mathbf{p}$ be a generic configuration of $n \geqslant d+2$ points. Let $\mathbf{v} = m_G^{\mathbb{E}}(\mathbf{p})$, where G is an ordered graph (with n vertices and m edges) that is generically locally rigid in $\mathbb{R}^d$.*

Suppose there is a configuration $\mathbf{q}$, also of n points, along with an ordered graph H (with n vertices and m edges) that is generically globally rigid and such that $\mathbf{v} = m_H^{\mathbb{E}}(\mathbf{q})$.

Then there is a vertex relabeling of H such that $G = H$. Moreover, under this vertex relabeling, up to congruence, $\mathbf{q} = \mathbf{p}$.

Proof sketch. The derivation of Theorem 1.5 from Theorem 3.4 works nearly unmodified. Both G and H are generically locally rigid by hypothesis and so $M_{d,G}$ and $M_{d,H}$ are of the same dimension. The configuration $\mathbf{p}$ maps to a generic point in the intersection of $M_{d,G}$ and $M_{d,H}$, so the two measurement varieties are equal. When applying 3.4, we rely on the generic global rigidity of H instead of G. The rest of the proof then goes through unchanged. $\square$

In this version, we only need to assume that G is generically locally rigid, instead of generically globally rigid. Theorem 7.6 then tells us that H (whose generic global rigidity can be tested in a realization setting) and $\mathbf{q}$ certify that the input problem is, in fact, well posed, and that we have found its solution. This version still makes some assumptions on G and the number of vertices in H.

This motivates the following question.

QUESTION 7.7. *Is the following claim true:*

In any fixed dimension $d \geqslant 2$, let $\mathbf{p}$ be a generic configuration of $n \geqslant d+2$ points. Let $\mathbf{v} = m^{\mathbb{E}}_G(\mathbf{p})$, where G is an ordered graph (with n vertices and m edges). Let $\mathbf{p}_S$ be the subconfiguration of $\mathbf{p}$ indexed by the vertices within the support of G.

Suppose there is a configuration $\mathbf{q}$, of n' points, with no two points coincident, along with an ordered generically globally rigid graph H (with n' vertices and m' edges) such that $\mathbf{v} = m^{\mathbb{E}}_H(\mathbf{q})$.

Then there is a vertex relabeling of $\mathbf{p}_S$ such that, up to congruence, $\mathbf{q} = \mathbf{p}_S$. Moreover, under this vertex relabeling, $G = H$.

A positive answer to this question would mean that, under the assumption that $\mathbf{p}$ is generic, such an $(H, \mathbf{q})$ would be a certificate that we have correctly realized the measured subconfiguration of $\mathbf{p}$.

The difficulty for this question is that we do not know how to rule out the possibility that $M_{d,G} \subsetneq M_{d,H}$. This is related to the issues mentioned at the end of Section 7.1.

There is one special case of note. The claim of this question is true when H is the complete graph K_{d+2}, (see [10, Proposition 4.23]). This is due to the fact that, aside from K_{d+2}, any other graph G with $N := \binom{d+1}{2}$ distinct edges has the property that every subset of edges is independent. Hence, the measurement variety, $M_{d,G}$, of G must be equal to all of $\mathbb{C}^N$, and so it cannot be a subset of $M_{d,H}$. Applying this idea iteratively, it can be shown that the claim of the question remains true if H allows for trilateration [10]. This fact allows one to apply trilateration to an unlabeled set of measurements, as is done in [20], without any assumptions on G or n.

7.3. Matrix completion.

A variant of global rigidity is ‘matrix completion’, which asks whether all the entries of an $m \times n$ matrix A of (low) rank r can be determined by a subset of its entries (at known positions). (See [28] for complete definitions and background.)

The algebraic setup (see [22]) takes A as a point on the determinantal variety of $m \times n$ matrices of rank at most r, and the observation process is the projection onto coordinates corresponding to the entries. The closure of the image of this projection corresponds to the measurement variety of a framework. For complex

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matrix completion, a result of [22] says that whether an observation pattern has a unique completion is a generic property. This means it makes sense to ask whether our results also hold in the matrix completion setting.

The following rank 3 examples are from [23] (a preprint version of [22]).

$$\begin{pmatrix} \star & \star & \star & \star & \star \\ \star & \star & \star & \star & \star \\ \star & \star & \star & \star & \star \\ \star & \star & \star & ? & ? \\ \star & \star & \star & ? & ? \\ \star & \star & \star & ? & ? \end{pmatrix} \qquad \begin{pmatrix} \star & \star & \star & \star & ? \\ \star & \star & \star & \star & ? \\ \star & \star & ? & \star & \star \\ \star & \star & ? & \star & \star \\ ? & \star & \star & \star & \star \\ ? & \star & \star & \star & \star \end{pmatrix}$$

It is shown there that, for each of these, the projection onto the known entries (labeled '$\star$') is dominant. Hence, they both have the same 'measurement varieties'. Additionally, if the underlying matrix is generic, there is exactly one way to fill in the unknown entries (labeled '?'), so they are also 'globally rigid'.

Importantly, they are *not* related by row and column permutations, so we have a counterexample to the straightforward translation of our main results to the matrix completion setting. (What goes wrong is that the stress criterion for global rigidity is not necessary for matrix completion. This was first observed in [28].)

On the other hand, as noted in [10, Remark 4.20], the matrix completion analogue of Boutin and Kemper's result for complete graphs is straightforward. Clarifying the relationship between unlabeled matrix completion and unlabeled rigidity would be interesting.

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Appendix A. Algebraic geometry background

DEFINITION A.1. A (complex embedded affine) *variety* (or *algebraic set*), V, is a (not necessarily strict) subset of $\mathbb{C}^N$, for some N, that is defined by the simultaneous vanishing of a finite set of polynomial equations with coefficients in $\mathbb{C}$ in the variables $x_1, x_2, \ldots, x_N$ which are associated with the coordinate axes of $\mathbb{C}^N$. We say that V is *defined over* $\mathbb{Q}$ if it can be defined by polynomials with coefficients in $\mathbb{Q}$.

A variety is homogeneous if its ideal is finitely generated by homogeneous polynomials. This is the same as the set V being a cone with its vertex at 0.

A variety can be stratified as a union of a finite number of complex analytic submanifolds of $\mathbb{C}^N$. A variety V has a well defined (maximal) *dimension* $\mathrm{Dim}(V)$, which will agree with the largest D for which there is a standard-topology open subset of V, that is a D-dimensional complex analytic submanifold of $\mathbb{C}^N$.

The set of polynomials that vanish on V form a radical ideal $I(V)$, which is generated by a finite set of polynomials.

A variety V is *reducible* if it is the proper union of two varieties V_1 and V_2. Otherwise it is called *irreducible*. A variety has a unique decomposition as a finite proper union of its maximal irreducible subvarieties called *components*.

Any (strict) subvariety W of an irreducible variety V must be of strictly lower dimension.

A subset W of a variety V is called *Zariski closed* if W is a variety.

DEFINITION A.2. The *Zariski tangent space* at a point $\mathbf{x}$ of a variety V is the kernel of the Jacobian matrix of a set of generating polynomials for $I(V)$ evaluated at $\mathbf{x}$.

A point $\mathbf{x}$ of an irreducible variety V is called (algebraically) *smooth* in V if the dimension of the Zariski tangent space equals the dimension of V. Otherwise $\mathbf{x}$ is called (algebraically) *singular* in V.

A smooth point $\mathbf{x}$ in an irreducible variety V has a standard-topology neighborhood in V that is a complex analytic submanifold of $\mathbb{C}^N$ of dimension $\mathrm{Dim}(V)$.

The *locus* of singular points of V is denoted by $\mathrm{Sing}(V)$. The singular locus is itself a strict subvariety of V.

DEFINITION A.3. A *constructible set* S is a set that can be defined using a finite number of varieties and a finite number of Boolean set operations. We say that S is *defined over* $\mathbb{Q}$ if it can be defined by polynomials with coefficients in $\mathbb{Q}$.

S has a well defined (maximal) *dimension* $\mathrm{Dim}(S)$, which will agree with the largest S for which there is a standard-topology open subset of S, that is a D-dimensional complex analytic submanifold of $\mathbb{C}^N$.

The *Zariski closure* of S is the smallest variety V containing it. The set S has the same dimension as its Zariski closure V. If S is defined over $\mathbb{Q}$, then so too is V (this can be shown using the fact that S is invariant to elements of the absolute Galois group of $\mathbb{Q}$).

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The image of a variety V under a polynomial map is a constructible set S. If V is defined over $\mathbb{Q}$, then so too is S [2, Theorem 1.22]. If V is irreducible, then so too is the Zariski closure of S. (We say that S is *irreducible*.)

The following can be found in [24, Proposition 10.1].

LEMMA A.4. *An irreducible constructible set S contains a Zariski open subset of its Zariski closure V. Thus $V \setminus S$ is contained in a subvariety W of V. If S is defined over $\mathbb{Q}$, there is such a W that is as well.*

DEFINITION A.5. A point in an irreducible variety or constructible set, V defined over $\mathbb{Q}$ is called *generic* if its coordinates do not satisfy any algebraic equation with coefficients in $\mathbb{Q}$ besides those that are satisfied by every point in V.

The set of generic points has full measure in V.

When V is an irreducible variety and defined over $\mathbb{Q}$, all of its generic points are smooth.

A generic real configuration in $\mathbb{R}^d$ (as in Definition 1.2) is also a generic point in $\mathbb{C}^N$, considered as a variety, as in the current definition.

LEMMA A.6. *Let $V \subseteq W$ be an inclusion of varieties where W and V are irreducible and W is defined over $\mathbb{Q}$. Suppose that V has at least one point $\mathbf{y}$ which is generic in W (over $\mathbb{Q}$). Then the points in V which are generic in W are Zariski dense in V.*

Proof. Let ϕ be a nonzero algebraic function on W defined over $\mathbb{Q}$. Consider the Zariski open subset set $X_\phi := \{\mathbf{x} \in V \mid \phi(\mathbf{x}) \neq 0\}$. This is nonempty due to our assumption about the point $\mathbf{y}$. Thus, from the irreducibility of V, this X_ϕ is Zariski dense in V.

The set of points in V which are generic in W is defined as the intersection of these open and dense X_ϕ as ϕ ranges over the countable set of possible ϕ.

When U is any Zariski open and dense subset of V, then $V \setminus U$ is contained in a strict subvariety of V. From irreducibility and dimension considerations then, U must contain a standard-topology open and dense subset of the smooth locus of V. (In fact, using [25, Theorem 1, Page 58], we can see that U is standard-topology open and dense in all of V.)

As the smooth locus of V under the standard topology is a Baire space, a countable intersection of such subsets is standard-topology dense in the smooth locus of V. Thus, again from irreducibility and dimension considerations, this intersection is Zariski dense in all of V. $\square$

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Lemma A.7. *Let V be an irreducible variety and f a polynomial map, both defined over $\mathbb{Q}$. Then the image of a generic point in V is generic in $f(V)$.*

Lemma A.8. *Let V be an irreducible variety, f be a polynomial map $f : V \to \mathbb{C}^m$ all defined over $\mathbb{Q}$. Let $W := f(V)$. If $\mathbf{y}$ is generic in W, there is a point in $f^{-1}(\mathbf{y})$ that is generic in V.*

Proof. Let ϕ be a nonzero algebraic function on V defined over $\mathbb{Q}$. We start by showing that there is a point $\mathbf{x} \in f^{-1}(\mathbf{y})$ so that $\phi(\mathbf{x}) \neq 0$. Consider the constructible set $X_\phi := \{\mathbf{x} \in V \mid \phi(\mathbf{x}) \neq 0\}$. This is Zariski dense in V due to irreducibility, so its image $f(X_\phi)$ is Zariski dense in W. Therefore, from Lemma A.4, $Y_\phi := W \setminus f(X_\phi)$ is contained in some proper subvariety T of W defined over $\mathbb{Q}$.

But then since $\mathbf{y}$ is generic it cannot be in T, so $\mathbf{y}$ is in the image of X_ϕ, so there is an $\mathbf{x} \in f^{-1}(\mathbf{y})$ such that $\phi(\mathbf{x}) \neq 0$, as desired.

Let $Z_\phi = \{\mathbf{x} \in f^{-1}(\mathbf{y}) \mid \phi(\mathbf{x}) = 0\}$. We have shown that Z_ϕ is a proper subset of $f^{-1}(\mathbf{y})$ for any nonzero algebraic function ϕ on V, defined over $\mathbb{Q}$. It follows that for any finite collection of ϕ_i, the union of Z_{ϕ_i} is still a proper subset of $f^{-1}(\mathbf{y})$ (as we can consider the product of ϕ_i). But there are only countably many possible ϕ overall, and a countable union of algebraic subsets covers an algebraic set if and only if some finite collection of them do. (Proof: this is true for each irreducible component, as a proper algebraic subset has measure zero, and there are only finitely many irreducible components.) Thus the union of Z_ϕ does not cover $f^{-1}(\mathbf{y})$, that is, there is a generic point in $f^{-1}(\mathbf{y})$. □

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