J. Austral. Math. Soc. **23** (Series A) (1977), 105–111.

ON CERTAIN SUBALGEBRAS OF A DUAL B^*-ALGEBRA

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(Received 25 September 1975)

Abstract

Let A be a dual B^*-algebra and A_p the p-class in A. We show that the conjugate space of A_1 is A^{**}, the second conjugate space of A. We also obtain a three lines theorem for A_p ($1 \leqq p \leqq \infty$).

1. Introduction

Let H be a (complex) Hilbert space, $LC(H)$ the algebra of all compact operators on H and $L(H)$ the algebra of all continuous linear operators on H. Then $LC(H)$ is a simple dual B^*-algebra and every simple dual B^*-algebra is of this form. Also the second conjugate space of $LC(H)$ is $L(H)$. The class C_p ($0 < p \leqq \infty$) of compact operators in $LC(H)$ has many interesting properties and has been studied in various articles (e.g., see Gohberg and Krein (1969) and McCarthy (1967)). In Wong (1974), a similar class of spaces A_p ($0 < p \leqq \infty$) in an arbitrary dual B^*-algebra A was introduced and studied. The purpose of this paper is to establish three more results for A_p.

It is well known that the conjugate space of C_1 is $L(H)$ (see Schatten (1960), p. 47, Theorem 2). In this paper, we generalize this result to an arbitrary dual B^*-algebra A. In fact, we show that the conjugate space of A_1 is A^{**}, the second conjugate space of A. By identifying A_1 as a subspace of A^{***}, we prove that for every F in A^{***}, $F = G + H$ uniquely with $G \in A_1$, $H \in A^\perp$ and $\|F\| = \|G\| + \|H\|$. For the case A is a simple algebra, this result was proved by Dixmier (see Schatten (1960), Theorem 5). We also obtain a three lines theorem for A_p ($1 \leqq p \leqq \infty$) which is a generalization of Gohberg and Krein (1969), Theorem 13.1.

2. Notation and preliminaries

In this paper, all algebras and linear spaces under consideration are over the field of complex numbers. Definitions not explicitly given are taken from Rickart's book (Rickart (1960)).

NOTATION 1. If B is a Banach space, then B^* and B^{**} will be its conjugate space and second conjugate space respectively. Also B^{***} will be the conjugate space of B^{**}.

NOTATION 2. In this paper, A will denote a dual B^*-algebra with norm $\|\cdot\|$.

REMARK. It is well known that A^{**} is a W^*-algebra containing A as a $*$-subalgebra (see Sakai (1971), Theorem 1.17.2). Since A is a dual algebra, by Wong (1973), Theorem 3.1, A is a two-sided ideal of A^{**}.

Let b be a nonzero normal element in A and $\{e_\tau\}$ a maximal orthogonal family of hermitian minimal idempotents in A such that $e_\tau b = be_\tau = k_\tau e_\tau$, where k_τ is a constant. Then it is shown in Wong (1974), that the set $\{k_n\} = \{k_\tau : k_\tau \neq 0\}$ is countable and independent of the choice of $\{e_\tau\}$ and

$$b = \sum_\tau e_\tau b = \sum_n k_n e_n, \tag{2.1}$$

where $e_n \in \{e_\tau\}$ with $e_n b = k_n e_n$. (2.1) is called a spectal representation of b.

Now suppose A is a nonzero element in A and $a^*a = \Sigma_n r_n e_n$ is a spectral representation of a^*a. Since a^*a is a positive element, $r_n > 0$. Put $k_n = \sqrt{r_n}$ and define

$$|a|_p = \left(\sum_n k_n^p\right)^{1/p} \qquad (0 < p < \infty)$$

and

$$|a|_\infty = \max\{k_n : n = 1, 2, \cdots\}.$$

For $a = 0$, we define $|a|_p = 0$ $(0 < p \leqq \infty)$. Let

$$A_p = \{a \in A : |a|_p < \infty\} \qquad (0 < p \leqq \infty).$$

It was shown in Wong (1974) that for $1 \leqq p \leqq \infty$, A_p is a dual A^*-algebra which is a dense two-sided ideal of A, $A = A_\infty$ and $A_1 = \{ab : a, b \in A_2\}$. Let $b, c \in A_2$ and $\{f_\tau\}$ a maximal orthogonal family of hermitian minimal idempotents in A. Then $f_\tau c^* b f_\tau = m_\tau f_\tau$ for some constant m_τ. By Wong (1974), Theorem 4.1, $\Sigma_\tau m_\tau$ is absolutely summable and independent of $\{f_\tau\}$ and A_2 is a proper H^*-algebra with the inner product defined by $(b, c) = \Sigma_\tau m_\tau$. For each $a \in A_1$, let $a = c^*b$ with $b, c \in A_2$. Define

$$\operatorname{tr}(a) = (b, c) = \sum_\tau m_\tau. \tag{2.2}$$

Then by Wong (1974), Lemma 4.4, $\operatorname{tr}(a) = \Sigma_\tau (af_\tau, f_\tau)$ and $|\operatorname{tr}(a)| \leqq |a|_1$.

3. The conjugate spaces of A_1 and A^{**}

A bounded linear operator T on A is called a right centralizer if $T(ab) = (T(a))b$ for all a, b in A. The set of all right centralizers on A is denoted by $R(A)$.

LEMMA 3.1. *As normed linear spaces, A^{**} and $R(A)$ are isometrically isomorphic.*

PROOF. Let $\{x_\tau\}$ be an approximate identity for A. For each T in $R(A)$, let T° be a weak limit point of $\{Tx_\tau\}$ in A^{**}. Then by a similar argument in the proof of Wong (1971), Lemma 2.1, we can show that T° is unique and $T^\circ a = T(a)$ for all a in A.

Conversely, let $T^\circ \in A^{**}$. Since $T^\circ a \in A$ for all a in A, it follows that the mapping $T: a \to T^\circ a$ is a right centralizer on A, clearly $\| T \| \leqq \| T^\circ \|$. Since for all f in A^*,

$$| T^\circ(f) | = |\lim_\tau f(Tx_\tau) | \leqq \| f \| \, \| T \|,$$

it follows that $\| T^\circ \| \leqq \| T \|$ and so they are equal. The lemma now follows.

For each right centralizer T on A and x in A_1, by Wong (1974), Lemma 3.6, $Tx \in A_1$ and $| Tx |_1 \leqq \| T \| \, | x |_1$. We define

$$F_T(x) = \operatorname{tr}(Tx) \qquad (x \in A). \tag{3.1}$$

LEMMA 3.2. *For each T in $R(A)$, F_T is an element in A_1^* with $\| F_T \| = \| T \|$. Conversely, for each F in A_1^*, there exists some T in $R(A)$ such that $F = F_T$.*

PROOF. Let T be a right centralizer on A. Since for all x in A_1,

$$| F_T(x) | = | \operatorname{tr}(Tx) | \leqq | Tx |_1 \leqq \| T \| \, | x |_1,$$

it follows that $F_T \in A_1^*$ and $\| F_T \| \leqq \| T \|$. We show that $\| F_T \| \geqq \| T \|$. In fact, let $a \in A$ and $\{e_\tau\}$ a maximal orthogonal family of hermitian minimal idempotents in A such that $(Ta)(Ta)^* e_\tau = e_\tau (Ta)(Ta)^* e_\tau$. Then

$$\begin{aligned} \| e_\tau (Ta)(Ta)^* e_\tau \| &= \operatorname{tr}(e_\tau (Ta)(Ta)^* e_\tau) \text{ (by Wong (1974), Lemma 4.4)} \\ &= \operatorname{tr}(T(a(Ta)^* e_\tau)) \leqq \| F_T \| \, | a(Ta)^* e_\tau |_1 \\ &\leqq \| F_T \| \, \| a \| \, \| Ta \| \, | e_\tau |_1 \text{ (by Wong (1974), Lemma 3.6)} \\ &= \| F_T \| \, \| a \| \, \| Ta \|, \end{aligned} \tag{3.2}$$

where the last equality in (3.2) follows from the fact that $| e_\tau |_1 = 1$ (Wong

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(1974), Lemma 3.1). Since $\{e_\tau\}$ is maximal and commutes with $(Ta)(Ta)^*$, it follows easily that

$$(3.3)\qquad \begin{aligned}\| Ta \|^2 = \|(Ta)(Ta)^*\| &= \sup_\tau \| e_\tau (Ta)(Ta)^* e_\tau \| \\ &\leqq \| F_T \| \, \| a \| \, \| Ta \|.\end{aligned}$$

Therefore it follows immediately from (3.3) that $\| T \| \leqq \| F_T \|$ and so they are equal.

Conversely, let F be an element in A_1^*. Then by Saworotnow (1970), Theorem 2, F gives a right centralizer S on A_2 such that $\operatorname{tr}(Sx) = F(x)(x \in A_1)$ and $\| S \| = \| F \|$. We show that S can be extended to a right centralizer T on A. In fact, let a be an element in the socle of A. Then $a \in A_2$. Let $\{f_\tau\}$ be a maximal orthogonal family of hermitian minimal idempotents in A such that $(Sa)(Sa)^* f_\tau = f_\tau (Sa)(Sa)^* f_\tau$. By (3.2) and (3.3), we see that $\| Sa \|^2 \leqq \| F \| \, \| a \| \, \| Sa \|$ and so $\| Sa \| \leqq \| F \| \, \| a \|$. Since the socle of A is dense in A, it follows easily that S admits an extension T in $R(A)$ and this completes the proof.

Now we have a generalization of Schatten (1960), Theorem 2.

THEOREM 3.3. *Let A be a dual B^*-algebra. Then the conjugate space of A_1 is A^{**}.*

PROOF. This result immediately follows from Lemma 3.1 and Lemma 3.2.

In Wong (1974), p. 367, we had shown that $A^* = A_1$. Hence Schatten (1960), Theorem 3 holds for an arbitrary dual B^*-algebra.

Let $A^\perp$ be the subspace of A^{***} which vanishes identically on $A \subset A^{**}$. If a is in A_1, then the expression $\operatorname{tr}(ax)(x \in A)$ gives a linear functional in A^* (see Wong (1974)). Since $aT \in A_1 (T \in A^{**})$, the expression $\operatorname{tr}(aT)$ also gives a linear functional in A^{***}. Thus we can identify A_1 as a subspace of A^{***}. If $a \in A^\perp \cap A_1$, then $\operatorname{tr}(ax) = 0$ for all $x \in A$ and so $\operatorname{tr}(aa^*e) = \operatorname{tr}(eaa^*e) = 0$ for any hermitian minimal idempotent $e \in A$. Hence $\| eaa^*e \| = \operatorname{tr}(eaa^*e) = 0$ and so $ea = 0$. Since e is arbitrary, it follows that $a = 0$. Consequently $A^\perp \cap A_1 = (0)$.

The following theorem is a generalization of a result by Dixmier. The argument used here is similar to that given in the proof of Schatten (1960), Theorem 5.

THEOREM 3.4. *Let f be a continuous linear functional on A^{**}. Then $F = G + H$ uniquely with $G \in A_1, H \in A^\perp$ and $\| F \| = \| G \| + \| H \|$.*

PROOF. By the proof of Schatten (1960), Theorem 5, it is sufficient to show that $\| F \| \geqq \| G \| + \| H \|$. Let $\delta > 0$ be given. Write $G = WQ$, where

$W \in R(A)$ with $\|W\| = 1$ and $Q = (G^*G)^{1/2}$ see Wong (1974). By Lemma 3.1, we can assume that $W \in A^{**}$. Let $Q = \Sigma_j k_j e_j = \Sigma_\tau e_\tau Q$ be a spectral representation of Q (see (2.1)). By Wong (1974), Lemma 3.1, $|Q|_1 = |G|_1 = \Sigma_j k_j$. Since $\{e_\tau\}$ is maximal, $\Sigma_\tau e_\tau A$ is dense in A and so we can choose some $a \in \Sigma_\tau e_\tau A$ with $\|a\| \leqq 1$ such that

$$G(a) > \|G\| - \delta. \tag{3.4}$$

Write $a = \Sigma_{j=1}^m e_\tau a_j$ with $a_j \in A$ and $e_{\tau_j} \in \{e_\tau\}$. Choose n so large that $\Sigma_{j>n}^j k_j > \delta$ and choose $b° \in A^{**}$ with $\|b°\| \leqq 1$ such that

$$H(b°) > \|H\| - \delta. \tag{3.5}$$

Let $E = \{e_j\}_{j=1}^n \cup \{e_{\tau_j}\}_{j=1}^m$. Then $E \subset \{e_\tau\}$ and $e_\tau E = (0)$ if $e_\tau \not\in E$. Let $P = \Sigma\{e : e \in E\}$ and $b = (1-P)b°(1-P)$. Since $-\|1-P\| = 1$, $\|b\| \leqq 1$. Since $a = Pa$, we have

$$\|a + b\| = \max(\|a\|, \|b\|) \leqq 1. \tag{3.6}$$

Also

$$Gb = WQb = W\Big(\sum_j k_j e_j b\Big) = W\Big(\sum_{j>n} k_j e_j\Big) b. \tag{3.7}$$

By identifying A_1 as a subspace of A^{***}, it follows from (3.7) that

$$\begin{aligned} G(b) = \operatorname{tr}(Gb) \leqq |Gb|_1 &\leqq \Big|\sum_{j>n} k_j e_j\Big|_1 \\ &= \sum_{j>n} k_j < \delta. \end{aligned} \tag{3.8}$$

Since a and $b - b°$ are in A, we have $H(a) = 0$ and $H(b) = H(b°)$. It follows now easily from (3.4), (3.5), (3.6) and (3.8) that $\|F\| \geqq \|G\| + \|H\|$ and this completes the proof.

4. A three lines theorem for A_p $(1 \leqq p \leqq \infty)$

For $1 \leqq p \leqq \infty$, it was shown in Wong (1974) that A_p can be identified with A_q^* $(p^{-1} + q^{-1} = 1)$. In fact, for each F in A_q^*, there exists some a in A_p such that $F(x) = \operatorname{tr}(ax)(x \in A_q)$ and $\|F\| = |a|_p$.

In this section, S_A will denote the socle of A. By Wong (1974), Theorem 3.9, S_A is dense in A_p $(1 \leqq p \leqq \infty)$.

Let $a(\neq 0)$ be a positive element in A with a spectral representation $a = \Sigma_n k_n e_n$. Then $k_n > 0$. Let z be a complex number. Define a^z $(\operatorname{Re} z \geqq 0)$ by $a^z = \Sigma_n k_n^z e_n$, where $k_n^z = e^{z \ln(k_n)}$. If $a \in S_A$, then it is clear that A^z is always well defined.

LEMMA 4.1. *Let $a \in A^{**}$ and $p^{-1} + q^{-1} = 1$ $(1 \leqq p < \infty)$. Then $a \in A_p$ if and only if*

$$\sup\{|\operatorname{tr}(as)|/|s|_q : s \in S_A\} < \infty. \tag{4.1}$$

PROOF. If $a \in A_p$, then by Wong (1974), Lemma 4.2,

$$|\operatorname{tr}(as)| \leqq |as|_1 \leqq |a|_p |s|_q \qquad (s \in S_A).$$

Hence (4.1) holds. Conversely, suppose (4.1) holds. Then the functional $F(s) = \operatorname{tr}(as)$ is continuous on S_A (with norm $|\,.\,|_q$). Since S_A is dense in A_q, it follows that F can be extended to a linear functional in A_q^*. Hence by Wong (1974), Theorem 6.2, there exists some b in A_p such that $F(x) = \operatorname{tr}(bx)$ for all $x \in A_q$. Thus $\operatorname{tr}(as) = \operatorname{tr}(bs)$ for all $s \in S_A$. Let e be a hermitian minimal idempotent in A. Since $a^*e \in S_A$, we have

$$\operatorname{tr}(eaa^*e) = \operatorname{tr}(aa^*e) = \operatorname{tr}(ba^*e) = \operatorname{tr}(eba^*e). \tag{4.2}$$

Since by Wong (1974), Lemma 4.4, $eaa^*e = (\operatorname{tr}(eaa^*e))e$ and $eba^*e = (\operatorname{tr}(eba^*e))e$, it follows from (4.2) that $eaa^*e = eab^*e$. Similarly $eab^*e = ebb^*e$. Consequently, $e(a-b)(a-b)^*e = 0$ and so $e(a-b) = 0$. Since e is arbitrary, we see easily that $A(a-b) = 0$. Thus $A^{**}(a-b) = 0$ and so $a = b \in A_p$. This completes the proof.

REMARK. Some arguments in the above proof are similar to those in the proof of Gohberg and Krein (1969), Lemma 12.1.

Let E_A be the set of all hermitian minimal idempotents in A. For each $e \in E_A$, $a \in A$ and $T \in A^{**}$, let $k(e, T, a)$ be the constant such that $k(e, T, a)e = eTae$.

DEFINITION. *Let G be a region in the complex plane. We call $\{T_z \in A^{**} : z \in G\}$ holomorphic in the region G, if for any $e \in E_A$ and $a \in A$, the scalar function $z \to k(e, T_z, a)$ is holomorphic in G.*

REMARK. It is easy to see that if A is a simple algebra, then the above definition is equivalent to that given in Gohberg and Krein (1969), since $A = LC(H)$ for some Hilbert space H and $e \in E_A$ has the form $e = x \otimes x$ with x in H.

Now we have the main result in this section which is a generalization of Gohberg and Krein (1969), Theorem 13.1.

THEOREM 4.2. *Let $\{T_z \in A^{**} : u \leqq \operatorname{Re} z \leqq v\}$ $(u < v)$ be holomorphic. Suppose that $T_z \in A_{r_1}$ $(1 \leqq r_1 < \infty)$ on the line $z = u + iy$ $(-\infty < y < \infty)$ and $T_z \in A_{r_2}$ $(r_1 < r_2 \leqq \infty)$ on the line $z = v + iy$ $(-\infty < y < \infty)$. If $|T_{u+iy}|_{r_1} \leqq C_1$, $|T_{v+iy}|_{r_2} \leqq C_2$ $(-\infty < y < \infty)$ and*

$$\ln | k(e, T_z, a | \leqq N_{e,a} \exp(k_{e,a} | \operatorname{Im} z |),$$

where $u < \operatorname{Re} z < v$, $e \in E_A$, $a \in A$ *and* $0 \leqq k_{e,a} < \pi/(v-u)$, *then, for all* $z = x + iy$ $(u < x < v, -\infty < y < \infty)$, $T_z = T_{x+iy} \in A_r$ *and* $| T_{x-iy} |_r \leqq C_1^{1-t}x C_2^{t_x}$, *where* $t_x = (x-u)/(v-u)$ *and* $r^{-1} = r_1^{-1} + t_x(r_2^{-1} - r_1^{-1})$.

Proof. Let $s \in S_A$ with $| s |_{r'} = 1$ $((r')^{-1} + r^{-1} = 1)$ and $G = (s^*s)^{1/2}$. Then we can write $s = WG$ and $G = W^*s$, where $W, W^* \in R(A)$ (see Wong (1974)). By Lemma 3.1, we can assume that W and W^* are in A^{**}. Hence $G \in S_A$ and so it has a spectral representation $G = \sum_{j=1}^n k_j e_j$. Consider the function

$$f(z) = \operatorname{tr}(T_a W G^{c+dz}) \qquad (u \leqq \operatorname{Re} z \leqq v),$$

where

$$c + dz = r'((v-z)/(v-u)r_1' + (z-u)/(v-u)r_2'),$$

$(r_1'^{-1} + r_j^{-1} = 1;\ j = 1,2)$. Then it is easy to see that $f(z) = \sum_{j=1}^n k_j$, where $k_j = k(e_j, T_z, WG^{c+dz})$. Hence $f(z)$ is holomorphic on $u \leqq \operatorname{Re} z \leqq v$. Now by using Lemma 4.1 and the argument in the proof of Gohberg and Krein (1969), Theorem 13.1, we can prove the theorem.

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https://doi.org/10.1017/S1446788700017390 Published online by Cambridge University Press