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THE GENERALIZED EQUATIONS OF BISYMMETRY ASSOCIATIVITY AND TRANSITIVITY ON QUASIGROUPS

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1. **Introduction.** The generalized equations of bisymmetry, associativity and transitivity are, respectively,

$$(x\,1\,y)\,2\,(z\,3\,u) = (x\,4\,z)\,5\,(y\,6\,u) \tag{1}$$

$$(x\,1\,y)\,2\,z = x\,3\,(y\,4\,z) \tag{2}$$

$$(x\,1\,z)\,2\,(y\,3\,z) = x\,4\,y. \tag{3}$$

The numbers $1, 2, 3, \ldots, 6$ represent binary operations and x, y, z and u are taken freely from certain sets.

We shall be concerned with the cases in which x, y, z, and u are from the same set and each operation is a quasigroup operation. Under these conditions the solution of all three equations is known [1], [2]; equations (1) and (3) having been reduced to the form of (2) and a solution of (2) being given. We wish to present a new approach to these equations which we feel has the advantages that the equations may be resolved independently, the motivation behind the proof is clear, and the method lends itself to application on algebraic structures weaker than quasigroups. (Details of these generalizations will be given elsewhere.)

2. **The generalized equation of bisymmetry.**

THEOREM 1. *If (1) holds for all $x, y, z, u \in G$, and each $(G, i)(i=1, 2, \ldots, 6)$ is a quasigroup, then every $(G, i)(i=1, 2, \ldots, 6)$ is isotopic to the same abelian group.*

We shall require the following result, a proof of which may be found in [1];

THEOREM 0. *If the Thomsen condition holds in a quasigroup, then that quasigroup is isotopic to an abelian group.*

The Thomsen condition is said to hold in a quasigroup if every array

$$x_1y_2 = x_2y_1,$$

$$x_1y_3 = x_3y_1,$$

implies that

$$x_3y_2 = x_2y_3.$$

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Proof of Theorem 1. We shall show that the Thomsen condition holds in $(G, 1)$. Let us suppose that

$$x_1\,1\,y_2 = x_2\,1\,y_1, \quad x_1, x_2, y_1, y_2 \in G \tag{4}$$

$$x_1\,1\,y_3 = x_3\,1\,y_1, \quad x_3, y_3 \in G. \tag{5}$$

Substituting the expressions of (4) into (1) and equating the right-hand sides of that equation we find,

$$(x_1\,4\,z)\,5\,(y_2\,6\,u) = (x_2\,4\,z)\,5\,(y_1\,6\,u). \tag{6}$$

$(G, 4)$, $(G, 6)$ are quasigroups so we can choose $u_1, u_2, z_1, z_2 \in G$ such that,

$$x_1\,4\,z_1 = x_3\,4\,z_2 = v \text{ (say)}$$

$$y_1\,6\,u_1 = y_3\,6\,u_2 = w \text{ (say)}$$

and using these relationships in (6) we have, by first putting $z=z_1$, $u=u_1$,

$$(x_3\,4\,z_2)\,5\,(y_2\,6\,u_1) = (x_2\,4\,z_1)\,5\,(y_3\,6\,u_2).$$

This is equivalent to

$$(x_3\,1\,y_2)\,2\,(z_2\,3\,u_1) = (x_2\,1\,y_3)\,2\,(z_1\,3\,u_2). \tag{7}$$

However,

$$v\,5\,w = (x_3\,4\,z_2)\,5\,(y_1\,6\,u_1) = (x_1\,4\,z_1)\,5\,(y_3\,6\,u_2),$$

and this is equivalent to

$$(x_3\,1\,y_1)\,2\,(z_2\,3\,u_1) = (x_1\,1\,y_3)\,2\,(z_1\,3\,u_2).$$

$(G, 2)$ is cancellative and

$$x_3\,1\,y_1 = x_1\,1\,y_3$$

so

$$z_2\,3\,u_1 = z_1\,3\,u_2.$$

Consequently from (7) we obtain,

$$x_3\,1\,y_2 = x_2\,1\,y_3.$$

Thus we have shown that the Thomsen condition holds in $(G, 1)$, so by Theorem 0, $(G, 1)$ is isotopic to an abelian group.

The isotopies between the (G, i) $(i=1, 2, \ldots, 6)$ are established as follows.

We fix $x=a$, $y=b$ in (1), which becomes,

$$(a\,1\,b)\,2\,(z\,3\,u) = (a\,4\,z)\,5\,(b\,6\,u).$$

Left or right translation by a fixed element under a quasigroup operation is a bijection, so $(G, 3)$, $(G, 5)$ are isotopic.

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Similarly, the symmetry of equation (1) gives us that the pairs, $(G, 2)$ and $(G, 4)$, $(G, 2)$, and $(G, 6)$, $(G, 1)$, and $(G, 5)$ are isotopic.

We now complete the proof by showing $(G, 1)$ and $(G, 4)$ are isotopic.

Define bijections f and g by

$$y\,6\,u = c \Leftrightarrow u = f(y), \quad c \in G$$

$$z\,3\,u = c \Leftrightarrow z = g(u)$$

which is possible as $(G, 3)$, $(G, 6)$ are quasigroups.

Choose $u, z \in G$ such that

$$y\,6\,u = c$$

$$z\,3\,u = c.$$

This gives

$$(x\,1\,y)\,2\,c = (x\,4\,g(f(y)))\,5\,c$$

which establishes that $(G, 1)$ and $(G, 4)$ are indeed isotopic.

3. **The generalized equation of associativity**

THEOREM 2. *If* (2) *holds for all* $x, y, z \in G$ *and the* $(G, i)(i=1, 2, 3, 4)$ *are quasigroups, then each* $(G, i)(i=1, 2, 3, 4)$ *is isotopic to the same group.*

We shall utilize the following result [1];

THEOREM 00. *If the Reidemeister condition holds in a quasigroup then that quasigroup is isotopic to a group.*

The Reidemeister condition is said to hold in a quasigroup if every array of the form

$$x_1y_2 = x_2y_1,$$

$$x_3y_2 = x_4y_1,$$

$$x_2y_3 = x_1y_4,$$

implies that

$$x_3y_4 = x_4y_3.$$

Proof of Theorem 2. Let us assume that

$$x_1\,1\,y_2 = x_2\,1\,y_1,\ x_2\,1\,y_3 = x_1\,1\,y_4, \quad x_1, x_2, x_3, x_4 \in G$$

$$x_3\,1\,y_2 = x_4\,1\,y_1 \qquad\qquad y_1, y_2, y_3, y_4 \in G$$

If we substitute these into (2) we get

$$x_1\,3\,(y_2\,4\,z) = x_2\,3\,(y_1\,4\,z) \tag{8}$$

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(9) $$x_2\,3\,(y_3\,4\,z) = x_1\,3\,(y_4\,4\,z)$$

(10) $$x_3\,3\,(y_2\,4\,z) = x_4\,3\,(y_1\,4\,z).$$

We now choose $z_1, z_2 \in G$ such that

(11) $$y_2\,4\,z_1 = y_4\,4\,z_2,$$

and put $z=z_1$ in (10) to give,

$$x_3\,3\,(y_2\,4\,z_1) = x_4\,3\,(y_1\,4\,z_1)$$

i.e.

(12) $$x_4\,3\,(y_1\,4\,z_1) = x_3\,3\,(y_4\,4\,z_2).$$

However $z=z_1$ in (8) and $z=z_2$ in (9) yield

$$x_1\,3\,(y_2\,4\,z_1) = x_2\,3\,(y_1\,4\,z_1)$$

$$x_2\,3\,(y_3\,4\,z_2) = x_1\,3\,(y_4\,4\,z_2),$$

and as from (11),

$$x_1\,3\,(y_2\,4\,z_1) = x_1\,3\,(y_4\,4\,z_2),$$

it follows that,

$$x_2\,3\,(y_3\,4\,z_2) = x_2\,3\,(y_1\,4\,z_1)$$

and consequently,

$$y_3\,4\,z_2 = y_1\,4\,z_1.$$

Hence (12) may be written

$$x_3\,3\,(y_4\,4\,z_2) = x_4\,3\,(y_3\,4\,z_2)$$

which is equivalent to

$$(x_3\,1\,y_4)\,2\,z_2 = (x_4\,1\,y_3)\,2\,z_2.$$

$(G, 2)$ is cancellative, so,

$$x_3\,1\,y_4 = x_4\,1\,y_3.$$

This shows that the Reidemeister condition holds in $(G, 1)$ and, therefore, that $(G, 1)$ is isotopic to a group follows from Theorem 00.

The isotopies between the (G, i) $(i=1, 2, 3, 4)$ are established in the manner of Theorem 1.

4. The generalized equation of transitivity

THEOREM 3. *If (3) holds for all $x, y, z \in G$ and the (G, i) $(i=1, 2, 3, 4)$ are quasi-groups then each (G, i) $(i=1, 2, 3, 4)$ is isotopic to the same group.*

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Proof. The Reidemeister condition is shown to hold in $(G, 1)$.

Suppose that,

$$x_1 \, 1 \, z_2 = x_2 \, 1 \, z_1 \tag{13}$$

$$x_1 \, 1 \, z_4 = x_2 \, 1 \, z_3, \quad x_1, x_2, x_3, x_4 \in G \tag{14}$$

$$x_3 \, 1 \, z_2 = x_4 \, 1 \, z_1, \quad z_1, z_2, z_3, z_4 \in G. \tag{15}$$

We will show that

$$x_3 \, 1 \, z_4 = x_4 \, 1 \, z_3.$$

Choose $y_1, y_2 \in G$ such that

$$x_1 \, 4 \, y_1 = x_2 \, 4 \, y_2.$$

We then have

$$\begin{aligned}(x_1 \, 1 \, z_2) \, 2 \, (y_1 \, 3 \, z_2) &= (x_2 \, 1 \, z_1) \, 2 \, (y_2 \, 3 \, z_1) \\ &= (x_1 \, 1 \, z_4) \, 2 \, (y_1 \, 3 \, z_4) = (x_2 \, 1 \, z_3) \, 2 \, (y_2 \, 3 \, z_3),\end{aligned}$$

which implies the following set of equations,

$$y_1 \, 3 \, z_2 = y_2 \, 3 \, z_1 \tag{16}$$

$$y_1 \, 3 \, z_4 = y_2 \, 3 \, z_3. \tag{17}$$

From (15) and (16), we see that,

$$(x_3 \, 1 \, z_2) \, 2 \, (y_1 \, 3 \, z_2) = (x_4 \, 1 \, z_1) \, 2 \, (y_2 \, 3 \, z_1)$$

i.e.

$$x_3 \, 4 \, y_1 = x_4 \, 4 \, y_2. \tag{18}$$

Now,

$$x_3 \, 4 \, y_1 = (x_3 \, 1 \, z_4) \, 2 \, (y_1 \, 3 \, z_4)$$

and

$$x_4 \, 4 \, y_2 = (x_4 \, 1 \, z_3) \, 2 \, (y_2 \, 3 \, z_3).$$

However, (17) and (18), together with the cancellativity of $(G, 2)$, give,

$$x_3 \, 1 \, z_4 = x_4 \, 1 \, z_3.$$

Hence the Reidemeister condition holds in $(G, 1)$ and, therefore $(G, 1)$ is isotopic with a group.

The isotopies are established as in the manner of the proof of Theorem 1.

References

1. J. Aczél, *Quasigroups, nets and nomograms*, Advances in Math. **1**, Fasc. 3, 1965.

2. ———, *Theory of functional equations and its applications*, Academic Press, New York, 1966.

3. J. Aczél, V. D. Belousov, and M. Hosszú, *Generalised associativity and bisymmetry on quasigroups*, Acta Math. Acad. Sci. Hungar **11** (1960), 127–136.

4. F. Radó, *Équations fonctionnelles caractérisant les nomogrammes avec trois échelles rectilignes*, Mathematica (Cluj) **1** (24), 1 (1959), 143–166.

5. ———, *Sur quelques équations fonctionnelles avec plusieures fonctions à deux variables*, Mathematica (Cluj) **1** (24), 2 (1959), 321–339.

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