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RESEARCH ARTICLE

Biharmonic almost complex structures

Weiyong He

University of Oregon, Eugene, Oregon, 97403, United States; E-mail: whe@uoregon.edu.

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Abstract

This project uses methods in geometric analysis to study almost complex manifolds. We introduce the notion of *biharmonic almost complex structure* on a compact almost Hermitian manifold and study its regularity and existence in dimension four. First, we show that there always exists smooth energy-minimizing biharmonic almost-complex structures for any almost Hermitian four manifold. Then, we study the existence problem where the homotopy class is specified. Given a homotopy class $[\tau]$ of an almost complex structure, using the fact $\pi_4(S^2) = \mathbb{Z}_2$, there exists a canonical operation p on the homotopy classes satisfying $p^2 = \mathrm{id}$ such that $p([\tau])$ and $[\tau]$ have the same first Chern class. We prove that there exists an energy-minimizing biharmonic almost complex structure in the companion homotopy classes $[\tau]$ and $p([\tau])$. Our results show that, When M is simply connected, there exists an energy-minimizing biharmonic almost complex structure in the homotopy classes with the given first Chern class.

Contents

1. Introduction

In this paper, we study the energy-minimizing and critical almost complex structures, as a continuation of [5], with respect to the energy functional

$$\mathcal{E}_2(J) = \int_M |\Delta_g J|^2 dv,\ J \in \mathcal{J}_g, \tag{1.1}$$

where $\mathcal{J}_g$ is the space of almost complex structures which are compatible with g. We call these objects *biharmonic almost complex structures*, as these objects are tensor-valued versions of biharmonic maps. The first result of the paper is the following.

Theorem 1.1. *A $W^{2,2}$-biharmonic almost complex structure on (M^4, g) is smooth. Moreover, on any compact almost Hermitian manifold (M^4, g), there always exist minimizers of the energy functional $\mathcal{E}_2(J)$ which are smooth biharmonic almost complex structures. All such energy minimizers form a compact set.*

The existence of energy-minimizing almost complex structures is a standard practice of calculus of variations. The proof of smooth regularity in Theorem 1.1 is inspired by the theory of biharmonic maps in Chang-Wang-Yang [1]. There are subtle differences due to the nature of tensor-valued functions. We will present the smooth regularity for a much more general system in a paper together with Jiang [6].

Our second result concerns the existence of energy-minimizing almost complex structures in a fixed homotopy class. The topology of M will play an important role in the following. Let A denote the set of homotopy classes of almost complex structures on M, so the first Chern class gives a map

$$c_1 : A \to H^2(M, \mathbb{Z}).$$

Donaldson [3, Section 6] defined a map $p : A \to A$ with $p^2 = \mathrm{id}$ and $c_1 \circ p = c_1$ using the fact that $\pi_4(S^2) = \mathbb{Z}_2$. First, we discuss the case that M is simply connected. If M is not spin, then $\sigma = p(\sigma)$ (see [11]), and it is uniquely determined by its first Chern class. If M is spin, then the pair, σ and $p(\sigma)$, is uniquely determined by its first Chern class.

Theorem 1.2. *Let (M^4, g) be a compact, simply connected almost Hermitian four manifold. If M is nonspin, then every homotopy class contains an energy-minimizing biharmonic almost complex structure. If M is spin, there exists an energy-minimizing biharmonic almost complex structure in the pair of homotopy classes $(\sigma,\ p(\sigma))$.*

Theorem 1.2 can be stated as, for a compact simply connected almost Hermitian four manifold, given its first Chern class c_1, there exists an energy-minimizing biharmonic almost complex structure in the homotopy classes (possibly one or two) determined by c_1. When M is not necessarily simply connected, we have the following.

Theorem 1.3. *Given a pair of homotopy classes $(\sigma, p(\sigma))$ on (M, g), there exists an energy-minimizing biharmonic almost complex structure in the pair.*

As a comparison, despite much progress in the study of biharmonic maps and polyharmonic maps in the last two decades, the general existence result remains very limited. We briefly discuss the new input in the proof of Theorem 1.3. Recall the classic [12] on the existence of harmonic two-spheres. Sacks-Uhlenbeck [12] considered a perturbation elliptic system for harmonic maps and one technical core is the bubble analysis. The system for biharmonic almost complex structures is fourth order, and a natural perturbed biharmonic system becomes much more complicated. We are not able to prove a regularity result for such a system. This obstructs us from adopting Sacks-Uhlenbeck's approach using a perturbed biharmonic system. Instead, we analyze an energy-minimizing sequence directly in a fixed homotopy class. We get a weak limit in $W^{2,2}$. Using the special structure of the almost complex structure, we can argue that the weak limit satisfies the elliptic system weakly for biharmonic almost complex structures, and hence it is smooth by [6, Theorem 2, Corollary 1]. The main difficulty, as in other similar situations, is to understand what exactly happens if the convergence fails to be strongly in $W^{2,2}$. A major technical result is an ϵ-regularity for an energy-minimizing sequence in a fixed homotopy class. We prove that if the convergence of a weakly convergent energy-minimizing subsequence fails to be strongly in $W^{2,2}$, then there must be energy concentration around finitely many isolated points. The ϵ regularity for a minimizing sequence is very different from the classical ϵ-regularity in the theory of harmonic maps since there is no elliptic system to be dealt with.

A technical tool we develop in the paper is an extension theorem for almost complex structures in $W^{2,2}$, with precisely controlled behavior in the neck region. The classical extension theorems developed by Schoen-Uhlenbeck [13], Luckhaus [10] and Lin [9] played an essential role in the setting of $W^{1,p}$ maps, which do not extend to $W^{2,2}$ (see Simon [14, Section 2.6, 2.7]). Motivated by these classical methods in the harmonic maps, we analyze the *defect measure* as in Lin [9]. We prove an ϵ-regularity theorem for a minimizing sequence in $W^{2,2}$ for defect measure, by modifying techniques in Schoen-Uhlenbeck [13] and Lin [9] in a rather subtle way. The method developed should be useful in energy-minimizing problems in the setting of $W^{k,2}$ for $k \geq 2$.

Recently, together with He-Jiang-Lin [7], we apply this method to study the existence of biharmonic maps and polyharmonic maps. Among others, we prove the ϵ-regularity holds for an energy-minimizing sequence for a polyharmonic functional, and a weak limit is a polyharmonic map. Given the target manifold having trivial homotopy, we prove the existence of an energy-minimizing m-harmonic map in each homotopy class, which solves a longstanding open question for polyharmonic maps.

Theorem 1.3 does not answer what would precisely happen if we restrict to each homotopy class. We will discuss an intuitive conjectural picture regarding the existence problem in the pair $(\sigma, p(\sigma))$, which is closely related to the study of extrinsic biharmonic maps.

2. Existence of an energy minimizer

In this section we prove the existence of an energy-minimizing biharmonic almost complex structure on (M, g) and derive the Euler-Lagrange equation. We recall the definition of the Sobolev spaces of almost complex structures.

Definition 2.1. Given an almost Hermitian manifold (M, g) with compatible almost complex structures in $\mathcal{J}_g$, we define $W^{k,p}(\mathcal{J}_g)$ to be the subspace of $W^{k,p}(T^*M \otimes TM)$ consisting of those sections $J \in W^{k,p}(T^*M \otimes TM)$, satisfying the compatible condition almost everywhere,

$$J^2 = -id, g(J\cdot, \cdot) + g(\cdot, J\cdot) = 0. \tag{2.1}$$

We have the following.

Theorem 2.2. *Let $(M, g, \mathcal{J}_g)$ be a compact Hermitian manifold with compatible almost complex structures. Then there exists an energy minimizer of $\mathcal{E}_2(J)$ in $W^{2,2}(\mathcal{J}_g)$, satisfying the Euler-Lagrangian equation in the weak sense as in (2.5). Moreover, energy minimizers form a sequential compact set in $W^{2,2}$.*

Proof. We only prove the existence of $W^{2,2}$ energy minimizer of $\mathcal{E}_2(J)$ over $W^{2,2}(\mathcal{J}_g)$. We will derive the Euler-Lagrange equation later. This is a standard practice of calculus of variations. Take an energy-minimizing sequence $J_k \in W^{2,2}(\mathcal{J}_g)$, such that

$$\mathcal{E}_2(J_k) \to \inf_{J \in W^{2,2}(\mathcal{J}_g)} \mathcal{E}_2(J).$$

Since $J_k \in W^{2,2}(TM \otimes T^*M)$, it follows from Kondrachov compactness that a subsequence, still denoted by J_k, converges strongly in $W^{1,2}(T^*M \otimes TM)$ and weakly in $W^{2,2}(T^*M \otimes TM)$. Denote the limit by $J_0 \in W^{2,2}(TM \otimes T^*M)$. The strong convergence in $W^{1,2}$ implies, in particular, that J_k converges to J_0 almost everywhere; therefore, J_0 satisfies the compatible condition (2.1) almost everywhere. Hence, $J_0 \in W^{2,2}(\mathcal{J}_g)$. The weak convergence in $W^{2,2}$ implies that

$$\mathcal{E}_2(J_0) \le \liminf \mathcal{E}_2(J_k) = \inf_{J \in W^{2,2}(\mathcal{J}_g)} \mathcal{E}_2(J).$$

This forces that $\mathcal{E}_2(J_0) = \min \mathcal{E}_2(J)$, and J_0 is an energy minimizer. Moreover, this also implies that the convergence $J_k \to J_0$ is strongly in $W^{2,2}$ and that the set of all energy minimizers is compact. □

Straightforward computation gives the Euler-Lagrange equation in smooth setting

$$\Delta^2 J + J(\Delta^2 J)J = 0. \tag{2.2}$$

An equivalent form is $[\Delta^2 J, J] = 0$. By $J^2 = -id$, we can rewrite the equation as,

$$\Delta^2 J = Q(J, \nabla J, \nabla^2 J, \nabla^3 J), \tag{2.3}$$

where Q is given by

$$Q = J\Delta J\Delta J + J\nabla_p J\nabla_p \Delta J + J\nabla_p \Delta J\nabla_p J + J\Delta(\nabla_p J\nabla_p J). \tag{2.4}$$

We need to be more careful when working in $W^{2,2}(\mathcal{J}_g)$ to derive equations for weak solutions. For any $A \in TM \to TM$, it is convenient to define its adjoint A^* as follows.

$$g(A^*\cdot, \cdot) = g(\cdot, A\cdot).$$

Hence, g-compatibility can be written as $A + A^* = 0$. For any $J \in \mathcal{J}_g$, the tangent space $\mathcal{S}_J$ of J can be described as the endomorphisms $S : TM \to TM$ as

$$\mathcal{S}_J = \{S : SJ + JS = 0, S + S^* = 0\}.$$

Clearly, $S \in \mathcal{S}_J$, and then JS is in $\mathcal{S}_J$ as well. The adjoint can be directly carried over for Sobolev spaces. Necessary properties of the adjoint endomorphism are summarized in [6, Proposition 4.1]. For any $S \in \mathcal{S}_J \cap L^\infty$, it is direct to check that $J(t) = J\exp(tSJ)$ gives a path in $\mathcal{J}_g \cap L^\infty$ for $|t| << 1$, with $\partial_t J(0) = S$.

Proposition 2.3. *A critical point of $\mathcal{E}_2(J)$ on $W^{2,2}(\mathcal{J}_g)$ satisfies the Euler-Lagrange equation (2.3) in the following weak sense. For any $T \in W^{2,2}(T^*M \otimes TM) \cap L^\infty$, we have*

$$\int_M (\Delta J - J\nabla J\nabla J, \Delta T)dv + \int_M (A, T)dv + \int_M (B, \nabla T)dv = 0, \tag{2.5}$$

where we write

$$\begin{aligned} A =& J\Delta J\Delta J + \nabla_p J\nabla_p J\Delta J - \Delta J\nabla_p J\nabla_p J + \nabla_p J\Delta J\nabla_p J \\ B =& \nabla J\Delta JJ + J\Delta J\nabla J. \end{aligned}$$

Proof. Assume that $T \in W^{2,2}(\Gamma(T^*M \otimes TM)) \cap L^\infty$. Denote $R = T + JTJ$ and $S = R - R^*$. Then S satisfies $SJ + JS = 0, S + S^* = 0$. Taking variation in the form $J(t) = J\exp(tSJ)$ for small $|t| << 1$, we have $\int_M (\Delta J, \Delta S)dv = 0$. Since J is g-compatible, this implies $\int_M (\Delta J, \Delta R)dv = 0$. In other words, we have

$$\int_M (\Delta J, \Delta T)dv + \int_M (\Delta J, \Delta(JTJ))dv = 0.$$

For $J \in W^{2,2}$, we compute

$$\Delta(JTJ) = J(\Delta T)J + \Delta JTJ + JT\Delta J + 2\nabla J\nabla TJ + 2\nabla JT\nabla J + 2J\nabla T\nabla J.$$

Since J is g-compatible, it is straightforward to check that

$$\begin{aligned} (\Delta J, J\Delta TJ) =& (J\Delta JJ, \Delta T) \\ (\Delta J, \Delta JTJ) =& (\Delta J\Delta JJ, T). \end{aligned}$$

Hence, we get that

$$\int_M (\Delta J + J\Delta JJ, \Delta T) + (A_0, T) + 2(B, \nabla T) = 0, \tag{2.6}$$

where we have

$$\begin{aligned} A_0 =&\Delta J\Delta JJ + J\Delta J\Delta J + 2\nabla_p J\Delta J\nabla_p J,\\ B =&\nabla J\Delta JJ + J\Delta J\nabla J. \end{aligned}$$

However, we use $\Delta(J^2) = 0$ to conclude that

$$\begin{aligned} &\Delta JJ + J\Delta J + 2\nabla_p J\nabla_p J = 0\\ &J\Delta JJ = \Delta J - 2J\nabla_p J\nabla_p J. \end{aligned}$$

We compute that

$$\Delta J\Delta JJ = J\Delta J\Delta J + 2\nabla_p J\nabla_p J\Delta J - 2\Delta J\nabla_p J\nabla_p J.$$

Hence, we get that

$$A_0 = 2\big(J\Delta J\Delta J + \nabla_p J\nabla_p J\Delta J - \Delta J\nabla_p J\nabla_p J + \nabla_p J\Delta J\nabla_p J\big).$$

We denote $A_0 = 2A$, with

$$A = J\Delta J\Delta J + \nabla_p J\nabla_p J\Delta J - \Delta J\nabla_p J\nabla_p J + \nabla_p J\Delta J\nabla_p J.$$

Together with (2.6), we have

$$\int_M (\Delta J - J\nabla J\nabla J, \Delta T) + (A, T) + (B, \nabla T) = 0. \tag{2.7}$$

This completes the proof. □

Proposition 2.3 is equivalent to (2.2) for $J \in W^{2,2}$ when (2.2) is understood as a weak solution in the form that

$$\int_M (\Delta J, \Delta T)dv + \int_M (\Delta J, \Delta(JTJ))dv = 0. \tag{2.8}$$

An almost complex structure $J \in W^{2,2}(\mathcal{J}_g)$ is called a *weak biharmonic almost complex structure* if it satisfies (2.8) or its equivalent form (2.5). The equivalent descriptions below will be useful as well.

Proposition 2.4. *A weak biharmonic almost complex satisfies the following.*

$$\int_M (\Delta J, \Delta(TJ - JT))dv = 0, \tag{2.9}$$

where $T \in \Gamma(TM \otimes T^*M)$. Or equivalently, we have the following.

$$\int_M (\Delta J, \Delta TJ - J\Delta T)dv + 2\int_M (\Delta J, \nabla T\nabla J - \nabla J\nabla T)dv = 0. \tag{2.10}$$

As a consequence, the weak limit of a sequence of biharmonic almost complex structure with bounded $W^{2,2}$ norm is biharmonic.

Proof. To derive (2.9), we take $T = \tilde{T}J$ in (2.8) and observe that

$$\int_M (\Delta J, (\Delta J)T)dv = \int_M (\Delta J, T\Delta J)dv. \tag{2.11}$$

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Similarly, we have

$$(\Delta J, T\Delta J) = -(\Delta J\Delta J, T).$$

This gives (2.10). Suppose a sequence of weak biharmonic almost complex structures J_k, with bounded $W^{2,2}$ norm, converges weakly in $W^{2,2}$ to J_0 and strongly in $W^{1,2}$. By passing to the limit, J_0 satisfies (2.10). More precisely, for any fixed smooth T, we have

$$\int_M (\Delta(J_k - J_0), \Delta T J_0 - J_0\Delta T)dv \to 0.$$

The strong convergence $J_k \to J_0$ in L^2 implies that

$$\int_M (\Delta J_k, \Delta T(J_k - J_0) - (J_k - J_0)\Delta T)dv \to 0.$$

Together, this implies that

$$\int_M (\Delta J_k, (\Delta T)J_k - J_k\Delta T)dv \to \int_M (\Delta J_0, (\Delta T)J_0 - J_0\Delta T)dv.$$

Similarly, we have

$$\int_M (\Delta J_k, \nabla T\nabla J_k - \nabla J_k\nabla T)dv \to \int_M (\Delta J_0, \nabla T\nabla J_0 - \nabla J_0\nabla T)dv.$$

This completes the proof. □

He-Jiang [6, Theorem 2, Corollary 1] proved a smooth regularity for a general elliptic system. As a result, we have the following.

Theorem 2.5. *A $W^{2,2}$ weak solution of biharmonic almost complex structure in dimension four is smooth.*

3. Biharmonic almost complex structures in a homotopy class

A fundamental problem in the theory of harmonic maps is finding harmonic maps in a fixed homotopy class. Similarly, we would like to ask the same question for biharmonic almost complex structures. We shall see that the topology of M plays a very important role. The following famous example about almost complex structures on a $K3$ surface constructed by Donaldson [3] serves as an important example. We recall relevant discussions. A compatible almost complex structure J on an oriented Riemannian four-manifold M can be considered as a section of the associated $SO(4)/U(2)$-bundle over M (the sphere bundle of Λ^2_+, known as the 'twistor space'). Let A denote the set of homotopy classes of almost complex structures, so the first Chern class gives a map

$$c_1 : A \to H^2(M, \mathbb{Z}).$$

Donaldson [3, Section 6] defined a map $p : A \to A$ with $p^2 = \mathrm{id}$ and $c_1 \circ p = c_1$ as follows. For $\sigma \in A$, $p(\sigma)$ agrees with σ outside a small ball in M, and over this ball, the two compare by the nonzero element of

$$[S^4, SO(4)/U(2)] = [S^4, S^2] \cong \mathbb{Z}/2.$$

We need the following general result about σ and $p(\sigma) \in A$, which we have learned from Teichner through mathoverflow [11].

Lemma 3.1. *If M is simply connected and nonspin, then the first Chern class determines a unique homotopy class of an almost complex structure. When M is simply connected and spin, the first Chern class determines a pair of homotopy classes σ and $p(\sigma)$.*

The main result in this section is to prove Theorem 1.3. The proof consists of several steps. Let $J_k \in \sigma$ be a minimizing sequence such that $\mathcal{E}_2(J_k) \to E_0$. A direct computation shows that J_k has uniformly bounded $W^{2,2}$ norm. Hence, we can assume that J_k converges to J_0 weakly in $W^{2,2}$ and strongly in $W^{1,2}$, in particular $J_0 \in W^{2,2}(\mathcal{J}_g)$. First, we have the following.

Lemma 3.2. *The limit J_0 is a smooth biharmonic almost complex structure.*

Proof. We only need to prove that J_0 is a weak biharmonic almost complex structure satisfying (2.10). Let J_k be a minimizing sequence and J_0 be its weak limit in $W^{2,2}(\mathcal{J}_g)$. For any fixed smooth $T \in \Gamma(TM \otimes T^*M)$, consider

$$S_t = t(T + J_k T J_k) - t(T + J_k T J_k)^*.$$

Note that for $|t|$ sufficiently small, S_t has small L^∞ norm and we construct

$$J_k(t) = J_k \exp(S_t) \in \mathcal{J}_g.$$

We compute, for $|t|$ sufficiently small,

$$E(J_k^t) = \int_M |\Delta J_k^t|^2 dv = E(J_k) + 4t \int_M (\Delta J_k, \Delta(J_k T - T J_k)) dv + O(t^2), \tag{3.1}$$

where the term $O(t^2)$ denotes the terms of higher order in t. We have $|O(t^2)| \le Ct^2$ for a uniformly bounded constant C (assuming t is small). A quick way to see (3.1) is to write the (matrix) expansion of $J_k(t)$ as

$$J_k(t) = J_k + t(J_k T - T J_k) - t(J_k T - T J_k)^* + O(t^2).$$

Since J_k is a minimizing sequence, hence when $k \to \infty$, we have

$$\liminf_{k\to\infty} E(J_k^t) - E(J_k) \ge 0.$$

Hence, for $|t|$ sufficiently small,

$$\limsup_k 4t \int_M (\Delta J_k, \Delta(J_k T - T J_k)) dv \ge \liminf_k 4t \int_M (\Delta J_k, \Delta(J_k T - T J_k)) dv \ge 0.$$

In particular, this implies that

$$\limsup_k (\Delta J_k, \Delta(J_k T - T J_k)) dv = \liminf_k \int_M (\Delta J_k, \Delta(J_k T - T J_k)) dv = 0.$$

Similar as in (2.10) and (2.11), we use the fact

$$\int_M (\Delta J_k, \Delta J_k T) dv = \int_M (\Delta J_k, T \Delta J_k) dv$$

to conclude that

$$\lim_k \int_M (\Delta J_k, J_k \Delta T - \Delta T J_k) dv + 2 \int_M (\Delta J_k, \nabla J_k \nabla T - \nabla T \nabla J_k) dv = 0.$$

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Since J_k converges to J_0 weakly in $W^{2,2}$ and strongly in $W^{1,2}$, the above implies that

$$\int_M (\Delta J_0, J_0\Delta T - \Delta T J_0)dv + 2\int_M (\Delta J_0, \nabla J_0 \nabla T - \nabla T \nabla J_0)dv = 0.$$

By Proposition 2.4, J_0 is a weak biharmonic almost complex structure. Hence it is smooth by regularity results in [6]. □

An important question is to understand the homotopy class of J_0. To proceed, we need the following fact about the first Chern class.

Lemma 3.3 (Wood [16]). *Given a compact almost Hermitian manifold (M, g, J, ω), then the first Chern class can be represented by the following 2-form γ such that*

$$2\pi\gamma = \mathcal{R}(\omega) + \chi, \ \textit{with}\ \chi(X, Y) := \frac{1}{4}\omega(\nabla_X J, \nabla_Y J), \tag{3.2}$$

where $\mathcal{R}(\omega)$ is the curvature operator acting on the Kähler form ω.

Lemma 3.3 has the following important consequence.

Lemma 3.4. *The first Chern class $c_1(J_0) = c_1(J_k)$ for $J_k \in \sigma$.*

Proof. The first Chern class $c_1(J)$ is the deformation invariant, and hence it remains the same in the homotopy class σ. By Lemma 3.3, the first Chern class is represented by the 2-form γ. We write γ_k for J_k and γ_0 for J_0. Since J_k converges strongly to J_0 in $W^{1,2}$, then for any smooth two form ζ, (3.2) implies that

$$\lim_{k\to\infty} \int_M \gamma_k \wedge \zeta = \int_M \gamma_0 \wedge \zeta.$$

It follows that $[\gamma_k] = [\gamma_0] \in H^2(M, \mathbb{Z})$. □

Hence, the fist Chern class remains unchanged for the weak limit. If M is simply connected and nonspin, Lemma 3.1 implies that J_0 is in the homotopy class σ.

Lemma 3.5. *If M is simply connected and nonspin, then every homotopy class contains an energy-minimizing biharmonic almost complex structure.*

If M is simply connected and spin, the situation is more complicated. By Lemma 3.1, J_0 is either in the class σ or in $p(\sigma)$. In the former case, J_0 is an energy-minimizing biharmonic almost complex structure in σ; in the later case, we prove that there is an energy-minimizing biharmonic almost complex structure in $p(\sigma)$. To achieve this, we consider the homotopy class σ and $p(\sigma)$ simultaneously. We consider $\inf_{J\in\sigma} \mathcal{E}_2(J)$ and $\inf_{J\in p(\sigma)} \mathcal{E}_2(J)$. We assume that $\inf_{J\in\sigma} \mathcal{E}_2(J) \geq \inf_{J\in p(\sigma)} \mathcal{E}_2(J)$. A minimizing sequence J_k in $p(\sigma)$ will have a weak limit J_0 and $\mathcal{E}_2(J_0) \geq \inf_{J\in p(\sigma)} \mathcal{E}_2(J)$, since J_0 is either in σ or $p(\sigma)$. This will force that J_0 is an energy-minimizing biharmonic almost complex structure.

Lemma 3.6. *If M is simply connected and spin, then for each first Chern class c of an almost complex structure, at least one homotopy class (among two homotopy classes corresponding to c) contains an energy-minimizing biharmonic almost complex structures.*

When M is not necessarily simply connected, Lemma 3.1 does not hold anymore. We need more precise control of weak convergence to obtain the following.

Lemma 3.7. *Let J_k be an energy-minimizing sequence of $\mathcal{E}_2(J)$ in the homotopy class σ. Then the weak limit J_0 (of a convergent subsequence) lies in either σ or $p(\sigma)$.*

The key is to prove a version of ϵ-regularity for the minimizing sequence J_k. As a consequence, J_k converges strongly in $W^{2,2}$ to J_0 except around finitely many isolated points. This will imply that the homotopy class of J_0 is either σ or $p(\sigma)$.

Now we are ready to state and prove the ϵ-regularity for the minimizing sequence J_k (see Lemma 3.8 and Lemma 3.9). Fix a sufficiently small positive number ϵ_0, which depends only on (M, g) and will be specified later. Let ι be the injectivity radius of (M, g). Suppose a minimizing sequence $J_k \in \sigma$ converges weakly to J_0 in $W^{2,2}$ and strongly in $W^{1,2}$. For $r \in (0, \iota)$, $p \in M$ and $J \in W^{2,2}$, denote

$$\begin{aligned} E(r,p) &= \int_{B_r(p)} |\Delta J|^2 dv \\ F(r,p) &= \int_{B_r(p)} (|\nabla^2 J|^2 + |\nabla J|^4) dv. \end{aligned} \tag{3.3}$$

We write $E_0(r,p), F_0(r,p), E_k(r,p), F_k(r,p)$ correspondingly for J_0 and J_k. Set $\mathcal{S}_r = \{p \in M : \liminf_{k\to\infty} F_k(r,p) \geq \epsilon_0\}$. Clearly, $\mathcal{S}_r \subset \mathcal{S}_s$ for $r < s$. Denote

$$\mathcal{S} := \cap_{r>0}\mathcal{S}_r = \lim_{r\to 0} \mathcal{S}_r.$$

We introduce the measures which are all totally bounded,

$$\begin{cases} d\mu_k = (|\nabla^2 J_k|^2 + |\nabla J_k|^4) dv \\ d\mu_0 = (|\nabla^2 J_0|^2 + |\nabla J_0|^4) dv \\ d\xi_k = |\Delta J_k|^2 dv \\ d\xi_0 = |\Delta J_0|^2 dv. \end{cases} \tag{3.4}$$

By passing to a subsequence, μ_k converges weakly to a positive Radon measure μ, and ξ_k converges weakly to a positive Radon measure ξ. By Fatou's lemma, there exist positive Radon measures ν and λ (called the *defect measure* [8, 9]), such that

$$\begin{cases} d\mu = d\nu + d\mu_0 \\ d\xi = d\lambda + d\xi_0. \end{cases}$$

Certainly, J_k converges to J_0 strongly in $W^{2,2}$ if and only if either $\nu \equiv 0$ or $\lambda \equiv 0$. But ν and λ are not necessarily the same in general. The interplay between two defect measures ν and λ makes our discussions below more complicated than the theory of the harmonic maps, where only the measure $|\nabla u|^2 dv$ comes to play. We have the following.

Lemma 3.8. *The support of ν equals $\mathcal{S}$, which contains at most finitely many points.*

Proof. First, it is straightforward to see that $\mathcal{S}$ is contained in the support of ν. Since J_0 is smooth, we have for any $x \in M$, $\lim_{r\to 0} \mu_0(B_r(x)) = 0$. If $x \in \mathcal{S}$, then

$$\lim_{r\to 0} \nu(B_r(x)) = \lim_{r\to 0}[\mu(B_r(x)) - \mu_0(B_r(x))] \geq \epsilon_0.$$

This shows that $\mathcal{S}$ is contained in the support of ν.

Next, we claim the following. For $r \in (0, r_0]$, where r_0 is a fixed, sufficiently small number, if $\mu(B_{2r}(p)) < \epsilon_0$, then $\nu \equiv 0$ in $B_{r/2}(p)$.

We sketch the idea of the proof briefly. If $\mu(B_{2r}(p)) < \epsilon_0$ for ϵ_0 sufficiently small, then after passing to a subsequence, we have

$$\int_{B_{2r}(p)} (|\nabla^2 J_k|^2 + |\nabla J_k|^4) dv < 2\epsilon_0$$

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for sufficiently large k. This implies that J_k is close to its average in B_{2r} and the same discussion holds for J_0. In particular, J_k and J_0, when restricted in B_r, are homotopy to each other. The key is to construct a new 'almost minimizing' sequence $\tilde{J}_k$ in the same homotopy class such that $\tilde{J}_k = J_0$ inside $B_{r/2}(p)$ and $\tilde{J}_k = J_k$ outside $B_r(p)$, while the behavior in the annulus region is precisely controlled. This would imply $\nu, \lambda \equiv 0$ in $B_{r/2}(p)$.

Suppose at the moment, the claim is established. If p is in the support of ν, then $\mu(B_{2r}(p)) \geq \epsilon_0$ for all sufficiently small r. Hence $p \in \mathcal{S}$. Since the total energy is bounded, it follows that $\mathcal{S}$ contains, at most, finitely many isolated points. We complete the proof given Lemma 3.9 below, where we establish the claim. □

First, we specify the choice of r_0. We can do the scaling $g_r = r^{-2}g$ for $r \leq r_0$. We choose r_0 sufficiently small, such that g_r is sufficiently close to the Euclidean metric in the ball $B_2(p)$ (we identify $B_2(p)$ with the Euclidean ball B_2 equipped with the metric g_r) such that

$$|g_{ij} - \delta_{ij}| + \sum_{k=1}^{4} |D^k g_{ij}| < \delta_0, \tag{3.5}$$

where δ_0 measures how close the metric g_r is with respect to the Euclidean metric in B_2. We also assume that $g_{ij}(0) = \delta_{ij}, \partial g_{ij}(0) = 0$.

Note that we choose r_0, δ_0 and ϵ_0 uniformly for any point $p \in M$. The constants r_0, δ_0 and ϵ_0 are all fixed. Since r will also be fixed and the energy functionals are scaling invariant, we consider (M, g_r) instead of (M, g). In other words, we can assume that, by scaling if necessary, g satisfies (3.5) in any geodesic ball $B_2(p) \subset M$ such that the injectivity radius of (M, g) is bigger than 2. We identify $(B_2(p), g)$ with the Euclidean ball (B_2, g) via the exponential map $\exp_p : T_pM \to M$. Over (B_2, g), the tangent bundle is trivial and we write $J : TM \to TM$ over (B_2, g) as a matrix-value function $J(x)$. We use ∇, ∇^2, etc. to denote covariant derivatives of with respect to g over B_2. We will also use D, D^2 to denote the Euclidean derivatives over B_2. After this choice of the scaling and local coordinates, an almost complex structure J over (B_2, g) is a matrix-valued function, which we still denote as J. We establish the main technical lemma.

Lemma 3.9. *Suppose $\mu(B_2(p)) \leq \epsilon_0$. Then $\nu \equiv 0$ in $B_1(p)$.*

The proof of this lemma involves a construction of a sequence of 'almost energy-minimizing' almost complex structures $\tilde{J}_k$ in σ such that

$$\tilde{J}_k(x) = \begin{cases} J_k(x), x \in M \setminus B_1(p) \\ J_0(x), x \in B_{1-j^{-1}}(p), \end{cases} \tag{3.6}$$

where $k \geq k_j = k(j)$ is sufficiently large depending on j. In the end, we will let $j \to \infty$ (and $k_j \to \infty$ accordingly) to get an almost energy-minimizing subsequence. Such a construction is a type of extension of an almost complex structure which equals J_k in $M \setminus B_1(p)$ and which equals J_0 in $B_{1-j^{-1}}(p)$. The construction happens in a small annulus $B_1(p) \setminus B_{1-j^{-1}}(p)$. The small energy condition $\mu(B_2(p)) \leq \epsilon_0$ plays a very important role. In particular, this implies that $\tilde{J}_k$ is still in σ, using a theorem of White [15].

The construction involves several delicate choices of small constants and cutoff functions. We shall first briefly explain the process of construction, leaving details to be proved below. We work on (B_2, g). Let $\psi_j : [0, \infty) \to [0, 1]$ be a smooth cutoff function depending on j such that

$$\psi_j(s) = \begin{cases} 1, s \geq 1 \\ 0, s \leq 1 - j^{-1} \end{cases}$$

with bounds $|\psi_j'| \leq 3j, |\psi_j''| \leq 10j^2$. For $x \in B_2$, we denote

$$J_{k,j}(x) = J_k(x) + (J_0(x) - J_k(x))(1 - \psi_j(|x|)).$$

Note that

$$J_{k,j}(x) = \begin{cases} J_k(x), |x| \geq 1 \\ J_0(x), |x| \leq 1 - j^{-1}. \end{cases} \tag{3.7}$$

We extend $J_{k,j}$ to M such that it equals J_k outside $B_1(p)$. Note that $J_{k,j}$ might not even be invertible for some points $|x| \in (1 - j^{-1}, 1)$ since the convergence of $J_k \to J_0$ does not imply the convergence in L^∞. To overcome this difficulty, we construct a smooth approximation of $J_{k,j}$ using a local average technique (a modification of *mollifier*).

Let $\phi(x) = \phi(|x|)$ be a nonnegative smooth radial cutoff function which is supported in $B = B_1$ with $\int_B \phi dx = 1$. For any given J and $\rho > 0$, we denote

$$J_\rho(x) = \int_{B_\rho(x)} \phi_\rho(y - x) J(y) dy = \int_B \phi(z) J(x + \rho z) dz, \tag{3.8}$$

where we use the notation, $\phi_\rho(x) = \rho^{-4}\phi\left(\frac{x}{\rho}\right)$. When $\rho = 0$, ϕ_ρ is the delta-function and $J_\rho(x) = J(x)$ (this is also clear from the second equality in (3.8)). Certainly, J_ρ is the smooth approximation of J, and J_ρ converges to J (in a certain norm depending on the regularity of J) when $\rho \to 0$. However, such a smooth approximation does not preserve (3.7) in general. Hence, we allow ρ to be dependent of $|x|$ and we write $\rho : [0, 2] \to [0, 1]$. Such a technique is a modification of Schoen-Uhlenbeck [12]. The support of ρ is contained in $(1 - j^{-1}, 1)$. In other words,

$$\rho(s) = 0, s \in [0, 1 - j^{-1}] \cup [1, 2]. \tag{3.9}$$

We choose $\rho(1 - (2j)^{-1}) = \max \rho = \bar{\rho}$, where $\bar{\rho}$ is a small positive number and it can be chosen such that $4C_0\bar{\rho}j^2 = 1$, where C_0 is a uniform constant, specified below. By choosing $\bar{\rho}$, we require the derivatives of ρ satisfying

$$|\rho'| + |\rho''| \leq 4C_0\bar{\rho}j^2 = 1. \tag{3.10}$$

For $x \in B_2$, we also denote $\rho(x) = \rho(|x|)$. The choice of the function ρ depends crucially on the cutoff function ψ_j, and this is the first key point in our construction.

Given such a function ρ, we construct, for $x \in B = B_1$,

$$J_{k,j,\rho(x)}(x) = \int_{B_{\rho(x)}(x)} \phi_\rho(y - x) J_{k,j}(y) dy = \int_B \phi(z) J_{k,j}(x + \rho(x)z) dz. \tag{3.11}$$

By the choice of $\rho(x)$, which is zero when $1 \leq |x| \leq 2$ or $|x| \leq 1 - j^{-1}$, $J_{k,j,\rho(x)}$ can be extended to M and satisfies (3.7). Similarly, we denote

$$\begin{aligned} J_{0,\rho(x)}(x) &= \int_{B_{\rho(x)}(x)} \phi_\rho(y - x) J_0(y) dy = \int_B \phi(z) J_0(x + \rho(x)z) dz. \\ J_{k,\rho(x)}(x) &= \int_{B_{\rho(x)}(x)} \phi_\rho(y - x) J_k(y) dy = \int_B \phi(z) J_k(x + \rho(x)z) dz. \end{aligned} \tag{3.12}$$

Note that $J_{0,\rho(x)}(x)$, $J_{k,\rho(x)}(x)$ and $J_{k,j,\rho(x)}(x)$ are neither almost complex structures nor compatible with the metric g in general. It is straightforward to see that $J_{0,\rho(x)}(x)$ is close to $J_0(x)$ given $\bar{\rho}$ is sufficiently small (J_0 is smooth and $J_{0,\rho}$ is a standard approximation). We will also show that $J_{k,\rho(x)}(x)$

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is close to an almost complex structure and it is almost compatible with the metric g, using the Poincare inequality and small energy assumption. One difficulty is to prove that $J_{k,j,\rho}$ is also close to an almost complex structure, with a suitable choice of ρ and $\bar{\rho}$ and $k = k(\bar{\rho}, j)$. The dependence of k on $\bar{\rho}$ and j is inevitable. In particular, we need to choose ρ depending on ψ_j.

Once we construct $J_{k,j,\rho}$, we use the technique in [5] to construct a unique almost complex structure $\tilde{J}_k$ using $J_{k,j,\rho}$, such that it is compatible with g. We assert that

$$\int_M |\nabla J_k - \nabla \tilde{J}_k|^4 dv \leq C\epsilon_0 \tag{3.13}$$

which implies that $\tilde{J}_k \in \sigma$ using a theorem of White [15, Theorem 2 and Section 6].

Here comes another essential point of the proof. By the construction, $\tilde{J}_k$ equals J_k outside B_1 and agrees with J_0 in $B_{1-j^{-1}}$. If $\tilde{J}_k$ is an energy-minimizing sequence in σ, this implies that $\lambda \equiv 0$ in B_1 and completes the proof. Unfortunately, we are not able to prove

$$\int_{B_1 \backslash B_{1-j^{-1}}} |\Delta \tilde{J}_k|^2 dv \to 0 \tag{3.14}$$

for $k \geq k_j$ and $j \to \infty$. Instead, we prove the following inequality approximately.

$$\int_{B_1 \backslash B_{1-j^{-1}}} |\Delta \tilde{J}_k|^2 dv \lessapprox C \int_{B_1 \backslash B_{1-j^{-1}}} (|\nabla^2 J_k|^2 + |\nabla J_k|^4) dv. \tag{3.15}$$

Given (3.15) and the fact that $\{J_k\}$ is an energy-minimizing sequence, we can obtain that

$$\lambda(B_1) \leq C\nu(\partial B_1). \tag{3.16}$$

Since the defect measure $\nu(\partial B_1)$ can be strictly positive on ∂B_1, this does not directly lead to the conclusion $\lambda \equiv 0$. However, the construction above actually works on any ball $B_r \subset B_2$, $r \in [1/4, 7/4]$ (replacing B_1 by B_r), and the arguments can be directly carried over. Hence, we will prove for $r \in [1/4, 7/4]$,

$$\lambda(B_r) \leq C\nu(\partial B_r). \tag{3.17}$$

In particular, we have for $r \in [3/2, 7/4]$,

$$\lambda(B_{3/2}) \leq C\nu(\partial B_r). \tag{3.18}$$

Since ν is a totally bounded positive Radon measure, then $\nu(\partial B_r) = 0$ for infinitely many r (actually $\nu(\partial B_r) > 0$ for at most countably many r). Hence, this proves that $\lambda \equiv 0$ in $B_{3/2}$. It is then a standard practice to prove that $\nu \equiv 0$ in B_1.

We state two versions of the Poincare inequality which are needed in the proof. For $f \in W^{1,2}(B_R)$, $B_R \subset \mathbb{R}^n$, denote $\underline{f}$ to be its average in the ball,

$$\underline{f} = \frac{1}{\mathrm{Vol}(B_R)} \int_{B_R} f(y) dy.$$

Then we have

$$R^{-n} \int_{B_R} |f - \underline{f}|^2 dy \leq CR^{2-n} \int_{B_R} |Df|^2 dy, \tag{3.19}$$

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where C is a uniform dimensional constant. Suppose ϕ is a cutoff function supported in B_R such that $\int_{B_R} \phi(y)dy = 1$. We denote

$$f_* = \int_{B_R} \phi(y) f(y) dy,$$

and then we have

$$R^{-n} \int_{B_R} |f - f_*|^2 dy \leq C R^{2-n} \int_{B_R} |Df|^2 dy. \tag{3.20}$$

We should mention that f can be taken as vector-valued and matrix-valued functions as a direct generalization. Now we carry out the details to prove Lemma 3.9.

Proof. **Step one:** the construction of an 'almost' almost complex structure $J_{k,j,\rho(x)}$.

Note that J is g-compatible if $J + J^* = 0$. With the localization over B_2, J is g-compatible if $(J + gJ^t g^{-1})(x) = 0$ holds as a matrix-valued equation for all $x \in B_2$, where J^t is the transpose of J. If J is compatible with g, then J_ρ is almost g-compatible if $\bar{\rho}$ is sufficiently small (we assume that $|J|$ is bounded, of course). We estimate

$$\begin{aligned} |(J_\rho + gJ_\rho^t g^{-1})(x)| =& \left| \int_{B_\rho(x)} \phi_\rho(y-x)(J(y) + g(x)J^t(y)g^{-1}(x))dy \right| \\ \leq& \rho^{-4} \int_{B_\rho} \left| J(y) + g(x)J^t(y)g^{-1}(x) \right| dy \\ \leq& \rho^{-4} \int_{B_\rho} \left| -g(y)J^t(y)g^{-1}(y) + g(x)J^t(y)g^{-1}(x) \right| dy \\ \leq& C\delta_0 \bar{\rho}, \end{aligned} \tag{3.21}$$

where we use the facts $|g(x) - g(y)| \leq C\delta_0\bar{\rho}$ for $|y - x| \leq \rho$ and

$$g(x)J^t(y)g^{-1}(x) - g(y)J^t(y)g^{-1}(y) = (g(x) - g(y))J^t(y)g^{-1}(x) + g(x)J^t(y)(g^{-1}(x) - g^{-1}(y))$$

It is clear that (3.21) holds for all the cases $J = J_0$, J_k and $J_{k,j}$.

Next we show that $J_{k,j,\rho}$ is 'almost' an almost complex structure in the sense that $|J_{k,j,\rho}J_{k,j,\rho} + \mathrm{id}|$ is very small pointwise. Now we specify ρ further and discuss the properties of $J_{k,j,\rho}$. For $|x| \in [0, 1 - j^{-1}] \cup [1, 2]$, since $\rho(x) = 0$, we have

$$J_{k,j,\rho(x)}(x) = J_{k,j}(x) = \begin{cases} J_k(x), |x| \in [1,2], \\ J_0(x), |x| \in [0, 1 - j^{-1}]. \end{cases} \tag{3.22}$$

Fix $\delta_1 > 0$ sufficiently small (which can be taken as j^{-1}). We write $(1 - j^{-1}, 1)$ as three subintervals

$$(1 - j^{-1}, 1 - j^{-1} + \delta_1 j^{-1}] \cup (1 - j^{-1} + \delta_1 j^{-1}, 1 - \delta_1 j^{-1}) \cup [1 - \delta_1 j^{-1}, 1).$$

The discussions in each subinterval are different. We choose ρ such that

$$\begin{cases} \rho(1 - (2j)^{-1}) = \bar{\rho} \\ \rho(1 - j^{-1} + \delta_1 j^{-1}) = \rho(1 - \delta_1 j^{-1}) = \delta_1 \bar{\rho} \\ \rho(s) < \delta_1 \bar{\rho}, s \in (1 - j^{-1}, 1 - j^{-1} + \delta_1 j^{-1}) \cup (1 - \delta_1 j^{-1}, 1) \\ \rho(s) \geq \delta_1 \bar{\rho}, s \in (1 - j^{-1} + \delta_1 j^{-1}, 1 - \delta_1 j^{-1}). \end{cases} \tag{3.23}$$

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Let $h(s)$ be a smooth nonnegative function supported over $[0,2]$, satisfying the following.

$$\begin{aligned} & h(1) = 1 = \max h, \text{and } h(s) = h(2-s), \\ & h(j^{-1}) = j^{-1},\ h'(s) \geq 0, s \in [0,1], \\ & |h^{'}| + |h^{''}| \leq C_0. \end{aligned} \tag{3.24}$$

Denote $\rho(s) = \bar{\rho}\ h(2+2j(s-1))$. Then $\rho(s)$ is supported in $[1-j^{-1}, 1]$, satisfying (3.23). We now choose $\bar{\rho}$ such that

$$|\rho^{'}| + |\rho^{''}| \leq 4\bar{\rho} j^2(|h^{'}| + |h^{''}|) \leq 4C_0\bar{\rho} j^2 = 1.$$

For $|x| \in (1-j^{-1}, 1-j^{-1}+\delta_1 j^{-1}]$, we have

$$|x+\rho(x)z| \leq 1 - j^{-1} + \delta_1 j^{-1} + \delta_1\bar{\rho}.$$

Using $\psi_j(1-j^{-1}) = 0$ and $|\psi_j^{'}| \leq 3j$, we get

$$\psi_j(|x+\rho(x)z|) \leq 3j(\delta_1 j^{-1} + \delta_1\bar{\rho}) \leq 4\delta_1.$$

We compute

$$|J_{k,j,\rho(x)}(x) - J_{0,\rho(x)}(x)| = \left| \int_B \phi(z)\psi_j(|x+\rho(x)z)|(J_0 - J_k)(x+\rho(x)z)dz \right| \leq 100\delta_1.$$

Similarly, for $|x| \in [1-\delta_1 j^{-1}, 1)$, we have

$$|J_{k,j,\rho(x)}(x) - J_{k,\rho(x)}(x)| \leq 100\delta_1. \tag{3.25}$$

We can estimate

$$|J_{0,\rho(x)}(x) - J_0(x)| \leq \int_B \phi(z)|J_0(x+\rho(x)z) - J_0(x)|dz \leq C_1\bar{\rho}, \tag{3.26}$$

where $C_1 = C_1(\max|\nabla J_0|)$ is a uniform constant. We derive

$$|J_{k,j,\rho(x)} - J_0(x)| \leq 100\delta_1 + C_1\bar{\rho}, |x| \in (1-j^{-1}, 1-j^{-1}+\delta_1 j^{-1}]. \tag{3.27}$$

This implies that $J_{k,j,\rho(x)}$ is close to an almost complex structure for $|x| \in (1-j^{-1}, 1-j^{-1}+\delta_1 j^{-1})$, provided that $100\delta_1 + C_1\bar{\rho}$ is sufficiently small. Since $|\nabla J_k|$ might not be uniformly bounded, we do not have an effective pointwise estimate on $|J_{k,\rho(x)}(x) - J_k(x)|$ as above. Instead, we apply the Poincare inequality (3.20) in the ball $B_\rho(x)$ to $J(y)$ with $J_* = J_\rho(x) = \int_{B_\rho(x)} \phi_\rho(y-x)J(y)dy$,

$$\rho^{-4}\int_{B_\rho(x)} |J(y) - J_\rho(x)|^2 dv \leq C\rho^{-2}\int_{B_\rho(x)} |DJ(y)|^2 dy, \tag{3.28}$$

Replacing J by J_k, we can get that

$$\rho^{-4}\int_{B_\rho(x)} |J_k(y) - J_{k,\rho(x)}(x)|^2 dy \leq C\rho^{-2}\int_{B_\rho(x)} |DJ_k(y)|^2 dy. \tag{3.29}$$

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By the Hölder inequality, we see that

$$\rho^{-2}\int_{B_\rho(x)}|DJ_k(y)|^2dy \le C\left(\int_{B_\rho(x)}|DJ_k|^4dy\right)^{\frac{1}{2}}.$$

Using the fact that $\nabla J = DJ + \partial g * J$ and (3.5), it follows that

$$\rho^{-4}\int_{B_\rho(x)}|J_k(y)-J_{k,\rho(x)}(x)|^2dy \le C\left(\int_{B_\rho(x)}|DJ_k|^4dy\right)^{\frac{1}{2}} \le C(\sqrt{\epsilon_0}+\sqrt{\delta_0}). \tag{3.30}$$

Hence, (3.30) implies that there are many ys in $B_\rho(x)$ such that

$$|J_k(y)-J_{k,\rho(x)}(x)| \le C(\sqrt[4]{\epsilon_0}+\sqrt[4]{\delta_0}).$$

In particular, this implies that

$$|J_{k,\rho(x)}(x)J_{k,\rho(x)}(x)+\mathrm{id}| \le C(\sqrt[4]{\epsilon_0}+\sqrt[4]{\delta_0}).$$

Using (3.25) and the above, we get that

$$|J_{k,j,\rho(x)}J_{k,j,\rho(x)}+\mathrm{id}| \le C(\delta_1+\sqrt[4]{\epsilon_0}+\sqrt[4]{\delta_0}),\ |x|\in[1-\delta_1 j^{-1},1). \tag{3.31}$$

Next we consider $|x|\in(1-j^{-1}+\delta_1 j^{-1},1-\delta_1 j^{-1})$, where $\rho(x)\ge\delta_1\bar\rho$. We compute

$$|J_{k,j,\rho(x)}(x)-J_{0,\rho(x)}(x)| \le \int_B \phi(z)|(J_0-J_k)(x+\rho(x)z)|dz. \tag{3.32}$$

We have

$$\begin{aligned}\int_B \phi(z)|(J_0-J_k)(x+\rho(x)z)|dz \le&\rho(x)^{-4}\int_{B_{\rho(x)}(x)}|J_0(y)-J_k(y)|dy\\ \le&C\rho^{-2}\|J_0-J_k\|_{L^2(B_{3/2})}.\end{aligned}$$

Since J_k converges to J_0 strongly in $W^{1,2}$ and $\rho\ge\delta_1\bar\rho$ for $|x|\in(1-j^{-1}+\delta_1 j^{-1},1-\delta_1 j^{-1})$, we can choose $k_0=k_0(\delta_1,\bar\rho)$ sufficiently large such that

$$C\rho^{-2}\|J_0-J_k\|_{L^2}\le\bar\rho.$$

Hence, we get, for $k\ge k_0$,

$$\rho^{-4}\int_{B_{\rho(x)}(x)}|J_0(y)-J_k(y)|dy\le\bar\rho. \tag{3.33}$$

Using (3.33) and (3.32), we have that, for $k\ge k_0$,

$$|J_{k,j,\rho(x)}(x)-J_{0,\rho(x)}(x)|\le\bar\rho.$$

This together with (3.26) implies that, for $k\ge k_0$,

$$|J_{k,j,\rho(x)}(x)-J_0(x)|\le C\bar\rho,$$

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and in particular, we have

$$|J_{k,\,j,\,\rho(x)}(x)J_{k,\,j\,,\rho(x)}(x)+\mathrm{id}|\leq C\bar{\rho}.$$

Step two: construction of almost complex structure $\tilde{J}_k$ by projecting $J_{k,\,j,\,\rho(x)}$ w.r.t g.

Next we construct a sequence $\tilde{J}_k(x)$ using $J_{k,\,j,\,\rho(x)}(x)$ by the technique we have used in [5, see (4.15), (4.16)]. We briefly recall the construction. In the following, we consider $k\geq k_j$ for each j. For $|x|\in[0,1-j^{-1}]\cup[1,2]$, $\rho=0$, and we have

$$\tilde{J}_k=J_{k,\,j,\,\rho}=J_{k,\,j}=\begin{cases}J_0, & |x|\leq 1-j^{-1}\\ J_k, & |x|\geq 1\end{cases}\tag{3.34}$$

Now consider $|x|\in(1-j^{-1},1)$. Let $S_g(x)$ and $A_g(x)$ be the g-symmetric and g-skew symmetric part of $J_{k,\,j,\,\rho(x)}(x)$, respectively. We have by (3.21)

$$|S_g(x)|=\frac{1}{2}|(J_{k,\,j,\,\rho(x)}(x)+g(x)J^t_{k,\,j,\,\rho(x)}(x)g^{-1}(x))|\leq C\delta_0\bar{\rho}.$$

It follows that

$$|A_g^2+\mathrm{id}|<C\sqrt[4]{\epsilon_0}+C\delta_0\bar{\rho}.$$

Note that $-A_g^2$ is g-symmetric and that it is close to the identity matrix pointwise; in particular, it is positive definite. Denote Q_g to be the g-symmetric matrix such that $Q_g^2=-A_g^2$. Note that Q_g is uniquely determined and it commutes with A_g. Denote $\tilde{J}_k(x)=Q_g^{-1}(x)A_g(x)$ for $x\in B_2$. Then $\tilde{J}_k(x)$ is a g-compatible almost complex structure in B_2. We extend $\tilde{J}_k$ to M by simply putting $\tilde{J}_k=J_k$ on $M\setminus B_1(p)$. Now we establish (3.13). We only need to consider over $B_1(p)$. Note that the L^∞ norm of $J_{k,\,j}, Q_g, A_g, Q_g^{-1}$ and $1/|Q_g^{-1}|$ are all uniformly bounded by a dimensional constant. Since $Q_g^2=-A_g^2$ and $|A_g^2+\mathrm{id}|\ll 1$, we have the following expansion of the matrix.

$$Q_g=\sqrt{\mathrm{id}-(\mathrm{id}+A_g^2)}=\sum_{l=0}^{\infty}\binom{1/2}{l}(\mathrm{id}+A_g^2)^l.\tag{3.35}$$

We can compute directly that $|\nabla Q_g|\leq C|A_g||\nabla A_g|\leq C|\nabla A_g|$. Hence, we obtain $|\nabla\tilde{J}_k|\leq C|\nabla A_g|$. We also need (using $\nabla g=0$)

$$|\nabla S_g|=|\nabla J_{k,j,\rho}+g\nabla J^t_{k,j,\rho}g^{-1}|\leq C|\nabla J_{k,j,\rho}|.$$

We compute

$$|\nabla J_{k,j,\rho}|\leq C\int_B\phi(z)|\nabla J_{k,j}(x+\rho(x)z)|(1+|\rho'|)dz.$$

Hence, we have

$$|\nabla J_{k,j,\rho}|^4\leq C\int_B|\nabla J_{k,j}(x+\rho(x)z)|^4dz.$$

Moreover, we have $|\nabla A_g|\leq|\nabla J_{k,j,\rho}|+|\nabla S_g|\leq C|\nabla J_{k,j,\rho}|$. It follows that

$$\int_B|\nabla\tilde{J}_k|^4dv\leq C\int_B\left(\int_B|\nabla J_{k,j}(x+\rho(x)z)|^4dz\right)dv_x\leq C\int_{B_{3/2}}|\nabla J_{k,j}|^4dv.$$

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Since $\nabla J_{k,j} = \nabla J_k + (1-\psi_j)(\nabla J_0 - \nabla J_k) - (J_0 - J_k)\nabla\psi_j$ and $|\nabla\psi_j| \le 3j$, it follows that

$$\int_{B_{3/2}} |\nabla J_{k,j}|^4 dv \le C\int_{B_2} (|\nabla J_k|^4 + |\nabla J_0|^4) dv + Cj^4 \int_B |J_0 - J_k|^4 dv. \tag{3.36}$$

Using the Sobolev inequality, we know that

$$\|J_0 - J_k\|_{L^4} \le C\|J_0 - J_k\|_{W^{1,2}}.$$

By choosing $k_j = k(j)$ sufficiently large such that for $k \ge k_j$, we can assume that

$$Cj^4 \int_{B_1} (|J_0 - J_k|^2 + |J_0 - J_k|^4 + |\nabla J_0 - \nabla J_k|^2) dv \le j^{-1} \le \epsilon_0. \tag{3.37}$$

This establishes (3.13) and hence $\tilde{J}_k$ and J_k are in the same homotopy class, for $k \ge k_j$.

Step three: the comparison of J_k and $\tilde{J}_k$ implies $\lambda(B_1) \le C\nu(\partial B_1)$.

Fix $\epsilon > 0$. Since J_k is an energy-minimizing sequence, for k sufficiently large we have,

$$\int_M |\Delta J_k|^2 dv \le \int_M |\Delta \tilde{J}_k|^2 dv + \epsilon.$$

By the construction of $\tilde{J}_k$, we get

$$\int_{B_1} |\Delta J_k|^2 dv \le \int_{B_1 \backslash B_{1-j^{-1}}} |\Delta \tilde{J}_k|^2 dv + \int_{B_{1-j^{-1}}} |\Delta J_0|^2 dv + \epsilon.$$

By taking $j \to \infty$ (hence $k \ge k_j \to \infty$), we get (since B_1 is open)

$$\lambda(B_1) + \int_{B_1} |\Delta J_0|^2 dv \le \liminf \int_{B_1 \backslash B_{1-j^{-1}}} |\Delta \tilde{J}_k|^2 dv + \int_{B_{1-j^{-1}}} |\Delta J_0|^2 dv + \epsilon.$$

Hence, we get

$$\lambda(B_1) \le \liminf_{j\to\infty} \int_{B_1 \backslash B_{1-j^{-1}}} |\Delta \tilde{J}_k|^2 dv + \epsilon.$$

Since $\epsilon > 0$ is arbitrary, we have established the estimate

$$\lambda(B_1) \le \liminf_{j\to\infty} \int_{B_1 \backslash B_{1-j^{-1}}} |\Delta \tilde{J}_k|^2 dv. \tag{3.38}$$

Now we need estimates as in (3.15) to control the right-hand side of (3.38). Recall that we have the unique decomposition $J_{k,j,\rho} = A_g + S_g$ and $\tilde{J}_k = Q_g^{-1} A_g$, where Q_g is the unique square root of $-A_g^2$. Using (3.35), we have

$$|\nabla Q_g| \le C|\nabla A_g|, |\Delta Q_g| \le C(|\Delta A_g| + |\nabla A_g|^2).$$

It follows that

$$|\Delta(Q_g^{-1} A_g)| \le C(|\Delta A_g| + |\nabla A_g|^2) \le C(|\Delta J_{k,j,\rho}| + |\nabla J_{k,j,\rho}|^2).$$

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Hence, we obtain

$$\int_{B_1\backslash B_{1-j^{-1}}} |\Delta \tilde{J}_k|^2 dv \le C \int_{B_1\backslash B_{1-j^{-1}}} \left(|\Delta J_{k,j,\rho}|^2 + |\nabla J_{k,j,\rho}|^4\right) dv. \tag{3.39}$$

We compute

$$\begin{aligned} |\nabla J_{k,j}| =& |\nabla J_k + (1-\psi_j)(\nabla J_0 - \nabla J_k) - \nabla \psi_j (J_0 - J_k)| \\ =& |\psi_j \nabla J_k + (1-\psi_j)\nabla J_0 - \nabla\psi_j (J_0 - J_k)| \\ \le& |\nabla J_k| + |\nabla J_0| + Cj|J_0 - J_k|. \end{aligned}$$

Similarly, we compute

$$\begin{aligned} |\Delta J_{k,j}| =& |\Delta J_k + \Delta[(1-\psi_j)(J_0 - J_k)]| \\ \le& |\nabla^2 J_k| + |\nabla^2 J_0| + Cj^2(|J_0 - J_k| + |\nabla J_0 - \nabla J_k|). \end{aligned}$$

Write $y = x + \rho(x)z$. Then we have

$$\left|\frac{\partial y_i}{\partial x_j}\right| \le C(1 + |\rho^{'}|), \left|\frac{\partial^2 y_i}{\partial x_j \partial x_k}\right| \le C(1 + |\rho^{'}| + |\rho^{''}|).$$

Using $|\rho^{'}| + |\rho^{''}| \le 1$, $|\psi_j^{'}| + |\psi_j^{''}| \le 20j^2$, we can then get

$$\begin{aligned} |\nabla J_{k,j,\rho}| =& \left|\int_B \phi(z)\nabla_x J_{k,j}(x + \rho(x)z)dz\right| \\ \le& C\int_B \phi(z)|\nabla_z J_{k,j}|(1 + |\rho^{'}|)dz \\ \le& C\int_{B_1} \phi(z)(|\nabla J_k| + |\nabla J_0| + j|J_0 - J_k|)dz, \end{aligned}$$

where the function is evaluated at $y = x+\rho(x)z$. For any open set $U \subset B_2$, denote $U_{\bar\rho} = \{x : \mathrm{dist}(x, U) < \bar\rho\}$. We have

$$\begin{aligned} \int_U |\nabla J_{k,j,\rho}|^4 dv_x \le& C\int_U \left(\int_{B_1} \phi(z)(|\nabla J_k|(y) + |\nabla J_0|(y) + j|J_0 - J_k|(y))dz\right)^4 dv_x \\ \le& C\int_{U_{\bar\rho}} (|\nabla J_k|^4 + |\nabla J_0|^4 + j^4|J_0 - J_k|^4)dv, \end{aligned} \tag{3.40}$$

where we have used a standard technique to estimate the L^p norm of mollifier approximation (see [4, Lemma 7.2, (7.15)]). Similarly, we compute

$$\begin{aligned} |\Delta J_{k,j,\rho}| =& \left|\int_B \phi(z)\Delta_x\big(J_{k,j}(x + \rho(x)z)\big)dz\right| \\ \le& C\int_B \phi(z)\Big(|\nabla^2 J_k| + |\nabla^2 J_0| + j^2(|\nabla J_0 - \nabla J_k| + |J_0 - J_k|)\Big)dz. \end{aligned}$$

It follows that

$$\int_U |\Delta J_{k,j,\rho}|^2 dv_x \le C\int_{U_{\bar\rho}} \left(|\nabla^2 J_k|^2 + |\nabla^2 J_0|^2 + j^4|J_0 - J_k|^2 + j^4|\nabla J_0 - \nabla J_k|^2\right)dv_x. \tag{3.41}$$

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Take $U = B_1 \setminus B_{1-j^{-1}}$ in the above. We obtain,

$$\int_U (|\nabla J_{k,j,\rho}|^4 + |\nabla^2 J_{k,j,\rho}|^2) dv \le C \int_{U_{\bar\rho}} (|\nabla^2 J_k|^2 + |\nabla J_k|^4) dv + C(R_1 + R_2), \tag{3.42}$$

where the remainder terms read,

$$\begin{aligned} R_1 &= \int_{U_{\bar\rho}} (|\nabla J_0|^4 + |\Delta J_0|^2) dv, \\ R_2 &= j^4 \int_{U_{\bar\rho}} (|J_0 - J_k|^2 + |J_0 - J_k|^4 + |\nabla J_0 - \nabla J_k|^2) dv. \end{aligned}$$

Recall we assume that $k \ge k_j$, such that (3.37) holds. Hence, $R_2 \le j^{-1}$. Certainly when $j \to \infty$, the Lebesgue measure $U_{\bar\rho} \to 0$ and hence $R_1 \to 0$. By (3.42), we obtain

$$\liminf_{j\to\infty} \int_{B_1 \setminus B_{1-j^{-1}}} (|\nabla J_{k,j,\rho}|^4 + |\Delta J_{k,j,\rho}|^2) dv \le C \liminf_{j\to\infty} \int_{U_{\bar\rho}} (|\nabla^2 J_k|^2 + |\nabla J_k|^4) dv. \tag{3.43}$$

Hence, we obtain, by (3.39) and (3.43),

$$\liminf_{j\to\infty} \int_{B_1 \setminus B_{1-j^{-1}}} |\Delta \tilde J_k|^2 dv \le C \liminf_{j\to\infty} \int_{U_{\bar\rho}} (|\nabla^2 J_k|^2 + |\nabla J_k|^4) dv. \tag{3.44}$$

Note that $U = B_1 \setminus B_{1-j^{-1}}$, $\bar\rho = j^{-2}/10$. Fix $\epsilon > 0$. Denote

$$B_{1,\epsilon} = \{x : \mathrm{dist}(x, \partial B_1) < \epsilon\}.$$

For any j sufficiently large, $U_{\bar\rho} \subset \overline{B_{1,\epsilon}}$. Hence, we have

$$\lim_{k\to\infty} \int_{\overline{B_{1,\epsilon}}} (|\nabla^2 J_k|^2 + |\nabla J_k|^4) dv = \lim_{k\to\infty} \mu_k(\overline{B_{1,\epsilon}}) \le \mu(\overline{B_{1,\epsilon}}),$$

where we use the fact that μ_k converges to μ weakly and $\overline{B_{1,\epsilon}}$ is a closed set. Together with (3.44), we get that

$$\lim_{j\to\infty} \int_{B_1 \setminus B_{1-j^{-1}}} |\Delta \tilde J_k|^2 dv \le C\mu(\overline{B_{1,\epsilon}}). \tag{3.45}$$

Note that $\cap_{\epsilon>0} \overline{B_{1,\epsilon}} = \partial B_1$. We obtain $\lim_{\epsilon\to 0} \mu(\overline{B_{1,\epsilon}}) = \nu(\partial B_1)$. With (3.45), we obtain

$$\lim_{j\to\infty} \int_{B_1 \setminus B_{1-j^{-1}}} |\Delta \tilde J_k|^2 dv \le C\nu(\partial B_1). \tag{3.46}$$

Hence, we have obtained the desired estimate

$$\lambda(B_1) \le C\nu(\partial B_1). \tag{3.47}$$

If we replace B_1 by B_r, for $1 \le r \le \frac{3}{2}$, replace $B_1 \setminus B_{1-j^{-1}}$ by $B_r \setminus B_{r-j^{-1}}$ and apply the same arguments as in the proof (3.47), we obtain

$$\lambda(B_r) \le C\nu(\partial B_r). \tag{3.48}$$

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Since ν is totally bounded, this implies that $\lambda(B_r) \equiv 0$ for $r \leq 3/2$. This completes the proof of Lemma 3.9. □

With Lemma 3.9, we prove Lemma 3.7, which asserts that the homotopy class of J_0 is either σ or $p(\sigma)$, for a minimizing sequence J_k in σ.

Proof of Lemma 3.7. Lemma 3.9 implies that J_k converges to J_0 strongly in $W^{2,2}(M\backslash U_{r/2})$, where U is the collection of finitely many disjoint geodesic balls $B(p_i)$, $i = 1, \cdots N$ with radius $r/2$, for any sufficiently small positive number r. Moreover, for any point $p \in M\backslash U_r$, we have that

$$\int_{B_{r/2}(p)} |\nabla J_k|^4 + |\nabla^2 J_k|^2 \leq \epsilon_0.$$

Hence, we can construct $\tilde{J}_k$ such that

$$\tilde{J}_k(p) = J_0(p), p \in M\backslash U_{2r},$$
$$\tilde{J}_k(p) = J_k(p), p \in U_{r/2}$$

and

$$\int_M |\nabla J_k - \nabla\tilde{J}_k|^4 \leq C\epsilon_0.$$

This implies that $\tilde{J}_k$ is in σ. In short, we construct $\tilde{J}_k$ in the same homotopy class which coincides J_0 over $M\backslash U_{2r}$, where the convergence is strongly in $W^{2,2}$.

Given a homotopy class σ of almost complex structures on M, we recall the construction of $p(\sigma)$ (see Donaldson [3, Section 6]). A compatible almost complex structure J on an oriented Riemannian four-manifold M can be considered as a section of the associated $SO(4)/U(2)$-bundle over M (the sphere bundle of Λ^2_+, known as the 'twistor space'). Suppose there are two almost complex structures J_1 and J_2, which agree each other outside a small ball in M. Over this ball B, J_1 and J_2 compare by a map from S^4 to the fibre of the twistor bundle, hence defining an element in $[S^4, S^2] = \mathbb{Z}/2$. If σ is a homotopy class, $p(\sigma)$ agrees with σ outside a small ball, and over the ball the two compare by the nonzero element of $[S^4, S^2] = \mathbb{Z}/2$. Hence, if J_1 and J_2 agree outside a small ball, then either they are in the same homotopy class or their homotopy classes are related by the map p.

Applying this to $\tilde{J}_k$ and J_0 above, the homotopy classes differ by the composition p^k, for some $k \leq N$. Since $p \circ p = id$, it follows that J_0 is either in σ or $p(\sigma)$. □

Theorem 1.3 follows as an immediate consequence of Lemma 3.7.

Proof of Theorem 1.3. Given a pair of homotopy classes σ and $p(\sigma)$, choose a minimizing sequence J_k of bi-energy functional over σ and $p(\sigma)$ with respect to (M, g). By passing to a subsequence, we assume that J_k remains in homotopy class σ and it converges to a biharmonic almost complex structure J_0 weakly in $W^{2,2}$. Either J_0 is in σ or in $p(\sigma)$. Hence, there is an energy-minimizing biharmonic almost complex structure in the pair σ and $p(\sigma)$. Indeed, in this case, since J_k is a minimizing sequence over σ and $p(\sigma)$, and all J_k remains in σ, the limit J_0 has to be in σ as well since no extra energy can be concentrated. □

4. An intuitive conjectural picture

A $K3$ surface is a compact simply connected complex surface S with trivial canonical bundle. Let σ denote the homotopy class of the standard complex structures on S with $c_1 = 0$. S. Donaldson [3, Corollary 6.5] proved, using his polynomial invariants, the following.

Theorem 4.1 (Donaldson). *The homotopy class $p(\sigma)$ on S with $c_1 = 0$ does not contain any integrable representative.*

Theorem 1.3 does not specify precisely whether σ or $p(\sigma)$ contains an energy-minimizing biharmonic almost complex structure. There could be a couple options, as follows.

1. Both σ and $p(\sigma)$ contain an energy-minimizing biharmonic almost complex structure, which might or might not have the same energy.
2. Only one homotopy contains an energy-minimizing biharmonic almost complex structure, while the other does not.

We believe the following and Theorem 4.1 serves as an example.

Conjecture 4.2. *Given two homotopy classes σ and $p(\sigma)$ on M, exactly one homotopy class among the pair $(\sigma, p(\sigma))$ contains an energy-minimizing biharmonic almost complex structure. Suppose σ is the class containing a minimizer. Then $p(\sigma)$ does not contain a minimizer, and a minimizing sequence in $p(\sigma)$ converges weakly in $W^{2,2}$ to a minimizer in σ, which bubbles off a nonconstant biharmonic map from $\mathbb{R}^4$ to S^2 with a nontrivial relative homotopy class.*

In the process of energy-minimizing, the formation of a bubble gives a nonconstant extrinsic biharmonic map from $\mathbb{R}^4$ to S^2 by a blowup argument. But it does not seem to be straightforward to specify its relative homotopy class, even though it is intuitive that it should correspond to the nonzero element in $\pi_4(S^2)$ topologically. We believe the following.

Conjecture 4.3. *In the nontrivial (relative) homotopy class of $\mathbb{R}^4$ to S^2 corresponding to the nonzero element $\pi_4(S^2)$, there exists an energy-minimizing extrinsic biharmonic map. For maps from S^4 to S^2 in the homotopy class corresponding to nonzero element $\pi_4(S^2)$, there exists no energy-minimizing extrinsic biharmonic maps.*

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References

[1] S.-Y. A. Chang, L. Wang and P. C. Yang, 'A regularity theory of biharmonic maps', *Comm. Pure Appl. Math.* **52**(9) (1999), 1113–1137.
[2] S. K. Donaldson and P. B. Kronheimer, *The Geometry of Four-Manifolds*, Oxford Science Publications (Clarendon Press, Oxford, 1997).
[3] S. K. Donaldson, 'Polynomial invariants for smooth four-manifolds', *Topology* **29**(3) (1990), 257–315.
[4] D. Gilbarg and N. Trudinger, *Elliptic Partial Differential Equations of Second Order* (Springer-Verlag, Berlin, 1977).
[5] W. He, 'Energy-minimizing harmonic almost complex structures', arxiv.org/abs/1907.12211.
[6] W. He and R. Jiang, 'Polyharmonic almost complex structures', *J. Geom. Anal* **31**(12) (2021), 11648–11684.
[7] W. He, R. Jiang and L. Lin, 'Existence of polyharmonic maps in critical dimensions', arxiv.org/abs/1911.00849.
[8] F. H. Lin, 'Gradient estimates and blow-up analysis for stationary harmonic maps', *Ann. of Math.* **149**(3) (1999), 785–829.
[9] F. H. Lin, 'Mapping problems, fundamental groups and defect measures', *Acta Math. Sin.* **15**(1) (1999), 25–52.
[10] S. Luckhaus, 'Partial Hölder continuity for minima of certain energies among maps into a Riemannian manifold', *Indiana Univ. Math. J.* **37**(2) (1988), 349–367.
[11] P. Teichner, https://mathoverflow.net/questions/59820/a-question-on-classification-of-almost-complex-structures-on-4-manifolds.
[12] J. Sacks and K. Uhlenbeck, 'The existence of minimal immersions of 2-spheres', *Ann. of Math.* **113**(1) (1981), 1–24.

[13] R. Schoen and K. Uhlenbeck, ‘A regularity theory for harmonic maps’, *J. Differential Geom.* **17**(2) (1982), 307–335.
[14] L. Simon, *Theorems on Regularity and Singularity of Energy Minimizing Maps*, Lectures in Mathematics (Birkhäuser Verlag, Basel, 1996).
[15] B. White, ‘Infima of energy functionals in homotopy classes of mappings’, *J. Differential Geom.* **23**(2) (1986), 127–142.
[16] C. Wood, ‘Harmonic almost complex structures’, *Compositio Math.* **99**(2) (1995), 183–212.