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NOWHERE DENSE SUBSETS OF METRIC SPACES WITH APPLICATIONS TO STONE-ČECH COMPACTIFICATIONS

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Let X be a metric space. Assume either that X is locally compact or that X has no more than countably many isolated points. It is proved that if F is a nowhere dense subset of X, then it is regularly nowhere dense (in the sense of Katětov) and hence is contained in the topological boundary of some regular-closed subset of X. This result is used to obtain new properties of the remote points of the Stone-Čech compactification of a metric space without isolated points.

Let βX denote the Stone-Čech compactification of the completely regular Hausdorff space X. Fine and Gillman [**3**] define a point p of βX to be *remote* if p is not in the βX-closure of a discrete subset of X. The problem of characterizing the remote points of βX when X is a metric space has occupied the attention of a number of mathematicians. Plank [**8**] characterizes the remote points of βX in terms of zero-sets of X; in this paper we characterize the remote points of βX in terms of regular-closed subsets of X. Mandelker [**6**, Theorem 11.2] characterizes a remote point p of $\beta\mathbf{R}$ (where $\mathbf{R}$ is the space of real numbers) in terms of the unique z-ultrafilter converging to p; we extend this characterization to metric spaces without isolated points.

We will use, except where stated, the notation and terminology of the Gillman-Jerison text [**4**]. In particular, bd A denotes the topological boundary of a subset A of a topological space X, and $\mathbf{N}$ denotes the set of positive integers (used as an index set). If p is a point of the metric space X and if $\epsilon > 0$, then $S(p, \epsilon)$ will denote the set $\{x \in X: d(x, p) < \epsilon\}$. A subset A of a topological space is *regular-closed* if $A = \text{cl}(\text{int } A)$. The family of all regular-closed subsets of a space X is denoted by $R(X)$.

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1. Metric space properties. For $\epsilon > 0$, a subset A of a metric space is ϵ-*discrete* if for distinct points x and y of A, $d(x, y) \geqq \epsilon$. Note that ϵ-discrete subsets of metric spaces are closed. It is a consequence of Zorn's lemma that a metric space contains a maximal ϵ-discrete subset for each $\epsilon > 0$; evidently

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an ϵ-discrete subset A is maximal if and only if $S(y, \epsilon) \cap A \neq \emptyset$ for each point y of the metric space.

1.1 LEMMA. *Let A be a non-empty ϵ-discrete subset of a metric space X, let $0 < \delta < \epsilon/2$, and let $U = \bigcup \{S(x, \delta): x \in A\}$. Then*

$$\text{cl } U = \bigcup \{\text{cl } S(x, \delta): x \in A\}.$$

Proof. Clearly, $\bigcup \{\text{cl } S(x, \delta): x \in A\} \subseteq \text{cl } U$. Conversely, suppose $p \in \text{cl } U$ and put $\gamma = \epsilon - 2\delta$; then $\gamma > 0$. If x and y are distinct points of A such that $S(p, \gamma/2) \cap S(x, \delta) \neq \emptyset$ and $S(p, \gamma/2) \cap S(y, \delta) \neq \emptyset$, then $d(x, y) < \gamma + 2\delta = \epsilon$, which is a contradiction. Thus as $p \in \text{cl } U$, there exists a unique point $a \in A$ such that $S(p, \gamma/2) \cap S(a, \delta) \neq \emptyset$. Hence if V is any neighborhood of p, then $V \cap S(a, \delta) \neq \emptyset$ so $p \in \text{cl } S(a, \delta)$.

Lemma 1.1 is false for $\delta \geqq \epsilon/2$. To show this, let

$$X = \{(x, x/n): x \in \mathbf{R}, 0 \leqq x, n \in \mathbf{N}\}$$

with the metric d defined by:

$$\begin{aligned} d((x, x/n), (y, y/m)) &= |x - y| \text{ if } n = m \\ &= x + y \text{ if } n \neq m. \end{aligned}$$

Let $A = \{(1 + 1/n, (1 + 1/n)/n): n \in \mathbf{N}\}$. Then A is 2-discrete, but if $V = \bigcup \{S(x, 1): x \in A\}$, then $(0, 0) \in \text{cl } V - \bigcup \{\text{cl } S(x, 1): x \in A\}$.

In [**5**], Katětov calls a subset F of a topological space X *regularly nowhere dense* if $\text{cl } F = \text{cl } V \cap \text{cl } W$, where V and W are disjoint open subsets of X. If a set is regularly nowhere dense then it is evidently a subset of the boundary of some regular-closed set.

1.2 LEMMA. *In a metric space without isolated points, each nowhere dense subset is regularly nowhere dense.*

Proof. Let X be a metric space without isolated points and F a nowhere dense subset of X. Since cl F is also nowhere dense, we may assume that F is closed in X. For each $n \in \mathbf{N}$, put $G_n = \{x \in X: \text{d}(x, F) < 1/n\}$. Let A_n be a maximal $1/n$-discrete subset of $G_n - G_{n+1}$, and put $A = \bigcup \{A_n: n \in \mathbf{N}\}$. It easily follows that $F = \bigcap \{G_n: n \in \mathbf{N}\}$ and $\text{cl } G_{n+1} \subseteq G_n$ for each $n \in \mathbf{N}$.

We now show that $\text{cl } A - A = F$. Since

$$\text{cl } A = \bigcup \{A_i: 1 \leqq i \leqq n\} \cup \text{cl } (\bigcup \{A_i: i > n\}),$$

it follows that $\text{cl } A - A \subseteq \text{cl } G_{n+1}$ for each $n \in \mathbf{N}$; hence $\text{cl } A - A \subseteq F$. Conversely, let $p \in F$ and let $\epsilon > 0$. Choose $k \in \mathbf{N}$ such that $1/k < \epsilon$. Since $S(p, 1/k) \subseteq G_k$ and since F is nowhere dense, there is an $m \geqq k$ such that $S(p, 1/k) \cap (G_{m+1} - G_m) \neq \emptyset$. As A_m is a maximal $1/m$-discrete subset of $G_m - G_{m+1}$ and $1/m \leqq 1/k$, it follows that $a \in S(p, 1/k)$ for some $a \in A_m$; thus $p \in \text{cl } A$. Thus $F = \text{cl } A - A$ as claimed since $A \cap F = \emptyset$.

We next claim that A is a discrete subset of X. Let $x \in A_n$, and let δ be the smallest of $d(x, (X - G_n) \cup \text{cl } G_{n+2})$, $d(x, A_{n+1})$ and $1/n$. It is easily verified that $S(x, \delta) \cap A = \{x\}$.

Let $V = X - \mathrm{cl}\, A$. Evidently $F \cap V = \emptyset$; we now show that $F \subseteq \mathrm{cl}\, V$. Let $p \in F$ and let $\epsilon > 0$ be given. As $F = \mathrm{cl}\, A - A$, there is $n \in \mathbf{N}$ and $a \in A_n$ such that $a \in S(p, \epsilon)$. As A is discrete and disjoint from F, there exists $\delta > 0$ such that $S(a, \delta) \subseteq S(p, \epsilon)$, $S(a, \delta) \cap A = \{a\}$, and $S(a, \delta) \cap F = \emptyset$. Since a is not isolated, there exists $b \in S(a, \delta)$ such that $b \neq a$. Evidently $b \in V$, so $F \subseteq \mathrm{cl}\, V$.

Let B_n be a maximal $1/n$-discrete subset of $(G_n - G_{n+1}) \cap V$, and let $B = \bigcup \{B_n : n \in \mathbf{N}\}$. We claim that $\mathrm{cl}\, B - B = F$. Since

$$\mathrm{cl}\, B = \bigcup \{B_i : 1 \leqq i \leqq n\} \cup \mathrm{cl}\, (\bigcup \{B_i : i > n\}),$$

then $\mathrm{cl}\, B - B \subseteq \mathrm{cl}\, G_{n+1}$ for all $n \in \mathbf{N}$. Hence $\mathrm{cl}\, B - B \subseteq F$. Let $p \in F$. Since $B \cap F = \emptyset$, it suffices to show that $p \in \mathrm{cl}\, B$. Let $\epsilon > 0$ be given. There is $n \in \mathbf{N}$ such that $1/n < \epsilon$. Since $F \subseteq \mathrm{cl}\, V$, there is an $x \in V \cap S(p, 1/2n)$. Since $S(p, 1/2n) \subseteq G_{2n}$, there exists $k \geqq 2n$ such that $x \in G_k - G_{k+1}$. As B_k is a maximal $1/k$-discrete subset of $G_k - G_{k+1}$, there exists $b \in B_k \cap S(x, 1/2n)$. Now

$$d(p, b) \leqq d(p, x) + d(x, b) < 1/2n + 1/2n < \epsilon,$$

so $p \in \mathrm{cl}\, B$. Thus $\mathrm{cl}\, B - B = F$. Essentially the same proof as used above to show that A is a discrete subset of X can be used to show that B is a discrete subset of X. Also note that $A \cap \mathrm{cl}\, B = \emptyset$ and $B \cap \mathrm{cl}\, A = \emptyset$.

Let $p \in A_n$ for some $n \in \mathbf{N}$. Since A is discrete and $A \cap \mathrm{cl}\, B = \emptyset$, there exists $\delta(p) > 0$ such that $\delta(p) < 1/n$, $S(p, \delta(p)) \subseteq X - \mathrm{cl}\, B$, and $S(p, \delta(p)) \cap A = \{p\}$. Let $\epsilon(p) = \delta(p)/3$, $W_n = \bigcup \{S(p, \epsilon(p)) : p \in A_n\}$, and $W = \bigcup \{W_n : n \in \mathbf{N}\}$. Define $\epsilon(p)$ similarly for each $p \in B_n$. Let $U_n = \bigcup \{S(p, \epsilon(p)) : p \in B_n\}$ and $U = \bigcup \{U_n : n \in \mathbf{N}\}$. We now show that

$$\mathrm{cl}\, W = \bigcup \{\mathrm{cl}\, W_n : n \in \mathbf{N}\} \cup F. \tag{1}$$

Clearly, $\mathrm{cl}\, W_n \subseteq \mathrm{cl}\, W$ for each $n \in \mathbf{N}$. Since $A \subseteq W$ and $F \subseteq \mathrm{cl}\, A$, it follows that $\bigcup \{\mathrm{cl}\, W_n : n \in \mathbf{N}\} \cup F \subseteq \mathrm{cl}\, W$. Conversely, since

$$\mathrm{cl}\, W = \bigcup \{\mathrm{cl}\, W_i : 1 \leqq i \leqq 2n\} \cup \mathrm{cl}\, (\bigcup \{W_i : i > 2n\}) \tag{2}$$

it follows that

$$\mathrm{cl}\, W - \bigcup \{\mathrm{cl}\, W_i : i \in \mathbf{N}\} \subseteq \mathrm{cl}\, G_n \tag{3}$$

for all $n \in \mathbf{N}$; for, if $i > 2n$, then $x \in W_i$ implies that $d(x, p) < 1/i$ for some $p \in A_i$. As $A_i \subseteq G_i$, $d(p, F) < 1/i$, and so $d(x, F) < 2/i < 1/n$. It follows that $\mathrm{cl}\, (\bigcup \{W_i : i > 2n\}) \subseteq \mathrm{cl}\, G_n$, and (3) now follows from (2). As (3) holds for all $n \in \mathbf{N}$, we conclude that $\mathrm{cl}\, W - \bigcup \{\mathrm{cl}\, W_i : i \in \mathbf{N}\} \subseteq F$, and so $\mathrm{cl}\, W \subseteq \bigcup \{\mathrm{cl}\, W_i : i \in \mathbf{N}\} \cup F$. Thus (1) holds. A similar argument shows that

$$\mathrm{cl}\, U = \bigcup \{ \mathrm{cl}\, U_i : i \in \mathbf{N}\} \cup F. \tag{4}$$

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We now show that $\mathrm{cl}\, W \cap \mathrm{cl}\, U = F$. It suffices, in view of (1) and (4), to show that

$$[\bigcup \{\mathrm{cl}\, W_n : n \in \mathbf{N}\}] \cap [\bigcup \{\mathrm{cl}\, U_n : n \in \mathbf{N}\}] = \emptyset.$$

Assume that there is a point $x \in \mathrm{cl}\, W_n \cap \mathrm{cl}\, U_m$ for $n, m \in \mathbf{N}$. By Lemma 1.1 there is $p \in A_n$ and $q \in B_m$ such that $x \in \mathrm{cl}\, S(p, \epsilon(p)) \cap \mathrm{cl}\, S(q, \epsilon(q))$. Now

$$d(p, q) \leqq d(p, x) + d(q, x) \leqq \epsilon(p) + \epsilon(q) < \max \{\delta(p), \delta(q)\},$$

which is a contradiction. Thus $F = \mathrm{cl}\, W \cap \mathrm{cl}\, U$. As the above argument also shows that U and W are disjoint (open) subsets of X, it follows that F is regularly nowhere dense.

1.3 Corollary. *In a metric space without isolated points, each closed nowhere dense subset is the intersection of the closures of a countably infinite collection of pairwise disjoint open subsets.*

Proof. Throughout the proof, the notation introduced in the proof of Lemma 1.2 is used. Let $S_1 = W$, $T_1 = V$, and let C_n be a maximal $1/n$-discrete subset of $(T_1 - B) \cap (G_n - G_{n+1})$. Put $C = \bigcup \{C_n : n \in \mathbf{N}\}$; it follows (as in the proof that $\mathrm{cl}\, B - B = F$ in Lemma 1.2) that $\mathrm{cl}\, C - C = F$. Also, disjoint open sets S'_2 containing B and T'_2 containing C can be constructed with the property that $\mathrm{cl}\, S'_2 \cap \mathrm{cl}\, T'_2 = F$. Let $S_2 = S'_2 \cap T_1$ and $T_2 = T'_2 \cap T_1$; S_2 and T_2 are disjoint open sets. Since $B \subseteq S_2$, $C \subseteq T_2$, $F \subseteq \mathrm{cl}\, B$, and $F \subseteq \mathrm{cl} C$, then $\mathrm{cl}\, S_2 \cap \mathrm{cl}\, T_2 = F$. By induction we can obtain a family $\{S_n : n \in \mathbf{N}\}$ of pairwise disjoint open sets satisfying $F = \bigcap \{\mathrm{cl}\, S_n : n \in \mathbf{N}\}$.

We now generalize Lemma 1.2 in two different directions.

1.4 Lemma. *Let X be a metric space with a dense set of countably many isolated points. Then every nowhere dense subset of X is regularly nowhere dense.*

Proof. Let I denote the set of isolated points of X. If F is a nowhere dense subset of X then $F \cap I = \emptyset$. Our assumptions imply that X is separable (and hence second countable). Thus F is also separable. Let $\{p_n : n \in \mathbf{N}\}$ be a countable dense subset of F. We define, inductively, a subset $\{x(n, m) : n, m \in \mathbf{N} \text{ and } n \leqq m\}$ of I as follows.

(i) Choose $x(1, 1) \in I$ such that $d(x(1, 1), p_1) < 1$.

(ii) Choose $x(1, 2)$ and $x(2, 2) \in I$ such that $x(1, 1)$, $x(1, 2)$, and $x(2, 2)$ are distinct and $d(x(1, 2), p_1) < 1/2$, $d(x(2, 2), p_2) < 1/2$.

(iii) Suppose we have chosen a set $\{x(i, j) : 1 \leqq i \leqq j \leqq n\}$ of $n(n + 1)/2$ distinct points of I such that $d(x(i, j), p_i) < 1/j$. As I is dense in X we can find a set $\{x(i, n + 1) : 1 \leqq i \leqq n + 1\}$, contained in $I - \{x(i, j) : 1 \leqq i \leqq j \leqq n\}$, of $n + 1$ distinct points of I such that $d(x(i, n + 1), p_i) < 1/(n + 1)$ for $i = 1, \ldots, n + 1$.

Hence for each positive integer n we have a sequence S_n of distinct points $\{x(n, i) : i \geqq n\}$ with limit p_n. Note that $d(y, p_n) < 1/n$ for each $y \in S_n$ and

$n \in \mathbf{N}$. Put $G_1 = \{x(n, 2m): n, m \in \mathbf{N}$ and $2m \geqq n\}$ and $G_2 = \{x(n, 2m+1): n, m \in \mathbf{N}$ and $2m + 1 \geqq n\}$. We now claim that $\mathrm{cl}\, G_1 = G_1 \cup \mathrm{cl}\, F$. Since $p_n \in \mathrm{cl}\, G_1$ for each $n \in \mathbf{N}$ and $(p_n)_{n \in \mathbf{N}}$ is dense in F, it follows that $G_1 \cup \mathrm{cl}\, F \subseteq \mathrm{cl}\, G_1$. Conversely, if $x \in \mathrm{cl}\, G_1 - G_1$, choose a sequence $\{a_j: j \in \mathbf{N}\}$ of points of G_1 that converges to x. There are two possibilities. First, there may exist $n \in \mathbf{N}$ such that $a_j \in S_n$ for infinitely many values of j. It immediately follows that $x = p_n$ and so $x \in F$. Second, for each $n \in \mathbf{N}$ there may be only finitely many j for which $a_j \in S_n$. If so, let $\epsilon > 0$. Then $d(a_j, x) < \epsilon/2$ for infinitely many values of j, so we can find k and n in $\mathbf{N}$ such that $d(a_k, x) < \epsilon/2$, $a_k \in S_n$, and $1/n < \epsilon/2$. Then

$$d(x, p_n) < d(x, a_k) + d(a_k, p_n) < \epsilon/2 + 1/n < \epsilon.$$

It follows that $x \in \mathrm{cl}\, \{p_n: n \in \mathbf{N}\} = \mathrm{cl}\, F$. Thus $\mathrm{cl}\, G_1 = G_1 \cup \mathrm{cl}\, F$ as claimed. A similar argument shows that $\mathrm{cl}\, G_2 = G_2 \cup \mathrm{cl}\, F$, and so $\mathrm{cl}\, F = \mathrm{cl}\, G_1 \cap \mathrm{cl}\, G_2$. Thus F is regularly nowhere dense, as G_1 and G_2 are disjoint open sets. Note also that $\mathrm{cl}\, F = \mathrm{cl}\, G_1 - G_1 = \mathrm{cl}\, G_2 - G_2$.

1.5 THEOREM. *Let X be a metric space. If either*

(i) *X has no more than countably many isolated points, or*

(ii) *X is locally compact,*

then every nowhere dense subset of X is regularly nowhere dense.

Proof. Let F be a nowhere dense subset of X. First assume that X has no more than countably many isolated points. As before, denote the set of isolated points of X by I. Evidently $\mathrm{cl}[X - \mathrm{cl}\, I]$ has no isolated points. Now $\mathrm{cl}\, F \cap \mathrm{cl}[X - \mathrm{cl}\, I]$ is a nowhere dense subset of $\mathrm{cl}[X - \mathrm{cl}\, I]$; for, if not, there exists W open in X such that $\emptyset \neq W \cap \mathrm{cl}[X - \mathrm{cl}\, I] \subseteq \mathrm{cl}\, F$. Then $\emptyset \neq W \cap (X - \mathrm{cl}\, I) \subseteq \mathrm{cl}\, F$, which contradicts the fact that $\mathrm{int}(\mathrm{cl}\, F) = \emptyset$. Thus by Lemma 1.2 there exist disjoint open subsets U' and V' of $\mathrm{cl}[X - \mathrm{cl}\, I]$ such that

$$\mathrm{cl}\, F \cap \mathrm{cl}[X - \mathrm{cl}\, I] = \mathrm{cl}\, U' \cap \mathrm{cl}\, V'.$$

Put $U = U' \cap [X - \mathrm{cl}\, I]$ and $V = V' \cap [X - \mathrm{cl}\, I]$. Then U and V are open in X and it follows from [**4**, 0.12] that

$$\mathrm{cl}\, F \cap \mathrm{cl}[X - \mathrm{cl}\, I] = \mathrm{cl}\, U \cap \mathrm{cl}\, V. \tag{5}$$

It follows from the proof of Lemma 1.4 that there exist disjoint subsets T and W of I such that

$$\mathrm{cl}\, F \cap \mathrm{cl}\, I = \mathrm{cl}\, T - T = \mathrm{cl}\, W - W. \tag{6}$$

As $V \cap \mathrm{cl}\, I = \emptyset$, we have

$$\mathrm{cl}\, T \cap \mathrm{cl}\, V \subseteq \mathrm{cl}\, T - T \subseteq \mathrm{cl}\, F;$$

similarly, $\mathrm{cl}\, U \cap \mathrm{cl}\, W \subseteq \mathrm{cl}\, F$. Thus it follows from (5) and (6) that

$$\mathrm{cl}\, F = \mathrm{cl}\, (T \cup U) \cap \mathrm{cl}\, (V \cup W),$$

and of course $T \cup U$ and $V \cup W$ are disjoint. Hence F is a regularly nowhere dense subset of X.

Now assume that X is locally compact. As every metric space is paracompact (see, for example, [**1**, Theorem 9.5.3]), X is a locally compact, paracompact space. Hence X is a free union of a family $\{X_\alpha : \alpha \in \Sigma\}$ of locally compact, σ-compact metric spaces (see [**1**, Theorem 11.7.3]). Each X_α has at most countably many isolated points. If F is a nowhere dense subset of X, it follows from the above argument that for each $\alpha \in \Sigma$ there exist disjoint open subsets U_α and V_α of X_α such that

$$\mathrm{cl}_X F \cap X_\alpha = \mathrm{cl}_{X_\alpha} U_\alpha \cap \mathrm{cl}_{X_\alpha} V_\alpha.$$

Put $\bigcup_{\alpha \in \Sigma} U_\alpha = U$ and $\bigcup_{\alpha \in \Sigma} V_\alpha = V$. Then U and V are disjoint open subsets of X and $\mathrm{cl}_X F = \mathrm{cl}_X U \cap \mathrm{cl}_X V$. Thus F is regularly nowhere dense.

1.6 COROLLARY. *Let F be a closed nowhere dense subset of a metric space X. If either*

(i) *X has no more than countably many isolated points, or*

(ii) *X is locally compact,*

then F can be written as the intersection of the closures of a countably infinite family of pairwise disjoint open subsets of X.

Proof. If X has a dense, countably infinite set of isolated points, a simple modification of the proof of Lemma 1.4 gives the desired result. Now, making use of Corollary 1.3, proceed exactly as in the proof of Theorem 1.5.

1.7 *Remarks.* (1) Mandelker, in [**7**, Theorem 2.3], proved Lemma 1.2 for the special case in which X is the real line.

(2) Theorem 1.5 does not hold for more general classes of topological spaces. As an example, let $\mathfrak{N}$ denote the countable discrete space. Then $\beta\mathfrak{N}$ is extremally disconnected so $R(\beta\mathfrak{N})$ consists of the open-and-closed subsets of $\beta\mathfrak{N}$ (see [**4**, 1H and 6M]). Thus if $A \in R(\beta\mathfrak{N})$, then $\mathrm{bd}_{\beta\mathfrak{N}} A = \emptyset$. If $p \in \beta\mathfrak{N} - \mathfrak{N}$, then $\{p\}$ is a closed nowhere dense subset of $\beta\mathfrak{N}$, but $\{p\}$ is not regularly nowhere dense. Obviously any nondiscrete extremally disconnected space will yield a similar type of counterexample.

(3) The authors do not know if every nowhere dense subset of an arbitrary metric space is regularly nowhere dense.

2. Properties of the remote points of βX.

2.1 *Definition.* Let X be a completely regular Hausdorff space. A point $p \in \beta X$ is called a *remote point* of βX if p is not in the βX-closure of any discrete subspace of X.

Remote points were first defined and studied by Fine and Gillman [**3**], who proved that if the continuum hypothesis is assumed, then the set of remote points of $\beta\mathbf{R}(\beta\mathbf{Q})$ is dense in $\beta\mathbf{R} - \mathbf{R}(\beta\mathbf{Q} - \mathbf{Q})$ ($\mathbf{R}$ denotes the space of reals, $\mathbf{Q}$ the space of rationals), Let $T(\beta X)$ denote the set of remote points of βX. If A is closed in X, let $A^* = \mathrm{cl}_{\beta X} A - X$ (in particular, $\beta X - X = X^*$). Plank [**8**] has given the following characterization of $T(\beta X)$.

2.2 THEOREM [**8**, 5.3 and 5.5]. *Let X be a metric space without isolated points, and let $\mathbf{Z}(X)$ denote the family of zero-sets of X. Then*

$$\begin{aligned} T(\beta X) &= \bigcap \{X^* - \mathrm{bd}_{X^*} Z^*: Z \in \mathbf{Z}(X)\} \\ &= \bigcap \{X^* - (\mathrm{bd}_X Z)^*: Z \in \mathbf{Z}(X)\}. \end{aligned}$$

Assume the continuum hypothesis. If X is separable, then $T(\beta X)$ is dense in X^*.

Let X be a completely regular Hausdorff space. It is well-known (see, for example, [**9**, p. 66]) that $R(X)$ is a complete Boolean algebra under the following operations:

(1) $\bigvee_\alpha A_\alpha = \mathrm{cl}_X [\bigcup_\alpha A_\alpha]$;
(2) $\bigwedge_\alpha A_\alpha = \mathrm{cl}_X[\mathrm{int}_X \bigcap_\alpha A_\alpha]$;
(3) $A' = \mathrm{cl}_X(X - A)$ (A' denotes the Boolean-algebraic complement of A).

The following result is a portion of [**10**, 2.3, 2.7, 2.11].

2.3 THEOREM. *Let X be a metric space. Then*:
(i) *The map $A \to A^*$ is a Boolean algebra homomorphism from $R(X)$ into $R(X^*)$.*
(ii) $(\mathrm{bd}_X A)^* = \mathrm{bd}_{X^*} A^*$ *for each $A \in R(X)$.*

Let $[R(X)]^*$ denote the image of $R(X)$ under the above homomorphism. We can now obtain our characterization of $T(\beta X)$.

2.4 THEOREM. *Let X be a metric space without isolated points, and let $p \in X^*$. Then $p \in T(\beta X)$ if and only if p is not the X^*-boundary of any member of the Boolean subalgebra $[R(X)]^*$ of $R(X^*)$.*

Proof. Every closed subset of X is a zero-set of X, so

$$\{\mathrm{bd}_X A: A \in R(X)\} \subseteq \{\mathrm{bd}_X A: Z \in \mathbf{Z}(X)\}.$$

Thus by Theorems 2.2 and 2.3(ii) it follows that

$$\begin{aligned} T(\beta X) &\subseteq \bigcap \{X^* - (\mathrm{bd}_X A)^*: A \in R(X)\} \\ &= \bigcap \{X^* - \mathrm{bd}_{X^*} A^*: A \in R(X)\}. \end{aligned}$$

Conversely, by Lemma 1.2 if $Z \in \mathbf{Z}(X)$, then there exists $A \in R(X)$ such that $\mathrm{bd}_X Z \subseteq \mathrm{bd}_X A$. It immediately follows from Theorems 2.2 and 2.3 that

$$\bigcap \{X^* - \mathrm{bd}_{X^*} A^*: A \in R(X)\} \subseteq T(\beta X),$$

and our theorem is proved.

2.5 *Remark*. The characterization of $T(\beta X)$ given in Theorem 2.4 is similar to a characterization of the set of P-points of X^* if X is locally compact and realcompact (P-points are discussed in [**4**, 4L et seq.]). For such an X it is known that $\mathbf{Z}(X^*) \subseteq R(X^*)$ (see [**2**, 3.1]). It follows easily that the P-points of X^* are precisely those points of X^* that are not on the X^*-boundary of any member of the Boolean subalgebra of $R(X^*)$ generated by $\mathbf{Z}(X^*)$.

Let X be a nowhere locally compact, separable metric space (for example, the rationals or the irrationals). We can use Theorems 2.2 and 2.4 to give a characterization of $T(\beta X)$ that depends only on the topology of X^* and not on the topology of X, despite the fact that $T(\beta X)$ is defined in terms of the topology of X.

2.6 THEOREM. *Let X be a nowhere locally compact, separable metric space. If the continuum hypothesis is assumed, then $T(\beta X)$ can be characterized as the largest extremally disconnected dense subspace of X^*.*

Proof. If the continuum hypothesis is assumed, then by Theorem 2.2 $T(\beta X)$ is dense in X^*. It is proved in [**10**, 4.5] that our assumptions about X imply that $T(\beta X)$ is extremally disconnected. Let $p \in X^* - T(\beta X)$, and let E be a dense subspace of X^* that contains p. By Theorem 2.4 there exists $A \in R(X)$ such that $p \in \mathrm{bd}_{X^*}A^*$. As E is dense in X^*, $E \cap \mathrm{int}_{X^*}A^*$ and $E - A^*$ are disjoint nonempty open subsets of E. Since $A^* \in R(X^*)$ (see Theorem 2.3), it follows from the fact that E is dense in X^* that $p \in \mathrm{cl}_E(E \cap \mathrm{int}_{X^*}A^*) \cap \mathrm{cl}_E(E - A^*)$. Thus E contains two disjoint nonempty open subsets whose E-closures are not disjoint. It follows that E is not extremally disconnected, and the theorem is proved.

If X is not assumed to be nowhere locally compact, then Theorem 2.6 may fail; see [**10**, 4.6].

Finally, we generalize a result due to Mandelker [**7**]. Recall that Mandelker calls a z-filter $\mathscr{F}$ on X *round* if for every $Z \in \mathscr{F}$ there is $W \in \mathscr{F}$ and a cozero-set S of X such that $W \subseteq S \subseteq Z$ (see [**7**, p. 1]). The following result is due to Mandelker; see [**6**, Theorems 4.5 and 8.1].

2.7 THEOREM. *A prime z-filter on a perfectly normal, Hausdorff space is nonminimal if and only if it has a nowhere dense member.*

The following result generalizes [**7**, Theorem 2.4]. The method of proof is essentially the same as that employed in [**7**, Theorems 2.3 and 2.4]. If $p \in \beta X$, $\mathscr{A}(p)$ will denote the unique z-ultrafilter on X that converges to p.

2.8 THEOREM. *Let X be a metric space without isolated points. Then a prime z-filter $\mathscr{P}$ on X is round if and only if $\mathscr{P} = \mathscr{A}(p)$ for some $p \in T(\beta X)$.*

Proof. Let $\mathscr{P}$ be round. Suppose that $\mathscr{P}$ is strictly contained in the prime z-filter $\mathscr{Q}$. As $\mathscr{Q}$ is nonminimal, by Theorem 2.7 we can choose $F \in \mathscr{Q}$ such that $\mathrm{int}_X F = \emptyset$. By Lemma 1.2 there exists $A \in R(X)$ such that $F \subseteq \mathrm{bd}_X A = \mathrm{bd}_X(\mathrm{cl}_X(X - A))$. Now $A \cup \mathrm{cl}_X(X - A) = X$, so as every closed subset

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of a metric space is a zero-set it follows from the fact that $\mathscr{P}$ is prime that either $A \in \mathscr{P}$ or $\mathrm{cl}_X(X - A) \in \mathscr{P}$; say $A \in \mathscr{P}$. Now $\mathrm{bd}_X A \in \mathscr{Q}$ as $F \in \mathscr{Q}$ and $F \subseteq \mathrm{bd}_X A$. As $\mathscr{P}$ is round there exists $G \in \mathscr{P}$ such that $G \subseteq \mathrm{int}_X A$. Thus $\mathscr{Q}$ contains the disjoint zero-sets G and $\mathrm{bd}_X A$, which is a contradiction. Thus $\mathscr{P}$ is a z-ultrafilter, and since it is round it contains no nowhere dense set. Hence by Theorem 2.2 $\mathscr{P} = \mathscr{A}(p)$ for some remote point p.

Conversely, let $p \in T(\beta X)$ and let $Z \in \mathscr{A}(p)$. Then $\mathrm{bd}_X Z \notin \mathscr{A}(p)$ by Theorem 2.2. As $\mathscr{A}(p)$ is a z-ultrafilter, there exists $F \in \mathscr{A}(p)$ such that $F \cap \mathrm{bd}_X Z = \emptyset$. Hence $F \cap Z \in \mathscr{A}(p)$ and $F \cap Z \subseteq \mathrm{int}_X Z \subseteq Z$. Thus $\mathscr{A}(p)$ is round, since every open subset of a metric space is a cozero-set.

References

1. J. Dugundji, *Topology* (Allyn and Bacon, Mass., 1965).
2. N. J. Fine and L. Gillman, *Extensions of continuous functions in βN*, Bull. Amer. Math. Soc. *66* (1960), 376–381.
3. ——— *Remote points in βR*, Proc. Amer. Math. Soc. *13* (1962), 29–36.
4. L. Gillman and M. Jerison, *Rings of continuous functions*, University Series in Higher Mathematics (Van Nostrand, Princeton, New Jersey, 1960).
5. M. Katětov, *On the equivalence of certain types of extensions of topological spaces*, Časopis Pěst. Mat. Fys. *72* (1947), 101–106.
6. M. Mandelker, *Prime z-ideal structure of $C(R)$*, Fund. Math. *63* (1968), 145–166.
7. ——— *Round z-filters and round subsets of βX*, Israel J. Math. *7* (1969), 1–8.
8. D. L. Plank, *On a class of subalgebras of $C(X)$ with applications to $\beta X - X$*, Fund. Math. *64* (1969), 41–52.
9. R. Sikorski, *Boolean algebras*, 2nd ed., Ergebnisse der Mathematik und ihrer Grenzgebiete, Neue Folge, Band 25 (Academic Press, New York; Springer-Verlag, Berlin and New York, 1964).
10. R. G. Woods, *Co-absolutes of remainders of Stone-Čech compactifications*, Pacific J. Math. *37* (1971), 545-560.

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