Bull. Aust. Math. Soc. **108** (2023), 162–165
doi:10.1017/S0004972722001150

A REMARK ON THE GEOMETRIC INTERPRETATION OF THE A3W CONDITION FROM OPTIMAL TRANSPORT

CALE RANKIN

(Received 14 August 2022; accepted 10 September 2022; first published online 13 October 2022)

Abstract

We provide a geometric interpretation of the well-known A3w condition for regularity in optimal transport.

2020 *Mathematics subject classification*: primary 49K20; secondary 35J60.

Keywords and phrases: optimal transport, A3w, MTW condition.

1. Introduction

In optimal transport, a condition known as A3w is necessary for regularity of the optimal transport map. Here we provide a geometric interpretation of A3w. We use freely the notation from [4]. Let $c \in C^2(\mathbf{R}^n \times \mathbf{R}^n)$ satisfy A1 and A2 (see Section 2). Keeping in mind the prototypical case $c(x, y) = |x - y|^2$, we fix $x_0, y_0 \in \mathbf{R}^n$ and perform a linear transformation so that $c_{xy}(x_0, y_0) = -I$. Define coordinates

$$q(x) := -c_y(x, y_0), \tag{1.1}$$

$$p(y) := -c_x(x_0, y), \tag{1.2}$$

and denote the inverse transformations by $x(q), y(p)$. Write $c(q, p) = c(x(q), y(p))$ and let $q_0 = q(x_0)$ and $p_0 = p(y_0)$. We prove A3w is satisfied if and only if whenever these transformations are performed,

$$(q - q_0) \cdot (p - p_0) \geq 0 \Longrightarrow c(q, p) + c(q_0, p_0) \leq c(q, p_0) + c(q_0, p).$$

Heuristically, A3w implies that when $q - q_0$ 'points in the same direction' as $p - p_0$, it is cheaper to transport q to p and q_0 to p_0 than the alternative q to p_0 and q_0 to p. Thus, A3w represents compatibility between directions in the cost-convex geometry and the cost of transport.

A3w first appeared (in a stronger form) in [4]. It was weakened in [6] and a new interpretation was given in [2]. The impetus for the above interpretation is

This research is supported by ARC DP 200101084 and the Fields Institute for Research in Mathematical Sciences.

Lemma 2.1 in [1]. Our result can also be realised by a particular choice of c-convex function in the unpublished preprint [5].

2. Proof of result

Let $c \in C^2(\mathbf{R}^n \times \mathbf{R}^n)$ satisfy the following well-known conditions.

A1. For each $x_0, y_0 \in \mathbf{R}^n$, the mappings

$$x \mapsto c_y(x, y_0) \quad \text{and} \quad y \mapsto c_x(x_0, y)$$

are injective.

A2. For each $x_0, y_0 \in \mathbf{R}^n$, we have $\det c_{i,j}(x_0, y_0) \neq 0$.

Here, and throughout, subscripts before a comma denote differentiation with respect to the first variable, subscripts after a comma denote differentiation with respect to the second variable.

By A1, we define on $\mathcal{U} := \{(x, c_x(x, y)) : x, y \in \mathbf{R}^n\}$ a mapping $Y : \mathcal{U} \to \mathbf{R}^n$ by

$$c_x(x, Y(x, p)) = p.$$

The A3w condition, usually expressed with fourth derivatives but written here as in [3], is the following statement.

A3w. Fix x. The function

$$p \mapsto c_{ij}(x, Y(x, p))\xi_i\xi_j$$

is concave along line segments orthogonal to ξ.

To verify A3w, it suffices to verify the midpoint concavity, that is, whenever $\xi \cdot \eta = 0$, it follows that

$$0 \geq [c_{ij}(x, Y(x, p + \eta)) - 2c_{ij}(x, Y(x, p)) + c_{ij}(x, Y(x, p - \eta))]\xi_i\xi_j. \tag{2.1}$$

Finally, we recall that a set $A \subset \mathbf{R}^n$ is called c-convex with respect to y_0 provided $c_y(A, y_0)$ is convex. When the A3w condition is satisfied and $y, y_0 \in \mathbf{R}^n$ are given, the section $\{x \in \mathbf{R}^n : c(x, y) > c(x, y_0)\}$ is c-convex with respect to y_0 [3].

Now fix $(x_0, p_0) \in \mathcal{U}$ and $y_0 = Y(x_0, p_0)$. To simplify the proof, we assume $x_0, y_0, q_0, p_0 = 0$. Up to an affine transformation (replace y with $\tilde{y} := -c_{xy}(0, 0)y$), we assume $c_{xy}(0, 0) = -I$. Note that with q, p, as defined in (1.1), (1.2), this implies $\partial q/\partial x(0) = I$. Put

$$\tilde{c}(x, y) := c(x, y) - c(x, 0) - c(0, y) + c(0, 0),$$
$$\overline{c}(q, p) := \tilde{c}(x(q), y(p)).$$

THEOREM 2.1. *The A3w condition is satisfied if and only if whenever the above transformations are applied, the following implication holds:*

$$q \cdot p \geq 0 \Longrightarrow \overline{c}(q, p) \leq 0. \tag{2.2}$$

https://doi.org/10.1017/S0004972722001150 Published online by Cambridge University Press

PROOF. Observe by a Taylor series

$$\bar{c}(q,p) = -(q\cdot p) + \bar{c}_{ij}(\tau q, p)q_i q_j \tag{2.3}$$

for some $\tau \in (0,1)$. First, assume A3w and let $q\cdot p > 0$. By (2.3), we have $\bar{c}(-tq,p) > 0 > \bar{c}(tq,p)$ for $t > 0$ sufficiently small. If $\bar{c}(q,p) > 0$, then the c-convexity (in our coordinates, convexity) of the section

$$\{q : \bar{c}(q,p) > \bar{c}(q,0) = 0\}$$

is violated. By continuity, $\bar{c}(q,p) \le 0$ whenever $q\cdot p \ge 0$.

In the other direction, take nonzero q with $q\cdot p = 0$ and small t. By (2.2) and (2.3),

$$0 \ge \bar{c}(tq,p)/t^2 = \bar{c}_{ij}(t\tau q, p)q_i q_j.$$

This inequality also holds with $-p$. Moreover, $\bar{c}_{ij}(t\tau q, 0) = 0$. Thus,

$$0 \ge [\bar{c}_{ij}(t\tau q, p) - 2\bar{c}_{ij}(t\tau q, 0) + \bar{c}_{ij}(t\tau q, -p)]q_i q_j.$$

Sending $t \to 0$ and returning to our original coordinates, we obtain (2.1). □

REMARK 2.2. On a Riemannian manifold with $c(x,y) = d(x,y)^2$, for d the distance function, Loeper [2] proved A3w implies nonnegative sectional curvature. Our result expedites his proof. Let $x_0 = y_0 \in M$ and $u, v \in T_{x_0}M$ satisfy $u\cdot v = 0$ with $x = \exp_{x_0}(tu)$ and $y = \exp_{x_0}(tv)$. Working in a sufficiently small local coordinate chart, our previous proof implies that if A3w is satisfied,

$$d(x,y)^2 \le d(x_0,y)^2 + d(x_0,x)^2 = 2t. \tag{2.4}$$

The sectional curvature in the plane generated by u, v is the κ satisfying

$$d(\exp_{x_0}(tu), \exp_{x_0}(tv)) = \sqrt{2}t\Big(1 - \frac{\kappa}{12}t^2 + O(t^3)\Big) \quad \text{as } t \to 0, \tag{2.5}$$

whereby comparison with (2.4) proves the result. (See [7, Equation (1)] for (2.5).) We note Loeper proved his result using an infinitesimal version of (2.4).

Acknowledgements

My thanks to Jiakun Liu and Robert McCann for helpful comments and discussion.

References

[1] S. Chen and X.-J. Wang, 'Strict convexity and $C^{1,\alpha}$ regularity of potential functions in optimal transportation under condition A3w', *J. Differential Equations* **260**(2) (2016), 1954–1974.

[2] G. Loeper, 'On the regularity of solutions of optimal transportation problems', *Acta Math.* **202**(2) (2009), 241–283.

[3] G. Loeper and N. S. Trudinger, 'Weak formulation of the MTW condition and convexity properties of potentials', *Methods Appl. Anal.* **28**(1) (2021), 53–60.

[4] X.-N. Ma, N. S. Trudinger and X.-J. Wang, 'Regularity of potential functions of the optimal transportation problem', *Arch. Ration. Mech. Anal.* **177**(2) (2005), 151–183.

[5] N. S. Trudinger and X.-J. Wang, 'On convexity notions in optimal transportation', Preprint, 2008, http://web.archive.org/maths.anu.edu.au/files/note_on_convexity.pdf.

https://doi.org/10.1017/S0004972722001150 Published online by Cambridge University Press

[6] N. S. Trudinger and X.-J. Wang, 'On the second boundary value problem for Monge–Ampère type equations and optimal transportation', *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)* **8**(1) (2009), 143–174.
[7] C. Villani, 'Synthetic theory of Ricci curvature bounds', *Jpn. J. Math.* **11**(2) (2016), 219–263.

CALE RANKIN, Fields Institute for Research in Mathematical Sciences,
Toronto, ON M5T 3J1, Canada
e-mail: cale.rankin@utoronto.ca

https://doi.org/10.1017/S0004972722001150 Published online by Cambridge University Press