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CHARACTERISTIC SUBGROUPS OF RELATIVELY FREE GROUPS

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A simple new proof is given of a result of Vaughan-Lee which implies that if G is a relatively free nilpotent group of finite rank k and nilpotency class c with $c < k$ then the characteristic subgroups of G are all fully invariant. It is proved that the condition $c < k$ can be weakened to $c < k + p - 2$ when G has p-power exponent for some prime p. On the other hand it is shown that for each prime p there is a 2-generator relatively free p-group G which is nilpotent of class $2p$ such that the centre of G is not fully invariant.

1. INTRODUCTION

For each positive integer k let F_k be the free group of rank k freely generated by the set $\{x_1, \ldots, x_k\}$. The following result was proved in [7].

THEOREM 1. (Vaughan-Lee). *If C is a characteristic subgroup of F_k which contains the kth term $\gamma_k(F_k)$ of the lower central series of F_k then C is fully invariant.*

One interesting consequence of this result is that every formation of finite nilpotent groups is subgroup closed — see [6]. A simple proof of Theorem 1 is given in Section 2 below.

A corollary of Theorem 1 is that if G is a relatively free nilpotent group of finite rank k and nilpotency class c where $c < k$ then every characteristic subgroup of G is fully invariant. (See [5] for basic facts about relatively free groups.) The following stronger result will be proved in Section 3 for the case where G has prime-power exponent.

THEOREM 2. *Let p be a prime number and let G be a relatively free nilpotent group of p-power exponent with finite rank k and nilpotency class c, where $c < k+p-2$. Then every characteristic subgroup of G is fully invariant.*

In the case where $k = 2$ and $p = 2$ Theorem 2 is the best possible result of its sort because, as is well known, the relatively free group of rank 2 in the variety of groups of exponent 4 and class 2 has a characteristic subgroup which is not fully invariant (see, for example, [4, Section 9] and see also Section 6 below). I do not know how close Theorem 2 is to being best possible in general, but some information can be obtained

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by module-theoretic methods. This is illustrated in Sections 4 – 6. It is shown that for each odd prime p there is a 2-generator relatively free p-group which is nilpotent of class $2p-1$ and has a characteristic subgroup which is not fully invariant. The following result will also be proved.

Theorem 3. *For each prime p there is a 2-generator relatively free p-group G such that G is nilpotent of class $2p$ and the centre of G is not fully invariant.*

Similar examples of relatively free nilpotent groups of finite rank in which the centre is not fully invariant may possibly be known in the "folk-lore". A non-nilpotent example is certainly known; namely, the relatively free group of rank 2 in the variety generated by the non-abelian group of order 6. In contrast, every relatively free group of infinite rank has fully invariant centre.

2. Proof of Theorem 1

If F is a free group on a given free generating set X and $x \in X$ we write δ_x for the "deletion" endomorphism of F defined by $x\delta_x = 1$ and $y\delta_x = y$ for all $y \in X\setminus\{x\}$. Furthermore, for all $w \in F$, $w(1-\delta_x)$ denotes $w(w\delta_x)^{-1}$.

As shown in [7], Theorem 1 is an easy consequence of the following lemma (Lemma 1). We shall not repeat the deduction of Theorem 1 here but give a proof of the lemma which avoids the complexity of the proof in [7].

Lemma 1. *Let C be a characteristic subgroup of F_k which contains $\gamma_k(F_k)$ and let $w \in C$. Then $w\delta_{x_i} \in C$ $(1 \leqslant i \leqslant k)$.*

Proof: The result is trivial if $k=1$, so we assume that $k \geqslant 2$. By symmetry, it is enough to prove that $w\delta_{x_k} \in C$. Write $w = w(x_1, \ldots, x_k)$. Let F be the free group freely generated by $2k$ elements $x_1, \ldots, x_k, y_1, \ldots, y_k$ and write $Y = \{y_1, \ldots, y_k\}$. Let v be the element of F defined by

$$v = w\big(x_2x_1y_1^{-1}, x_3x_2y_2^{-1}, \ldots, x_kx_{k-1}y_{k-1}^{-1}, y_1y_2\ldots y_k\big)$$

and let
$$v^* = v(1-\delta_{y_1})(1-\delta_{y_2})\ldots(1-\delta_{y_k}).$$

Thus v^* is the product, in some order, of elements $v_S^{\varepsilon(S)}$ where S ranges over the subsets of Y, $\varepsilon(S) = (-1)^{|S|}$ and $v_S = v\Big(\prod_{y\in S}\delta_y\Big)$. Also, by [5, 33.38 and 33.42], $v^* \in \gamma_k(F)$.

Let $\xi\colon F \to F_k$ be the homomorphism defined by $x_i\xi = x_i$ and $y_i\xi = x_i$ for all i. Thus $v^*\xi \in \gamma_k(F_k)$ and so $v^*\xi \in C$. Also, $v^*\xi$ is the product of the elements $(v_S\xi)^{\varepsilon(S)}$. We shall prove (i) $v_Y\xi = (w\delta_{x_k})\alpha$ where α is an automorphism of F_k, and (ii) for all $S \neq Y$, $v_S\xi = w\beta$ where β is an automorphism of F_k (depending on S). It follows that $(v_S\xi)^{\varepsilon(S)} \in C$ for all $S \neq Y$; whence $(w\delta_{x_k})\alpha \in C$ and so $w\delta_{x_k} \in C$, as required.

To prove (i) note that

$$v_Y\xi = w(x_2x_1, x_3x_2, \dots, x_kx_{k-1}, 1) = (w\delta_{x_k})\alpha$$

where α is the automorphism of F_k defined by $x_k\alpha = x_k$ and $x_i\alpha = x_{i+1}x_i$ for $i < k$. To prove (ii), suppose $S \neq Y$ and, for $i = 1, \dots, k$, write $\mu_i = 1$ if $y_i \in S$ and $\mu_i = 0$ if $y_i \notin S$. Thus $v_S\xi = w\beta$ where β is the endomorphism of F_k defined by $x_k\beta = x_1^{1-\mu_1} \dots x_k^{1-\mu_k}$ and $x_i\beta = x_{i+1}x_i^{\mu_i}$ for $i < k$. We shall show that β is an automorphism of F_k. By the Hopf property of F_k (see [5, 41.52]) it is sufficient to show that β is surjective. Thus it is sufficient to show that $x_1, \dots, x_k$ all belong to the subgroup $\langle x_1\beta, \dots, x_k\beta\rangle$.

Let d be the smallest positive integer such that $y_d \notin S$. If $d < k$ then $x_d\beta = x_{d+1}$, $x_{d+1}\beta = x_{d+2}x_{d+1}^{\mu_{d+1}}, \dots, x_{k-1}\beta = x_kx_{k-1}^{\mu_{k-1}}$, and so $x_{d+1}, \dots, x_k \in \langle x_1\beta, \dots, x_k\beta\rangle$. But $x_k\beta = x_dx_{d+1}^{1-\mu_{d+1}} \dots x_k^{1-\mu_k}$. Hence $x_d \in \langle x_1\beta, \dots, x_k\beta\rangle$. If $d > 1$ then $x_{d-1}\beta = x_dx_{d-1}, \dots, x_1\beta = x_2x_1$, and so $x_{d-1}, \dots, x_1 \in \langle x_1\beta, \dots, x_k\beta\rangle$. ∎

3. Proof of Theorem 2

Let G be as in the statement of the theorem and let $\{a_1, \dots, a_k\}$ be a free generating set for G. The result is trivial if $k = 1$ or $c = 0$, so we assume that $k \geqslant 2$ and $c \geqslant 1$. Let Φ be the Frattini subgroup of G. Since G is a finite p-group, G/Φ is an elementary abelian group of order p^k and an endomorphism θ of G is an automorphism if and only if $\{(a_1\theta)\Phi, \dots, (a_k\theta)\Phi\}$ is a basis of G/Φ.

Let C be a characteristic subgroup of G. Let $u(x_1, \dots, x_k)$ be an element of F_k such that $u(a_1, \dots, a_k) \in C$, and let $g_1, \dots, g_k \in G$. It suffices to show that $u(g_1, \dots, g_k) \in C$. Let r be the rank of $\langle g_1\Phi, \dots, g_k\Phi\rangle$. Then, for some subset R of $\{1, \dots, k\}$ of cardinality r, $\langle g_1\Phi, \dots, g_k\Phi\rangle = \langle g_i\Phi \colon i \in R\rangle$. Suppose that $r \neq 0$, let σ be a permutation of $\{1, \dots, k\}$ such that $\{1, \dots, r\} = R\sigma$, and let $\tilde{u}(x_1, \dots, x_k) = u(x_{1\sigma}, \dots, x_{k\sigma})$. Then

$$\tilde{u}(a_1, \dots, a_k) = u(a_{1\sigma}, \dots, a_{k\sigma}) = u(a_1, \dots, a_k)\sigma^*,$$

where σ^* is an automorphism of G. Thus $\tilde{u}(a_1, \dots, a_k) \in C$. Also, $u(g_1, \dots, g_k) = \tilde{u}(g_{1\sigma^{-1}}, \dots, g_{k\sigma^{-1}})$ and, by choice of σ, $\langle g_{1\sigma^{-1}}\Phi, \dots, g_{r\sigma^{-1}}\Phi\rangle = \langle g_1\Phi, \dots, g_k\Phi\rangle$. Thus (by considering $\tilde{u}$ instead of u) we may reduce to the case where $R = \{1, \dots, r\}$.

Hence there are elements $t_{r+1}(x_1, \dots, x_r), \dots, t_k(x_1, \dots, x_r)$ of F_r and elements $f_{r+1}, \dots, f_k$ of Φ such that $g_i = t_i(g_1, \dots, g_r)f_i$ $(r+1 \leqslant i \leqslant k)$. Since $\{g_1\Phi, \dots, g_r\Phi\}$ is contained in a basis of G/Φ, there is an automorphism θ of G such that $a_i\theta = g_i$ $(1 \leqslant i \leqslant r)$. It is enough to prove that $u(g_1, \dots, g_k)\theta^{-1} \in C$. But

$$u(g_1, \dots, g_k)\theta^{-1} = u(a_1, \dots, a_r, t_{r+1}(a_1, \dots, a_r)h_{r+1}, \dots, t_k(a_1, \dots, a_r)h_k)$$

where $h_i = f_i\theta^{-1} \in \Phi$ $(r+1 \leqslant i \leqslant k)$. Let

$$w(x_1, \dots, x_k) = u(x_1, \dots, x_r, t_{r+1}(x_1, \dots, x_r)x_{r+1}, \dots, t_k(x_1, \dots, x_r)x_k).$$

Thus we wish to prove that $w(a_1, \dots, a_r, h_{r+1}, \dots, h_k) \in C$. Note that $w(a_1, \dots, a_k) = u(a_1, \dots, a_k)\tau$ where τ is an automorphism of G. Thus $w(a_1, \dots, a_k) \in C$ and it is enough to prove the following lemma, which also covers the case where $r = 0$. The proof is a modification of the proof of Lemma 1.

Lemma 2. *With G as in the statement of Theorem 2, let $\{a_1, \dots, a_k\}$ be a free generating set for G, let C be a characteristic subgroup of G, and let $w(x_1, \dots, x_k)$ be an element of F_k such that $w(a_1, \dots, a_k) \in C$. Then, for all $r \in \{0, 1, \dots, k\}$ and all $h_{r+1}, \dots, h_k \in \Phi$ (where Φ is the Frattini subgroup of G), $w(a_1, \dots, a_r, h_{r+1}, \dots, h_k)$ belongs to C.*

Proof: As before we may assume that $k \geqslant 2$ and $c \geqslant 1$. The result is trivially true if $r = k$. Thus, using downward induction on r, we may assume that $r < k$ and the result is true when r is replaced by $r+1$. Let F be the free group freely generated by a set $\{x_1, \dots, x_k\} \cup Y$ where Y consists of the $k+p-2$ elements $y_1, \dots, y_r, y_{r+1}^{(1)}, \dots, y_{r+1}^{(p-1)}, y_{r+2}, \dots, y_k$. Let $\xi\colon F \to G$ be the homomorphism defined by $x_i\xi = a_i$ for all i, $y_i\xi = a_i$ for all $i \neq r+1$, and $y_{r+1}^{(j)}\xi = a_{r+1}$ for all j. Let α be the automorphism of G defined by $a_i\alpha = a_{i+1}a_i$ $(1 \leqslant i \leqslant r)$ and $a_i\alpha = a_i$ $(r+1 \leqslant i \leqslant k)$, and write $h_i' = h_i\alpha$ $(r+1 \leqslant i \leqslant k)$. Choose elements $u_{r+1}, \dots, u_k$ of $\langle x_1, \dots, x_k\rangle$ such that $u_i\xi = h_i'$ $(r+1 \leqslant i \leqslant k)$. Let v be the element of F defined by

$$v = w\left(x_2x_1y_1^{-1}, \dots, x_{r+1}x_ry_r^{-1}, y_1\cdots y_ry_{r+1}^{(1)}\cdots y_{r+1}^{(p-1)}y_{r+2}\cdots y_ku_{r+1}, u_{r+2}, \dots, u_k\right)$$

and let $v^* = v\Big(\prod_{y\in Y}(1-\delta_y)\Big)$ where the elements of Y are taken in some arbitrary order. Thus, by [5, 33.38 and 33.42], $v^* \in \gamma_{k+p-2}(F)$, and so $v^*\xi \in \gamma_{k+p-2}(G) = \{1\}$. In particular, $v^*\xi \in C$. For each subset S of Y let $v_S = v\Big(\prod_{y\in S}\delta_y\Big)$. As in the proof of Lemma 1, it suffices to prove (i) $v_Y\xi = w(a_1, \dots, a_r, h_{r+1}, \dots, h_k)\alpha$, and (ii) for all $S \neq Y$, $v_S\xi \in C$.

To prove (i) note that

$$v_Y\xi = w(a_2a_1, \dots, a_{r+1}a_r, h'_{r+1}, \dots, h'_k) = w(a_1, \dots, a_r, h_{r+1}, \dots, h_k)\alpha,$$

as required. To prove (ii), suppose $S \neq Y$, write

$$\nu = \left|\{y_{r+1}^{(1)}, \dots, y_{r+1}^{(p-1)}\} \cap S\right|$$

and, for $i \in \{1, \dots, r, r+2, \dots, k\}$, write $\mu_i = 1$ if $y_i \in S$ and $\mu_i = 0$ if $y_i \notin S$. Thus $v_S\xi = w(b_1, \dots, b_k)$ where $b_1 = a_2 a_1^{\mu_1}, \dots, b_r = a_{r+1} a_r^{\mu_r}$,

$$b_{r+1} = a_1^{1-\mu_1} \dots a_r^{1-\mu_r} a_{r+1}^{p-1-\nu} a_{r+2}^{1-\mu_{r+2}} \dots a_k^{1-\mu_k} h'_{r+1},$$

$b_{r+2} = h'_{r+2}, \dots, b_k = h'_k$. It is enough to prove that $\langle b_1\Phi, \dots, b_{r+1}\Phi\rangle$ has rank $r+1$, for then there is an automorphism β of G such that $a_i\beta = b_i$ $(1 \leqslant i \leqslant r+1)$ which gives

$$v_S\xi = w(a_1, \dots, a_{r+1}, h'_{r+2}\beta^{-1}, \dots, h'_k\beta^{-1})\beta$$

and so $v_S\xi \in C$ by the inductive hypothesis.

Suppose first that $\{y_1, \dots, y_r\} \subseteq S$. Then

$$\langle b_1, \dots, b_{r+1}\rangle = \langle a_2 a_1, \dots, a_{r+1} a_r, a_{r+1}^{p-1-\nu} a_{r+2}^{1-\mu_{r+2}} \dots a_k^{1-\mu_k} h'_{r+1}\rangle$$

and either $\mu_i = 0$ for some $i \in \{r+2, \dots, k\}$ or $0 \leqslant \nu < p-1$. It follows easily that $\langle b_1\Phi, \dots, b_{r+1}\Phi\rangle$ has rank $r+1$.

Suppose finally that $\{y_1, \dots, y_r\}$ is not contained in S and let $N = \langle a_{r+2}, \dots, a_k\rangle\Phi$. It suffices to show that $\langle b_1 N, \dots, b_{r+1} N\rangle$ has rank $r+1$. But

$$\langle b_1 N, \dots, b_{r+1} N\rangle = \langle a_2 a_1^{\mu_1} N, \dots, a_{r+1} a_r^{\mu_r} N, a_1^{1-\mu_1} \dots a_r^{1-\mu_r} a_{r+1}^{p-1-\nu} N\rangle$$

where $\mu_i = 0$ for some $i \in \{1, \dots, r\}$. By the method of Lemma 1 we can show that $a_1 N, \dots, a_{r+1} N$ all belong to $\langle b_1 N, \dots, b_{r+1} N\rangle$, and this gives the required result.

4. Endomorphisms and modules

In the remainder of the paper we describe the construction of some relatively free p-groups with characteristic subgroups which are not fully invariant.

Let p be a prime number and k a positive integer, $k \geqslant 2$. Define subgroups $\lambda_c(F_k)$ of F_k for each positive integer c by $\lambda_1(F_k) = F_k$ and

$$\lambda_{c+1}(F_k) = \lambda_c(F_k)^p[\lambda_c(F_k), F_k]$$

(see [2, VIII.1.4 and VIII.1.5]). Thus each $\lambda_c(F_k)$ is a fully invariant subgroup of F_k, $\lambda_{c+1}(F_k) \subseteq \lambda_c(F_k)$ and $\lambda_c(F_k)/\lambda_{c+1}(F_k)$ is a finite elementary abelian p-group. We write $U_c = \lambda_c(F_k)/\lambda_{c+1}(F_k)$ and regard U_c as a vector space over the field $\mathrm{GF}(p)$ of p elements. Furthermore, we write $U = U_1$. Thus U has dimension k.

For each c let $\mathrm{End}(U_c)$ be the set of all linear transformations of U_c. It is usual to regard $\mathrm{End}(U_c)$ as a ring, but for our purposes here the additive structure of $\mathrm{End}(U_c)$ is irrelevant and we simply regard $\mathrm{End}(U_c)$ as a monoid under the operation of composition of functions. As usual we write $\mathrm{GL}(U_c)$ for the group of invertible elements of $\mathrm{End}(U_c)$. Every endomorphism of F_k induces on U_c an element of $\mathrm{End}(U_c)$. Since F_k is free, each element ζ of $\mathrm{End}(U)$ is induced by some endomorphism of F_k. This endomorphism induces an element of $\mathrm{End}(U_c)$ which, by [2, VIII.1.7a], depends only upon ζ and not on the choice of endomorphism of F_k. Thus we obtain a monoid homomorphism $\mathrm{End}(U) \longrightarrow \mathrm{End}(U_c)$ and, by restriction, a group homomorphism $\mathrm{GL}(U) \longrightarrow \mathrm{GL}(U_c)$. Hence U_c may be regarded as a module for $\mathrm{End}(U)$ and $\mathrm{GL}(U)$.

Note that $F_k/\lambda_{c+1}(F_k)$ is a finite relatively free p-group of rank k and nilpotency class at most c. (We shall see below that it has class exactly c.) Furthermore, U_c is a central subgroup of $F_k/\lambda_{c+1}(F_k)$. Suppose that V is a subgroup of U_c. Then it is straightforward to verify that V is fully invariant as a subgroup of $F_k/\lambda_{c+1}(F_k)$ if and only if it is an $\mathrm{End}(U)$-submodule of U_c and V is characteristic in $F_k/\lambda_{c+1}(F_k)$ if and only if it is a $\mathrm{GL}(U)$-submodule of U_c. Thus we shall investigate the submodule structure of U_c.

Let A be the free associative algebra (without identity element) over $\mathrm{GF}(p)$ on the free generating set $\{x_1, \ldots, x_k\}$. Thus $A = A_1 \oplus A_2 \oplus \ldots$ where, for each c, A_c is the subspace of A spanned by all monomials of degree c in $x_1, \ldots, x_k$. We identify U with A_1 in the obvious way. Thus A_1 is an $\mathrm{End}(U)$-module. Since A is free, the action of $\mathrm{End}(U)$ on A_1 can be extended to A so that each element of $\mathrm{End}(U)$ acts as an algebra endomorphism of A. Under this action A is an $\mathrm{End}(U)$-module and each A_c is a submodule.

The associative algebra A also carries the structure of a Lie algebra over $\mathrm{GF}(p)$, the Lie multiplication being the "commutator" operation defined by $[v, w] = vw - wv$ for all $v, w \in A$. Let L be the Lie subalgebra generated by $\{x_1, \ldots, x_k\}$. Then, as is well known, L is a free Lie algebra on $\{x_1, \ldots, x_k\}$ (see [3, Theorem 5.9]). Also $L = L_1 \oplus L_2 \oplus \ldots$ where $L_c = L \cap A_c$ for all c, and, in particular, $L_1 = A_1 = U$. It is easy to verify that L and the L_c are $\mathrm{End}(U)$-submodules of A.

The submodule structure of U_c is closely related to that of L. In explaining the connection we shall follow the presentation in [1]: see [1] for references to original sources.

For each positive integer c there is a certain group homomorphism $\phi_c\colon \lambda_c(F_k) \to A$ (with A regarded as a group under addition). In the case where p is odd these homomorphisms are determined by the following properties:

$$x_i\phi_1 = x_i \ (1 \leqslant i \leqslant k), \quad f^p\phi_{c+1} = f\phi_c,$$

and
$$[f, g]\phi_{c+1} = [f\phi_c, g\phi_1],$$

for all $c \geqslant 1$, $f \in \lambda_c(F_k)$ and $g \in \lambda_1(F_k) = F_k$. For $p = 2$ the only difference is in the formula for $f^p\phi_{c+1}$ with $c = 1$: this becomes

$$f^2\phi_2 = f\phi_1 + (f\phi_1)^2$$

for all $f \in \lambda_1(F_k)$.

The kernel of ϕ_c is equal to $\lambda_{c+1}(F_k)$ for all p, c (see [1]). Thus ϕ_c induces a vector space monomorphism $\tilde{\phi}_c\colon U_c \to A$. As noted in [1], $\tilde{\phi}_c$ is a $\mathrm{GL}(U)$-module monomorphism, and a similar proof shows that $\tilde{\phi}_c$ is in fact an $\mathrm{End}(U)$-module monomorphism. For $p \neq 2$ the image of $\tilde{\phi}_c$ is easily calculated to be $L_1 + \ldots + L_c$: thus U_c is isomorphic to $L_1 + \ldots + L_c$ as $\mathrm{End}(U)$-module. In the case $p = 2$, let E be the subspace of $A_1 + A_2$ spanned by the elements $x_i + x_i^2$ $(1 \leqslant i \leqslant k)$ and $[x_i, x_j]$ $(1 \leqslant i < j \leqslant k)$. Then the image of $\tilde{\phi}_1$ is L_1, the image of $\tilde{\phi}_2$ is E, and, for $c \geqslant 3$, the image of $\tilde{\phi}_c$ is $E + L_3 + \ldots + L_c$. Thus U_c is again determined up to isomorphism.

Clearly $\gamma_c(F_k) \subseteq \lambda_c(F_k)$ for all c. It is easy to prove by induction on c that $\gamma_c(F_k)\phi_c = L_c$ for all c. Hence $\gamma_c(F_k)$ is not contained in $\lambda_{c+1}(F_k)$ and the group $F_k/\lambda_{c+1}(F_k)$ has class exactly c.

In the remainder of this paper we take $k = 2$ and write z as an abbreviation for the element $[x_1, x_2]$ of L. Thus z spans L_2. Note that, for all $\zeta \in \mathrm{End}(U)$, $z\zeta = \det(\zeta)z$. Commutators in L will be written with a left-normed convention. Also, we shall need to use the well known fact that if v and w are elements of L such that $[v, w] = 0$ and $w \neq 0$ then v is a scalar multiple of w. This follows, for example, from [3, Theorem 5.10]. We shall now consider separately the cases where p is odd (Section 5) and $p = 2$ (Section 6).

5. Odd characteristic

Suppose that $k = 2$ and p is odd. We shall first show that $L_1 + \ldots + L_{2p-1}$ has a $\mathrm{GL}(U)$-submodule which is not an $\mathrm{End}(U)$-submodule. Let s_1 and s_2 be the elements of L_{2p-1} defined by $s_1 = [x_1, z, \ldots, z]$ and $s_2 = [x_2, z, \ldots, z]$. (Here and subsequently $z, \ldots, z$ will denote a sequence of $p-1$ copies of z.) Let V be the subspace of $L_1 + L_{2p-1}$ spanned by $x_1 + s_1$ and $x_2 + s_2$; that is, $V = \langle x_1 + s_1, x_2 + s_2\rangle$. Note that, for $\zeta \in \mathrm{GL}(U)$ and $i = 1, 2$,

$$s_i\zeta = (\det(\zeta))^{p-1}[x_i\zeta, z, \ldots, z] = [x_i\zeta, z, \ldots, z].$$

It follows that V is a $\mathrm{GL}(U)$-submodule of $L_1 + \ldots + L_{2p-1}$ isomorphic to L_1. But V is not an $\mathrm{End}(U)$-submodule because if η is the element of $\mathrm{End}(U)$ which satisfies $x_1\eta = x_1$ and $x_2\eta = 0$ on A_1 then $(x_1 + s_1)\eta = x_1 \notin V$. Thus U_{2p-1} has a $\mathrm{GL}(U)$-submodule which is not an $\mathrm{End}(U)$-submodule, and so the relatively free group $F_2/\lambda_{2p}(F_2)$ has a characteristic subgroup which is not fully invariant. As proved in Section 4, $F_2/\lambda_{2p}(F_2)$ has class $2p-1$. $\square$

We now move towards the proof of Theorem 3 (in the case where p is odd). Let

t_1, t_2, t_3, t_4 be the elements of L_{2p} defined by

$$t_1 = [x_1, z, \ldots, z, x_2] = [s_1, x_2], \quad t_2 = [x_2, z, \ldots, z, x_1] = [s_2, x_1],$$
$$t_3 = [x_1, z, \ldots, z, x_1] = [s_1, x_1], \quad t_4 = [x_2, z, \ldots, z, x_2] = [s_2, x_2].$$

LEMMA 3. *The elements t_1, t_2, t_3 and t_4 are linearly independent,*

$$\langle t_1, t_2, t_3, t_4\rangle \cap [L_{2p-1}, x_1] = \langle t_2, t_3\rangle$$

and
$$\langle t_1, t_2, t_3, t_4\rangle \cap [L_{2p-1}, x_2] = \langle t_1, t_4\rangle.$$

PROOF: We work inside A and use the fact that the monomials of A form a basis of A. Let $\theta: A \to A$ be the linear transformation which fixes $(x_2x_1)^{p-2}x_1^2x_2^2$ and $x_2^3x_1(x_1x_2)^{p-2}$ but maps all other monomials to 0. Then it is straightforward to verify that

$$t_1\theta = (p-1)(x_2x_1)^{p-2}x_1^2x_2^2$$

and
$$t_4\theta = x_2^3x_1(x_1x_2)^{p-2}.$$

But clearly $v\theta = 0$ for all $v \in [L_{2p-1}, x_1]$. It follows that t_1 and t_4 are linearly independent and

$$\langle t_1, t_4\rangle \cap [L_{2p-1}, x_1] = \{0\}.$$

Similarly, t_2 and t_3 are linearly independent and

$$\langle t_2, t_3\rangle \cap [L_{2p-1}, x_2] = \{0\}.$$

The result follows. $\square$

Let $W = \langle z + t_1, -z + t_2, t_3, t_4\rangle$. Then it is easy to verify that W is an $\mathrm{End}(U)$-submodule of $L_2 + L_{2p}$. Furthermore, the following result holds.

LEMMA 4. $V = \{v \in L : [v, x_i] \in W$ *for* $i = 1, 2\}$.

PROOF: Clearly $[v, x_1] \in W$ and $[v, x_2] \in W$ for all $v \in V$. Conversely, let v be an element of L such that $[v, x_1] \in W$ and $[v, x_2] \in W$. We shall prove that $v \in V$.

Write $v = v_1 + v_2 + \ldots$ where $v_j \in L_j$ for all j. Then, for $j \notin \{1, 2p-1\}$, $[v_j, x_1] = 0$ and $[v_j, x_2] = 0$. Hence $v_j = 0$ for all $j \notin \{1, 2p-1\}$. Thus $v \in L_1 + L_{2p-1}$.

Let $\{s_1, \ldots, s_n\}$ be any basis for L_{2p-1} where $s_1 = [x_1, z, \ldots, z]$ and $s_2 = [x_2, z, \ldots, z]$ as before. If $w \in L_{2p-1}$ and $[w, x_1] = 0$ then $w = 0$. Thus $[s_1, x_1], \ldots, [s_n, x_1]$ are linearly independent elements of L_{2p}. Since $v \in L_1 + L_{2p-1}$ we can write

$$v = \mu_1x_1 + \mu_2x_2 + \nu_1s_1 + \ldots + \nu_ns_n$$

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where $\mu_1, \mu_2, \nu_1, \ldots, \nu_n \in \mathrm{GF}(p)$. Thus

$$[v, x_1] = -\mu_2 z + \nu_1[s_1, x_1] + \ldots + \nu_n[s_n, x_1] \in W.$$

It follows that

$$\nu_1[s_1, x_1] + \ldots + \nu_n[s_n, x_1] \in \langle t_1, t_2, t_3, t_4\rangle \cap [L_{2p-1}, x_1].$$

But, by Lemma 3,

$$\langle t_1, t_2, t_3, t_4\rangle \cap [L_{2p-1}, x_1] = \langle [s_1, x_1], [s_2, x_1]\rangle.$$

Thus $\nu_j = 0$ for $j > 2$ and we can write

$$v = \mu_1 x_1 + \mu_2 x_2 + \nu_1 s_1 + \nu_2 s_2.$$

Hence

$$[v, x_1] = -\mu_2 z + \nu_1 t_3 + \nu_2 t_2 = (\nu_2 - \mu_2)z + \nu_1 t_3 + \nu_2(-z + t_2)$$

and so $(\nu_2 - \mu_2)z \in W$. But, by Lemma 3, $z \notin W$. Thus $\nu_2 = \mu_2$. Similarly, by consideration of $[v, x_2]$, $\nu_1 = \mu_1$. Thus

$$v = \mu_1(x_1 + s_1) + \mu_2(x_2 + s_2) \in V,$$

as required.

We shall now make use of the properties of the maps ϕ_c given in Section 4. Let M be the inverse image of V under ϕ_{2p-1} and let N be the inverse image of W under ϕ_{2p}. Thus

$$\lambda_{2p+1}(F_2) \subseteq N \subseteq \lambda_{2p}(F_2) \subseteq M \subseteq \lambda_{2p-1}(F_2).$$

Since W is an $\mathrm{End}(U)$-submodule of $L_1 + \ldots + L_{2p}$, $N/\lambda_{2p+1}(F_2)$ is a fully invariant subgroup of $F_2/\lambda_{2p+1}(F_2)$, and so N is a fully invariant subgroup of F_2. Let $G = F_2/N$. Then G is a 2-generator relatively free p-group which is nilpotent of class at most $2p$. But

$$(N\gamma_{2p}(F_2))\phi_{2p} = W + L_{2p} \neq W = N\phi_{2p}.$$

Thus $N\gamma_{2p}(F_2) \neq N$ and G has class exactly $2p$.

Since V is not an $\mathrm{End}(U)$-submodule of $L_1 + \ldots + L_{2p-1}$, $M/\lambda_{2p}(F_2)$ is not a fully invariant subgroup of $F_2/\lambda_{2p}(F_2)$. Thus M/N is not a fully invariant subgroup of G. To complete the proof of Theorem 3 (in the case where p is odd) we shall prove that $Z(G) = M/N$.

Let $f \in M$. Then $f\phi_{2p-1} \in V$ and so, by Lemma 4, $[f\phi_{2p-1}, x_i] \in W$ for $i = 1, 2$. Thus $[f, x_i]\phi_{2p} \in W$ and so $[f, x_i] \in N$ for $i = 1, 2$. Thus $M/N \subseteq Z(G)$.

Conversely, suppose that $f \in F_2$ and $[f, x_i] \in N$ for $i = 1, 2$. We first prove by induction on c that $f \in \lambda_c(F_2)$ for $c = 1, \ldots, 2p-1$. Clearly $f \in \lambda_1(F_2)$. Suppose

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$f \in \lambda_c(F_2)$ where $c < 2p-1$. Then, since $N \subseteq \lambda_{c+2}(F_2)$, $[f, x_i]\phi_{c+1} = 0$ for $i = 1, 2$, and so $[f\phi_c, x_i] = 0$ for $i = 1, 2$. Thus $f\phi_c = 0$ and so $f \in \lambda_{c+1}(F_2)$. Therefore $f \in \lambda_{2p-1}(F_2)$ and we can apply the map ϕ_{2p-1}. Thus, for $i = 1, 2$,

$$[f\phi_{2p-1}, x_i] = [f, x_i]\phi_{2p} \in W.$$

By Lemma 4 it follows that $f\phi_{2p-1} \in V$. Thus $f \in M$. Consequently $Z(G) \subseteq M/N$ and so $Z(G) = M/N$ as required. □

6. Characteristic 2

Suppose that $k = 2$ and $p = 2$. Let E be as defined in Section 4: thus E is the image of $\tilde{\phi}_2$. It is easily verified that the subspace of E spanned by the elements $x_1 + x_1^2 + z$ and $x_2 + x_2^2 + z$ is a $\mathrm{GL}(U)$-submodule but not an $\mathrm{End}(U)$-submodule. Thus $F_2/\lambda_3(F_2)$ has a characteristic subgroup which is not fully invariant — this is the example referred to in Section 1. But it does not seem possible to use this example to create a 2-generator relatively free 2-group of class 3 whose centre is not fully invariant. It seems necessary to go to a group of class 4. Thus we prove Theorem 3 in the case $p = 2$. The proof is similar to that for p odd, but rather easier. We omit some of the details.

Let $V_{(2)}$ be the subspace of $E + L_3$ spanned by the elements $x_1 + x_1^2 + z + [z, x_1]$ and $x_2 + x_2^2 + z + [z, x_2]$. It is easily verified that $V_{(2)}$ is a $\mathrm{GL}(U)$-submodule but not an $\mathrm{End}(U)$-submodule.

Let $W_{(2)}$ be the subspace of $E + L_3 + L_4$ spanned by

$$[z, x_1] + [z, x_1, x_1], \quad [z, x_2] + [z, x_2, x_2],$$

and

$$z + [z, x_1] + [z, x_2] + [z, x_1, x_2].$$

It is easily verified that $W_{(2)}$ is an $\mathrm{End}(U)$-submodule. Furthermore, it can be proved that

$$V_{(2)} = \{v \in E + L_3 : [v, x_i] \in W_{(2)} \text{ for } i = 1, 2\}.$$

Let $M_{(2)}$ be the inverse image of $V_{(2)}$ under ϕ_3 and let $N_{(2)}$ be the inverse image of $W_{(2)}$ under ϕ_4. Then $F_2/N_{(2)}$ is a 2-generator relatively free 2-group of nilpotency class 4 with centre $M_{(2)}/N_{(2)}$, which is not fully invariant.

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