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MULTIPLICITIES IN SYLOW SEQUENCES AND THE SOLVABLE RADICAL

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Abstract

A complete Sylow sequence, $\mathcal{P} = P_1, \ldots, P_m$, of a finite group G is a sequence of m Sylow p_i-subgroups of G, one for each p_i, where $p_1, \ldots, p_m$ are all of the distinct prime divisors of $|G|$. A product of the form $P_1 \cdots P_m$ is called a *complete Sylow product* of G. We prove that the solvable radical of G equals the intersection of all complete Sylow products of G if, for every composition factor S of G, and for every ordering of the prime divisors of $|S|$, there exist a complete Sylow sequence $\mathcal{P}$ of S, and $g \in S$ such that g is uniquely factorizable in $\mathcal{P}$. This generalizes our results in Kaplan and Levy ['The solvable radical of Sylow factorizable groups', *Arch. Math.* **85**(6) (2005), 490–496].

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1. Introduction

Let G be a finite group[1] and let $\pi(G) = \{p_1, \ldots, p_m\}$ denote the set of all distinct prime divisors of $|G|$. If A is a subgroup or a quotient group of G, and τ is a permutation of $1, \ldots, m$, then a *complete Sylow sequence* of A *of type* τ is a sequence of the form $\mathcal{P} = P_{\tau(1)}, \ldots, P_{\tau(m)}$, where each P_j is a Sylow p_j-subgroup of A and trivial Sylow subgroups of A (those for which p_j does not divide $|A|$) are ignored. Note that the length of such a sequence is $|\pi(A)|$. The corresponding product $\Pi(\mathcal{P}) = P_{\tau(1)} \cdots P_{\tau(m)}$ (which is a subset of A) is called a *complete Sylow product* of A *of type* τ. Whenever the precise ordering is immaterial, we just use the term complete Sylow sequence (product), and $\mathcal{P} = P_1, \ldots, P_m$ will refer to a fixed but otherwise arbitrary ordering of the primes in $\pi(G)$. We denote by $CSS(A)$ ($CSS_\tau(A)$) the set of all complete Sylow sequences of A (of type τ).

The earliest reference known to us that considers complete Sylow products is a paper by Miller [10]. Miller proves that if G is solvable, then $\Pi(\mathcal{P}) = G$ for every complete Sylow sequence $\mathcal{P}$ of G, and raises the question whether the reverse

[1] All groups considered in this paper are assumed to be finite.

implication is also true. The same claim and question are also discussed, much later and independently, by Hall in [5]. Thanks to the work of Thompson on the classification of N-groups [11, Corollary 3] the following solvability criterion is established: A group G is solvable if and only if $\Pi(\mathcal{P}) = G$ for every complete Sylow sequence $\mathcal{P}$ of G [1, Theorem 1], [8, Theorem A]. A closely related result already appears in [3] (see also [7, 17.14]).

In [8, 9] we found a connection between complete Sylow products of an arbitrary group and its solvable radical. For a given group G, we denote by $H_\tau(G)$ the intersection of all complete Sylow products of G of type τ, and by $H(G)$, the intersection of all complete Sylow products of G (obviously $H(G) \subseteq H_\tau(G)$). Let $R(G)$ denote the solvable radical of G. We proved that $H_\tau(G)$ and $H(G)$ are characteristic subgroups of G, and that $R(G) \leq H(G)$. We have also given certain sufficient conditions for equality. It is still an open question whether $H(G) = R(G)$ for every group G, or equivalently, whether $H(G)$ is always solvable.

In the present paper we consider these issues using the concept of multiplicity of an element in a Sylow sequence. This concept also appears in [10] under a different name.

DEFINITION 1.1. Let G be a group and let $\mathcal{P} = P_1, \ldots, P_m$ be a complete Sylow sequence of G. *A factorization of* $g \in G$ *in* $\mathcal{P}$ is a sequence $g_1, \ldots, g_m$ where $g_i \in P_i$ such that $g = g_1 \cdots g_m$. When convenient, we shall refer to an expression of the form $g = g_1 \cdots g_m$ as a factorization of $g \in G$. *The multiplicity of* g *in* $\mathcal{P}$ is the number of distinct factorizations of g in $\mathcal{P}$. This nonnegative integer will be denoted $m_{\mathcal{P}}(g)$.

Observe that, for any complete Sylow sequence $\mathcal{P}$ of G, we have $\sum_{g\in G} m_{\mathcal{P}}(g) = |G|$. Hence, $\Pi(\mathcal{P}) = G$ if and only if $m_{\mathcal{P}}(g) = 1$ for all $g \in G$. If such a $\mathcal{P}$ exists we say that G is *Sylow factorizable*. In [9] we have proved that if G is Sylow factorizable then $R(G) = H(G)$. However, it is known (see [6] and Proof of Proposition 1.5 here) that not all groups are Sylow factorizable. This motivates the examination of weaker conditions on multiplicities in Sylow sequences.

DEFINITION 1.2. Let G be a group and let τ be an arbitrary ordering of $\pi(G)$. Then G *is* τ*-unique1* if there exists a complete Sylow sequence $\mathcal{P}$ of type τ such that $m_{\mathcal{P}}(g) = 1$ for some $g \in G$. We shall say that G *is unique1* if[1] it is τ-unique1 for all τ.

Our first result shows that the $R(G) = H(G)$ question is related to the question of which simple groups are unique1.

THEOREM 1.3. *Let G be a group such that all composition factors of G are unique1. Then $R(G) = H(G)$.*

The question of whether a group G is unique1 is itself related to the question of whether its composition factors are unique1.

THEOREM 1.4. *The group property unique1 is closed under extensions.*

[1] The trivial group is defined to be unique1.

Determining which simple nonabelian groups are unique1 (simple abelian groups are trivially unique1) does not seem to be a particularly easy problem, and we shall not attempt to address it here in its general form. Nevertheless, the following very limited result may serve as an appetizer.

PROPOSITION 1.5. *All simple nonabelian groups G such that $|G|$ is divisible by exactly three primes, are unique1.*

Now we turn our attention to transformations which preserve the multiplicity of an element in a Sylow sequence.

DEFINITION 1.6. Let G be a group and let τ be some ordering of $\pi(G)$. Then

$$M_\tau(G) \stackrel{\text{def}}{=} \{g \in G \mid m_{\mathcal{P}}(gx) = m_{\mathcal{P}}(x), \text{ for all } x \in G, \text{ for all } \mathcal{P} \in CSS_\tau(G)\}$$
$$M(G) \stackrel{\text{def}}{=} \{g \in G \mid m_{\mathcal{P}}(gx) = m_{\mathcal{P}}(x), \text{ for all } x \in G, \text{ for all } \mathcal{P} \in CSS(G)\}.$$

REMARK 1.7. It is easy to verify that replacing gx by xg in Definition 1.6 yields the same two sets.

As a first application we rephrase the solvability criterion mentioned above.

THEOREM 1.8. *A group G is solvable if and only if $G = M_\tau(G)$ for some ordering τ of $\pi(G)$.*

PROOF. If G is solvable then $m_{\mathcal{P}}(x) = 1$ for all $x \in G$ and all $\mathcal{P} \in CSS_\tau(G)$, and therefore $G = M_\tau(G)$. In the other direction, let $\mathcal{P}$ be an arbitrary Sylow sequence of G of type τ. Then $\Pi(\mathcal{P})$ is not an empty set and hence $m_{\mathcal{P}}(x) > 0$ for some $x \in G$. Now $G = M_\tau(G)$ implies $m_{\mathcal{P}}(x) > 0$ for all $x \in G$ and hence $\Pi(\mathcal{P}) = G$. By [8, Theorem A], G is solvable. □

THEOREM 1.9. *Let G be a group and let τ be some ordering of $\pi(G)$. Then $M_\tau(G)$ and $M(G)$ are characteristic subgroups of G. Moreover, $R(G) \le M_\tau(G) \le H_\tau(G)$ and $R(G) \le M(G) \le H(G)$.*

We call a subgroup of $M(G)$ a *Sylow multiplicity preserving subgroup of G*. Thus, Theorem 1.9 allows us to view the solvable radical of G as a Sylow multiplicity preserving subgroup of G. Moreover, it is immediate from Theorem 1.9 that if G is such that $R(G) = H(G)$ then $R(G)$ is the unique maximal Sylow multiplicity preserving subgroup of G. Our last theorem shows that this already follows from assuming that G is τ-unique1.

THEOREM 1.10. *Let G be a group and let τ be some ordering of $\pi(G)$. If G is τ-unique1 then $M_\tau(G) = M(G) = R(G)$.*

REMARK 1.11. Note that the conclusion of Theorem 1.10 follows from the assumption of Theorem 1.3 (use Theorems 1.3 and 1.9). However, the assumption of Theorem 1.3 implies the assumption of Theorem 1.10, while we do not know whether the reverse implication holds.

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2. Proofs

2.1. Basic properties We summarize some useful concepts and results concerning multiplicities of elements in Sylow sequences, which will be used in the proofs of the main theorems.

DEFINITION 2.1. Let G be a group and let $\mathcal{P} = P_1, \ldots, P_m$ be a complete Sylow sequence of G. Let $b = b_1 \cdots b_m$ be a factorization of b in $\mathcal{P}$. We define the complete Sylow sequence $\mathcal{P}b_m^{-1} \cdots b_1^{-1}$ by

$$\mathcal{P}b_m^{-1} \cdots b_1^{-1} \stackrel{\text{def}}{=} P_1,\ P_2^{b_1^{-1}}, \ldots, P_i^{b_{i-1}^{-1}\cdots b_1^{-1}}, \ldots, P_m^{b_{m-1}^{-1}\cdots b_1^{-1}}.$$

REMARK 2.2. Since each b_i above takes any value in the group P_i, $\mathcal{P}b_m \cdots b_1$ is also well defined. One can verify the equality $\Pi(\mathcal{P}b_m^{-1} \cdots b_1^{-1}) = (\Pi\mathcal{P})b^{-1}$. We stress that the sequence $\mathcal{P}b_m^{-1} \cdots b_1^{-1}$ depends on the particular factorization chosen for b although its product depends only on b. Also note that a complete Sylow sequence $b_m^{-1} \cdots b_1^{-1}\mathcal{P}$ can be defined similarly, and that $\mathcal{P}b_m^{-1} \cdots b_1^{-1}$ is of the same type as $\mathcal{P}$.

LEMMA 2.3. *Let G be a group and let $\mathcal{P} = P_1, \ldots, P_m$ be a complete Sylow sequence of G. Let $a, b \in \Pi(\mathcal{P})$. Let $a = a_1 \cdots a_m$ and $b = b_1 \cdots b_m$ be factorizations of a and b in $\mathcal{P}$ respectively. Then*

$$(\mathcal{P}b_m^{-1} \cdots b_1^{-1})(a_m^{-1})^{b_{m-1}^{-1}\cdots b_1^{-1}} \cdots (a_2^{-1})^{b_1^{-1}} a_1^{-1} = \mathcal{P}(b_m^{-1}a_m^{-1}) \cdots (b_1^{-1}a_1^{-1}).$$

PROOF. Routine computation. □

LEMMA 2.4. *Let G be a group and let $\mathcal{P} = P_1, \ldots, P_m$ be a complete Sylow sequence of G. For any $x \in \Pi(\mathcal{P})$ and $y \in G$, and for any factorization $x = x_1 \cdots x_m$ of x in $\mathcal{P}$,*

$$m_{\mathcal{P}x_m^{-1}\cdots x_1^{-1}}(yx^{-1}) = m_{\mathcal{P}}(y).$$

PROOF. Let $y = y_1^{(i)} \cdots y_m^{(i)}$, $i = 1, 2$, be two distinct factorizations of y in $\mathcal{P}$. Then

$$yx^{-1} = (y_1^{(i)}x_1^{-1})(y_2^{(i)}x_2^{-1})^{x_1^{-1}} \cdots (y_m^{(i)}x_m^{-1})^{x_{m-1}^{-1}\cdots x_1^{-1}}, \quad i = 1, 2,$$

are two distinct factorizations of yx^{-1} in $\mathcal{P}x_m^{-1} \cdots x_1^{-1}$. Hence $m_{\mathcal{P}x_m^{-1}\cdots x_1^{-1}}(yx^{-1}) \geq m_{\mathcal{P}}(y)$ (including the cases $m_{\mathcal{P}}(y) = 0, 1$). However, since $1_G \in \Pi(\mathcal{P})$, we have $x^{-1} \in \Pi(\mathcal{P})x^{-1} = \Pi(\mathcal{P}x_m^{-1} \cdots x_1^{-1})$. In fact,

$$x^{-1} = x_1^{-1}(x_2^{-1})^{x_1^{-1}}(x_3^{-1})^{x_2^{-1}x_1^{-1}} \cdots (x_m^{-1})^{x_{m-1}^{-1}\cdots x_1^{-1}}$$

is a factorization of x^{-1} in $\mathcal{P}x_m^{-1} \cdots x_1^{-1}$. Hence, we can repeat the argument above with $\mathcal{P}$ replaced by $\mathcal{P}x_m^{-1} \cdots x_1^{-1}$, y replaced by yx^{-1} and x replaced by x^{-1}. We get

$$m_{\mathcal{P}x_m^{-1}\cdots x_1^{-1}(x_m)^{x_{m-1}^{-1}\cdots x_1^{-1}}\cdots(x_2)^{x_1^{-1}}x_1}(yx^{-1}x) \geq m_{\mathcal{P}x_m^{-1}\cdots x_1^{-1}}(yx^{-1}).$$

By Lemma 2.3, the left-hand side is $m_{\mathcal{P}}(y)$ and this concludes the proof. □

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Corollary 2.5. *Let G be a group and let $\mathcal{P}$ be a complete Sylow sequence of type τ of G. Let $g \in G$. If $m_{\mathcal{P}}(g) = 1$ then there exists a complete Sylow sequence $\mathcal{Q}$ of G of the same type τ such that $m_{\mathcal{Q}}(1_G) = 1$.*

Proof. Since $m_{\mathcal{P}}(g) = 1$, $g \in \Pi(\mathcal{P})$ and hence there exists a factorization $g = g_1 \cdots g_m$ of g in $\mathcal{P}$. Now, by taking in Lemma 2.4 $y = x = g$, we get $\mathcal{Q} = \mathcal{P} g_m^{-1} \cdots g_1^{-1}$. □

Lemma 2.6. *Let G be a group and let $\mathcal{P}$ be a complete Sylow sequence of G. Let $g \in G$ and suppose that $g = g_1^{(i)} \cdots g_m^{(i)}$, $i = 1, 2$, are two distinct factorizations of g in $\mathcal{P}$. Then there are at least three distinct values of $1 \le k \le m$ such that $g_k^{(1)} \neq g_k^{(2)}$.*

Proof. First observe that if $1_G = x_1 \cdots x_m$ is a nontrivial Sylow factorization of 1_G, then there are at least three distinct values of $1 \le k \le m$ such that $x_k \neq 1_G$. The claim of the lemma now follows from the fact that

$$1_G = (g_1^{(1)}(g_1^{(2)})^{-1})(g_2^{(1)}(g_2^{(2)})^{-1})^{(g_1^{(2)})^{-1}} \cdots (g_m^{(1)}(g_m^{(2)})^{-1})^{(g_{m-1}^{(2)})^{-1} \cdots (g_1^{(2)})^{-1}}$$

is a nontrivial Sylow factorization of 1_G. □

2.2. Proof of Theorem 1.3 The key result which relates the $R(G) = H(G)$ question to the unique1 property is formulated in the following lemma.

Lemma 2.7. *Let G be a group and let τ be an arbitrary ordering of $\pi(G)$. Let $N \trianglelefteq G$ be such that $\overline{G} = G/N$ is τ-unique1. Then $H_\tau(G) \cap N = H_\tau(N)$.*

Proof. Observe that $H_\tau(N) \le H_\tau(G) \cap N$ since every complete Sylow product of type τ of G contains a complete Sylow product of type τ of N. It therefore remains to prove $H_\tau(G) \cap N \le H_\tau(N)$. By assumption there exist a complete Sylow sequence $\overline{\mathcal{P}}$ of type τ of $\overline{G}$ and $\overline{g} \in \overline{G}$ such that $m_{\overline{\mathcal{P}}}(\overline{g}) = 1$. By Corollary 2.5 we can assume that $\overline{g} = 1_{\overline{G}}$. We have $\overline{\mathcal{P}} = P_1N/N, \ldots, P_mN/N$ where the P_i are Sylow p_i-subgroups of G. Denote $\mathcal{P} = P_1, \ldots, P_m$. Let $n_1, \ldots, n_m$ be m arbitrary elements of N (not necessarily p_i-elements). We claim that

$$(P_1^{n_1} \cdots P_m^{n_m}) \cap N = (P_1 \cap N)^{n_1} \cdots (P_m \cap N)^{n_m}.$$

The reason for this is as follows. The subset on the r.h.s. is easily seen to be contained in the subset on the left-hand side. (Here the assumption that $\overline{G}$ is τ-unique1 is not required.) For the reverse inclusion we make use of the assumption $m_{\overline{\mathcal{P}}}(1_{\overline{G}}) = 1$. Let $g \in (P_1^{n_1} \cdots P_m^{n_m}) \cap N$. Then $g = g_1 \cdots g_m$, with $g_i \in P_i^{n_i}$. Since also $g \in N$, its image under the canonical homomorphism $G \to \overline{G}$ is $1_{\overline{G}}$. Hence $\overline{g_1} \cdots \overline{g_m} = 1_{\overline{G}}$ where $\overline{g_i} \in P_i^{n_i}N/N = P_iN/N$ is the image of g_i. The assumption $m_{\overline{\mathcal{P}}}(1_{\overline{G}}) = 1$ now implies $\overline{g_i} = 1_{\overline{G}}$ for all $1 \le i \le m$. Hence $g_i \in P_i^{n_i} \cap N = (P_i \cap N)^{n_i}$ for all $1 \le i \le m$, concluding the proof of the reverse inclusion.

Note that the intersection of $(P_1 \cap N)^{n_1} \cdots (P_m \cap N)^{n_m}$ over all choices of $n_1, \ldots, n_m \in N$ is $H_\tau(N)$. On the other hand, the intersection of $P_1^{n_1} \cdots P_m^{n_m} \cap N$

over all choices of $n_1, \ldots, n_m \in N$ contains $H_\tau(G) \cap N$. We obtain $H_\tau(G) \cap N \leq H_\tau(N)$. □

PROOF OF THEOREM 1.3. Let τ be an arbitrary ordering of $\pi(G)$. If G is simple then $H_\tau(G) = H(G) = R(G)$ (see [8, remarks after Theorem B]). Henceforth, we assume that G is not simple and we prove the claim by induction on $|G|$. Since $R(G) \leq H(G) \leq H_\tau(G)$, it is sufficient to prove that $H_\tau(G)$ is solvable. Let N be a maximal normal subgroup of G. Then G/N is simple. Hence, either $H_\tau(G/N) = 1$ (if G/N is simple nonabelian) or $H_\tau(G/N) = G/N$ (if G/N is cyclic of prime order) and in both cases $H_\tau(G/N)$ is solvable. One can prove (see [8, Lemma 11]) that

$$H_\tau(G)/(H_\tau(G) \cap N) \cong H_\tau(G)N/N \leq H_\tau(G/N).$$

Hence $H_\tau(G)/(H_\tau(G) \cap N)$ is solvable. By assumption, the composition factor G/N of G is τ-unique1. Thus, by Lemma 2.7, $H_\tau(G) \cap N = H_\tau(N)$. However, $H_\tau(N)$ is solvable by induction. Since $H_\tau(G)/H_\tau(N)$ is also solvable, we get that $H_\tau(G)$ is solvable. □

2.3. Proofs of Theorem 1.4 and Proposition 1.5

PROOF OF THEOREM 1.4. Let G be an arbitrary group and let τ be an arbitrary ordering of $\pi(G)$. Let N be any nontrivial proper normal subgroup of G such that N and $\overline{G} = G/N$ are τ-unique1 for the same τ. We shall prove that G is τ-unique1. Suppose to the contrary that G is not τ-unique1. Because $\overline{G}$ is τ-unique1 we have a complete Sylow sequence $\overline{\mathcal{P}} = \overline{P}_1, \ldots, \overline{P}_m$ of $\overline{G}$ of type τ such that $m_{\overline{\mathcal{P}}}(1_{\overline{G}}) = 1$. Let $\mathcal{P} = P_1, \ldots, P_m$ be a complete Sylow sequence of G of type τ such that $\overline{P}_i = P_i N/N$. Let $\mathcal{Q} = Q_1, \ldots, Q_m$ be any complete Sylow sequence of G such that $Q_1 = P_1^{n_1}, \ldots, Q_m = P_m^{n_m}$ where $n_1, \ldots, n_m \in N$ are arbitrary. Note that any such sequence maps, under the canonical homomorphism $G \to \overline{G}$, to $\overline{\mathcal{P}} = \overline{P}_1, \ldots, \overline{P}_m$ of $\overline{G}$. Since G is not τ-unique1, there exists a nontrivial factorization $1_G = g_1 \cdots g_m$ in $\mathcal{Q}$ for any $\mathcal{Q}$ as above. In $\overline{G} = G/N$ we get $\overline{g_1} \cdots \overline{g_m} = 1_{\overline{G}}$ where $\overline{g_i} \in \overline{P}_i$ is the image of g_i. Now $m_{\overline{\mathcal{P}}}(1_{\overline{G}}) = 1$ implies $\overline{g_i} = 1_{\overline{G}}$ for all $1 \leq i \leq m$. Hence $g_i \in N$ and $g_i \in P_i^{n_i} \cap N = (P_i \cap N)^{n_i}$ for all $1 \leq i \leq m$. Thus 1_G is not uniquely factorizable in any complete Sylow sequence $((P_i \cap N)^{n_i})_{1 \leq i \leq m}$ of N. However, these are all of the complete Sylow sequences of N of type τ, contradicting the assumption that N is τ-unique1. □

Another property of unique1 which is worth observing is given in the following proposition.

PROPOSITION 2.8. *The property unique1 is inherited by normal subgroups.*

PROOF. Let G be a group with the property unique1 and let $N \trianglelefteq G$. Let τ be an arbitrary ordering of $\pi(G)$. Then, by Corollary 2.5, there exists a complete Sylow sequence $\mathcal{P} = P_1, \ldots, P_m$ of G of type τ such that $m_{\mathcal{P}}(1_G) = 1$. Hence $\mathcal{Q} = P_1 \cap N, \ldots, P_m \cap N$ satisfies $m_{\mathcal{Q}}(1_G) = 1$. □

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REMARK 2.9. Every group property α which is inherited by normal subgroups and extensions has a residual, namely, for every group G, the set $\{N \trianglelefteq G \mid G/N \text{ is } \alpha\}$ has a unique minimal element. Thus, the property unique1 has a residual.

LEMMA 2.10. *Let G be a group such that $|G|$ is divisible by exactly three primes. Let τ be any ordering of $\pi(G)$. If G is τ-unique1 then G is unique1.*

PROOF. Let $\mathcal{P} = P_1, P_2, P_3$ be a complete Sylow sequence of G of type τ. Suppose that $1_G = abc$ with $a \in P_1$, $b \in P_2$ and $c \in P_3$ all nontrivial. Then $1_G = abc = cab = bca$ and $1_G = c^{-1}b^{-1}a^{-1} = a^{-1}c^{-1}b^{-1} = b^{-1}a^{-1}c^{-1}$ are all nontrivial Sylow factorizations of 1_G, corresponding to all possible orderings of the P_i. Thus $m_{\mathcal{P}}(1_G) = 1$ if and only if $m_{\mathcal{Q}}(1_G) = 1$ where $\mathcal{Q} = P_{\sigma(1)}, P_{\sigma(2)}, P_{\sigma(3)}$ and σ is any ordering of $\pi(G)$. Now the claim follows from Corollary 2.5. □

PROOF OF PROPOSITION 1.5. There are exactly eight simple nonabelian groups G such that $|G|$ is divisible by exactly three primes [4]: $PSL(2, 5) \cong A_5$, $PSL(2, 7)$, $PSL(2, 8)$, $PSL(2, 17)$, $PSL(3, 3)$, A_6, $SU(4, 2) \cong O(5, 3)$ and $SU(3, 3)$. The first seven groups are Sylow factorizable. The Sylow factorizibility of the first six follows from the results of [6] and from the fact that Sylow factorizibility is inherited by normal subgroups and by quotient groups. The Sylow factorizibility of $SU(4, 2)$ can be proved as follows. First note that $|SU(4, 2)| = 2^6 \cdot 3^4 \cdot 5$. $SU(4, 2)$ has a maximal subgroup $M = V \rtimes A_5$ (see [2, p. 26]), where $V \cong \mathbb{Z}_2^4$. Since A_5 is equal to a complete Sylow product where the primes are ordered $(3, 2, 5)$, and $V \trianglelefteq M$, we get that M is equal to a complete Sylow product of the same type. Now, since $|G : M| = 3^3$, $SU(4, 2) = PM$, where P is a Sylow 3-subgroup of $SU(4, 2)$ containing a Sylow 3-subgroup of M. It follows that $SU(4, 2)$ is Sylow factorizable. Finally, $SU(3, 3)$ is not Sylow factorizable but it has a complete Sylow sequence $\mathcal{P}$ such that $|\Pi(\mathcal{P})| > \frac{1}{2}|G|$ [6]. Clearly, in such $\mathcal{P}$ we have at least one $g \in G$ for which $m_{\mathcal{P}}(g) = 1$. Thus, all eight groups are τ-unique1 for some τ. We now use Lemma 2.10 in order to deduce that each of these groups is unique1. □

2.4. Proofs of Theorems 1.9 and 1.10 Lemmas 2.11 and 2.12 below are part of the content of Theorem 1.9.

LEMMA 2.11. *Let G be a group and let τ be some ordering of $\pi(G)$. Then $M_\tau(G)$ and $M(G)$ are characteristic subgroups of G.*

PROOF. Clearly $1 \in M_\tau(G)$ so $M_\tau(G) \neq \phi$. Let $g_1, g_2 \in M_\tau(G)$. Let $\mathcal{P}$ be any complete Sylow sequence of G of type τ, and let $x \in G$ be arbitrary. Then $m_{\mathcal{P}}((g_1g_2)x) = m_{\mathcal{P}}(g_1(g_2x)) = m_{\mathcal{P}}(g_2x) = m_{\mathcal{P}}(x)$ and thus $g_1g_2 \in M_\tau(G)$. This proves that $M_\tau(G)$ is a subgroup of G. It is characteristic in G since any automorphism of G defines a bijection $CSS_\tau(G) \to CSS_\tau(G)$. The claims about $M(G)$ now follow from

$$M(G) = \bigcap_{\tau \in \mathrm{Sym}(\{1,\ldots,m\})} M_\tau(G). \qquad \square$$

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LEMMA 2.12. *Let G be a group and let τ be some ordering of $\pi(G)$. Then $M_\tau(G) \leq H_\tau(G)$ and $M(G) \leq H(G)$.*

PROOF. It is sufficient to prove that if $g \in M_\tau(G)$ and if $\mathcal{P}$ is any complete Sylow sequence of G of type τ, then $g \in \Pi(\mathcal{P})$. We have $1_G \in \Pi(\mathcal{P})$. Hence $m_{\mathcal{P}}(1_G) > 0$. Now $m_{\mathcal{P}}(g) = m_{\mathcal{P}}(g1_G) = m_{\mathcal{P}}(1_G) > 0$. Therefore $g \in \Pi(\mathcal{P})$. $\square$

LEMMA 2.13. *Let G be a group, $\pi(G) = \{p_1, \ldots, p_m\}$ and let $\mathcal{P} = P_1, \ldots, P_m$ be an arbitrary complete Sylow sequence of G of type τ. Fix $1 \leq j \leq m$, and $1 \neq n \in O_{p_j}(G)$. Then, for all $x \in G$, one can pair the factorizations of x and xn in $\mathcal{P}$ so that the pair members are identical in all but the jth factors, the jth factors differ by an element of $O_{p_j}(G)$, and each factorization of x and xn belongs to exactly one pair. It follows that $O_{p_j}(G) \leq M_\tau(G)$.*

PROOF. Let $x = x_1 \cdots x_j \cdots x_m$ be a factorization of x in $\mathcal{P}$. Then

$$xn = x_1 \cdots x_{j-1}(x_j n^{x_m^{-1}x_{m-1}^{-1}\cdots x_{j+1}^{-1}})x_{j+1} \cdots x_m,$$

is a factorization of xn in $\mathcal{P}$. This factorization differs from the given factorization of x in $\mathcal{P}$ only in the jth factor and the difference between the factors is

$$x_j^{-1}x_j n^{x_m^{-1}x_{m-1}^{-1}\cdots x_{j+1}^{-1}} = n^{x_m^{-1}x_{m-1}^{-1}\cdots x_{j+1}^{-1}} \in O_{p_j}(G)$$

(note that $1 \neq n$ implies $1 \neq n^{x_m^{-1}x_{m-1}^{-1}\cdots x_{j+1}^{-1}}$). We define a pairing of factorizations of x and xn in $\mathcal{P}$ by associating to the factorization of $x = x_1 \cdots x_j \cdots x_m$ of x in $\mathcal{P}$ the factorization

$$xn = x_1 \cdots x_{j-1}(x_j n^{x_m^{-1}x_{m-1}^{-1}\cdots x_{j+1}^{-1}})x_{j+1} \cdots x_m$$

of xn in $\mathcal{P}$. Since two distinct factorizations of the same element in $\mathcal{P}$ must differ by at least three factors (Lemma 2.6), it is clear that two distinct factorizations of x are paired with two distinct factorizations of xn. Thus, this pairing defines an injective function from the set of all factorizations of x in $\mathcal{P}$ to the set of all factorizations of xn in $\mathcal{P}$. Since $x = (xn)n^{-1}$, we have a bijection. Hence $m_{\mathcal{P}}(x) = m_{\mathcal{P}}(xn)$, implying (see Remark 1.7) that $n \in M_\tau(G)$. $\square$

DEFINITION 2.14. Let G be a group, $\pi(G) = \{p_1, \ldots, p_m\}$, $N \trianglelefteq G$, and let $\mathcal{P} = P_1, \ldots, P_m$ be a complete Sylow sequence of G. Set $\overline{G} = G/N$. For any $x \in G$ and any $A \leq G$ denote by $\overline{x}$ and $\overline{A}$, respectively, their images under the canonical homomorphism from G onto $\overline{G}$. Let $\overline{\mathcal{P}} = \overline{P_1}, \ldots, \overline{P_m}$. Let $x = x_1 \cdots x_m$ be a factorization of x in $\mathcal{P}$. The *image of the factorization $x = x_1 \cdots x_m$ in $\overline{G}$* is the factorization $\overline{x} = \overline{x_1} \cdots \overline{x_m}$ of $\overline{x}$ in $\overline{\mathcal{P}}$. Conversely, a *preimage of a factorization $\overline{x} = \overline{x_1} \cdots \overline{x_m}$ of $\overline{x}$ in $\overline{\mathcal{P}}$* is any factorization, $xn = y_1 \cdots y_m$ in $\mathcal{P}$, where $n \in N$, such that $\overline{y_i} = \overline{x_i}$ for all $1 \leq i \leq m$.

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LEMMA 2.15. *Let G be a group, $\pi(G) = \{p_1, \ldots, p_m\}$ and $\mathcal{P} = P_1, \ldots, P_m$ a complete Sylow sequence of G. Let $1 \le j \le m$ and let N be a normal p_j-subgroup of G. Using the notation of Definition 2.14 we have, for any $x \in G$,*

$$m_{\mathcal{P}}(x) = m_{\overline{\mathcal{P}}}(\overline{x}).$$

PROOF. If $m_{\mathcal{P}}(x) = 0$ then $x \notin \Pi(\mathcal{P})$. Since $\Pi(\mathcal{P})N = \Pi(\mathcal{P})$ (see [8, proof of Theorem B parts (c) and (d)]) we get $\overline{x} \notin \Pi(\overline{\mathcal{P}})$ and therefore $m_{\overline{\mathcal{P}}}(\overline{x}) = 0$. Now suppose that $m_{\mathcal{P}}(x) > 0$. The preimage of $\overline{x}$ is the set xN. Thus, any factorization of $\overline{x}$ in $\overline{\mathcal{P}}$ is the image of a factorization in $\mathcal{P}$ of an element of xN (the same factorization of $\overline{x}$ in $\overline{\mathcal{P}}$ can be the image of more than one factorization of more than one element of xN). Let $x = x_1^{(s)} \cdots x_m^{(s)}$, $1 \le s \le m_{\mathcal{P}}(x)$ be all of the $m_{\mathcal{P}}(x)$ distinct factorizations of x in $\mathcal{P}$. By Lemma 2.13, every factorization of xn in $\mathcal{P}$, where $n \in N$, can be paired with one of the $m_{\mathcal{P}}(x)$ factorizations of x in $\mathcal{P}$ so that the pair members are identical in all but the jth factors, and the jth factors differ by an element of N. Thus, when we vary n over N, the image in $\overline{\mathcal{P}}$ of all factorizations of xn which pair with the same sth factorization $x = x_1^{(s)} \cdots x_m^{(s)}$ of x in $\mathcal{P}$ is the same. On the other hand, two distinct factorizations of x in $\mathcal{P}$ (two distinct s values) must differ by at least three factors (Lemma 2.6) and hence their images in $\overline{\mathcal{P}}$ are distinct. The claim follows. $\square$

PROOF OF THEOREM 1.9. It remains to prove (see Lemmas 2.11 and 2.12) that $R(G) \le M_\tau(G)$ for an arbitrary ordering τ of $\pi(G) = \{p_1, \ldots, p_m\}$. We do this by induction on $|G|$. We can assume that $R(G) > 1$. Hence there exists $1 \le j \le m$ such that $O_{p_j}(G) > 1$. Let $\overline{G} = G/O_{p_j}(G)$ and, for all $x \in G$, let $\overline{x}$ denote its image under the canonical homomorphism $G \to \overline{G}$. For any $A \le G$ let $\overline{A} = AO_{p_j}(G)/O_{p_j}(G)$. Let $x \in G$ and let $y \in R(G)$. Let $\mathcal{P} = P_1, \ldots, P_m$ be a complete Sylow sequence of G of type τ. We have to prove that $m_{\mathcal{P}}(x \cdot y^{-1}) = m_{\mathcal{P}}(x)$. By Lemma 2.15, $m_{\mathcal{P}}(x) = m_{\overline{\mathcal{P}}}(\overline{x})$ and $m_{\mathcal{P}}(x \cdot y^{-1}) = m_{\overline{\mathcal{P}}}(\overline{x \cdot y^{-1}}) = m_{\overline{\mathcal{P}}}(\overline{x} \cdot \overline{y}^{-1})$. Since $\overline{R(G)} = R(\overline{G})$, we have $\overline{y} \in R(\overline{G})$ and by induction assumption $m_{\overline{\mathcal{P}}}(\overline{x} \cdot \overline{y}^{-1}) = m_{\overline{\mathcal{P}}}(\overline{x})$. The claim follows. $\square$

PROOF OF THEOREM 1.10. Set $N = M_\tau(G)$. By Theorem 1.9 it is sufficient to prove that N is solvable. By assumption, and by Corollary 2.5, there exists a complete Sylow sequence $\mathcal{P} = P_1, \ldots, P_m$ of type τ such that $m_{\mathcal{P}}(1_G) = 1$. It follows that $m_{\mathcal{P}}(h) = 1$ for all $h \in N$. Now set $\overline{G} = G/N$ and $\overline{\mathcal{P}} = (P_1N)/N, \ldots, (P_mN)/N$. We claim $m_{\overline{\mathcal{P}}}(1_{\overline{G}}) = 1$. Suppose to the contrary that $\overline{g_1} \cdots \overline{g_m} = 1_{\overline{G}}$ where $\overline{g_i} \in (P_iN)/N$ and at least one of the $\overline{g_i}$ is nontrivial. Let g_i be preimages of the $\overline{g_i}$ in P_i. Then $g_1 \cdots g_m = h$ for some $h \in N$. Since the $\overline{g_i}$ are not all trivial, there exists some g_i which is not in N. Since $m_{\mathcal{P}}(h) = 1$, the factorization $g_1 \cdots g_m = h$ is the only factorization of h in $\mathcal{P}$ and hence h is not factorizable in the Sylow sequence $N \cap P_1, \ldots, N \cap P_m$ of N. It follows that there exists some $\widetilde{h} \in N$ which has at least two distinct factorizations in $N \cap P_1, \ldots, N \cap P_m$ in contradiction to $m_{\mathcal{P}}(\widetilde{h}) = 1$. Thus, $m_{\overline{\mathcal{P}}}(1_{\overline{G}}) = 1$ and we can apply Lemma 2.7 and get $H_\tau(G) \cap N = H_\tau(N)$. By Theorem 1.9, we have $N \le H_\tau(G)$ and thus $N = H_\tau(N)$ implying [8, Theorem A] that N is solvable. $\square$

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