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ALGEBRAIC EXPANSIONS OF LOGICS

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Abstract. An algebraically expandable (AE) class is a class of algebraic structures axiomatizable by sentences of the form $\forall\exists! \bigwedge p = q$. For a logic L algebraized by a quasivariety $\mathcal{Q}$ we show that the AE-subclasses of $\mathcal{Q}$ correspond to certain natural expansions of L, which we call *algebraic expansions*. These turn out to be a special case of the expansions by implicit connectives studied by X. Caicedo. We proceed to characterize all the AE-subclasses of abelian ℓ-groups and perfect MV-algebras, thus fully describing the algebraic expansions of their associated logics.

§1. Introduction. The idea of expanding structures in a given language with new operations and relations definable in *some way* is pervasive in Algebra and Model Theory. If we focus on operations defined by systems of equations on algebraic structures we arrive at the notion of *Algebraic Expansions* [10]. Restricting to this kind of definability has the advantage of producing well-behaved expansions that can be studied with 'universal-algebraic' techniques (e.g., sheaf representations). We describe these expansions in more detail.

Let τ be an algebraic language. Given a class of τ-algebras $\mathcal{K}$ and a system of equations of the form

$$\begin{gathered} s_1(x_1, \dots, x_n, z_1, \dots, z_m) = t_1(x_1, \dots, x_n, z_1, \dots, z_m) \\ \vdots \\ s_k(x_1, \dots, x_n, z_1, \dots, z_m) = t_k(x_1, \dots, x_n, z_1, \dots, z_m) \end{gathered}$$

we can consider the class $\mathcal{A}$ of those algebras in $\mathcal{K}$ for which, given values for the x's, there are unique values for the z's such that all equalities hold. We say that $\mathcal{A}$ is an *Algebraically Expandable (AE) subclass* of $\mathcal{K}$ given that the members of $\mathcal{A}$ can be expanded with the operations defined by the system of equations. For example, let $\mathcal{K}$ be the class of $\{\to, 1\}$-subreducts of Boolean algebras, and consider the system of equations

$$\begin{gathered} z \to x_1 = 1, \\ z \to x_2 = 1, \\ ((x_1 \to z) \to (x_2 \to z)) \to (x_2 \to z) = 1. \end{gathered}$$

The class $\mathcal{A}$ in this case is the class of algebras in $\mathcal{K}$ where every two elements have a meet with respect to the ordering induced by $\to$. Expanding $\mathcal{A}$ with the

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meet operation yields a class term-equivalent to the class of generalized Boolean algebras.

In the setting of Abstract Logic expansions by new connectives are a common theme as well, in particular, expansions of a logic L with connectives determined in some way by L. As we know, there is a long-standing and fruitful interplay between Logic and Algebra, so it is natural to consider what, if any, is the logical counterpart of AE-classes. As we shall see, for the case of an algebraizable logic L which has as its equivalent algebraic semantics a quasivariety $\mathcal{Q}$, the AE-subclasses of $\mathcal{Q}$ are in correspondence with the family of a specific kind of expansions of L, which we call *algebraic expansions*. The notion of an algebraic expansion of a logic turns out to be quite natural, we think, and interestingly it falls into the general framework of expansions by implicit connectives studied by Caicedo in [5]. An immediate consequence is that algebraic expansions are again algebraizable. The algebraic expansions of L are naturally pre-ordered by morphisms that preserve the language of L. It turns out that (modulo the equivalence relation induced by the pre-ordering) this is a lattice. Furthermore, this lattice is dually isomorphic to the lattice of AE-subclasses of $\mathcal{Q}$ under inclusion.

Besides introducing the notion of algebraic expansions of a logic we analyze two particular cases: ℓ-groups and perfect MV-algebras. In both cases we obtain full descriptions of the AE-classes, and thus, of the algebraic expansions of their corresponding logics. We show that in both cases there is a continuum of expansions, and the lattices are isomorphic to $\mathbf{2}^\omega \oplus \mathbf{1}$ and $\mathbf{2}^\omega \oplus \mathbf{2}$, in the former and latter cases respectively.

In the next section we summarize all the basic definitions and properties of the theory of AE-classes needed for this article. In Section 3 we give the formal definition of algebraic expansion of a logic, and prove the fundamental results linking them with AE-classes (Theorems 3.1 and 3.2). In Section 4 we characterize the AE-classes of abelian ℓ-groups and the algebraic expansions of their corresponding logic. Finally, in Section 5, we translate the results from Section 4 to their analogs for perfect MV-algebras. This completely describes the algebraic expansions of the associated logic.

§2. Preliminaries. In this section we introduce fundamental definitions, establish notation, and present several basic facts needed in the sequel. We assume familiarity with basic Universal Algebra, Model Theory, and Abstract Algebraic Logic (see, e.g., [4, 19, 15], respectively).

2.1. Notation and basic definitions. Throughout this article algebras are considered as models of first-order languages without relations. For example, abelian ℓ-groups are algebras in the language $\tau_{\mathcal{G}} := \{+, -, 0, \vee, \wedge\}$. As is customary we use bold letters ($\mathbf{A}, \mathbf{B}, \mathbf{C}, \dots$) for algebraic structures and italic letters ($A, B, C, \dots$) for the underlying sets. For algebras $\mathbf{A}$ and $\mathbf{B}$ we write $\mathbf{A} \subseteq \mathbf{B}$ whenever $\mathbf{A}$ is a subalgebra of $\mathbf{B}$.

Given a structure $\mathbf{A}$ in a language τ and a term $t(x_1, \dots, x_n)$ in the same language, we write $t^{\mathbf{A}}(\bar{a})$ for the value of the term upon assigning elements $a_1, \dots, a_n$ from A to the variables $x_1, \dots, x_n$. We may omit the superscript $\mathbf{A}$ if there is no risk of confusion.

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Given a (first-order) formula φ, we say that φ is:

- an *identity* if it has the form $\forall \bar{x}(p(\bar{x}) = q(\bar{x}))$, where p and q are terms,
- a *quasi-identity* if it has the form $\forall \bar{x}(\alpha(\bar{x}) \rightarrow \beta(\bar{x}))$,[1] where α is a finite conjunction of term-equalities and β is a term-equality,
- *universal* if it has the form $\forall \bar{x}\psi$, where ψ is quantifier-free,
- *existential* if it has the form $\exists \bar{x}\psi$, where ψ is quantifier-free.

A *sentence* is a formula with no free variables. If Σ is a set of sentences, $\mathrm{Mod}(\Sigma)$ denotes the class of all models that satisfy the sentences in Σ.

Whenever we consider a class $\mathcal{K}$ of algebras, we assume that all algebras in $\mathcal{K}$ have the same language. Given a class $\mathcal{K}$ of algebras, we define the usual class operators:

- $\mathsf{I}(\mathcal{K})$ denotes the class of isomorphic images of members of $\mathcal{K}$,
- $\mathsf{H}(\mathcal{K})$ denotes the class of homomorphic images of members of $\mathcal{K}$,
- $\mathsf{S}(\mathcal{K})$ denotes the class of subalgebras of members of $\mathcal{K}$,
- $\mathsf{P}(\mathcal{K})$ denotes the class of direct products with factors in $\mathcal{K}$,
- $\mathsf{P_U}(\mathcal{K})$ denotes the class of ultraproducts with factors in $\mathcal{K}$.

If O is one of the above operators and $\mathcal{K} = \{\mathbf{A}_1, \dots, \mathbf{A}_n\}$, we write $\mathsf{O}(\mathbf{A}_1, \dots, \mathbf{A}_n)$ instead of $\mathsf{O}(\mathcal{K})$.

Let $\mathcal{K}$ be a class of algebras of the same language. We say that $\mathcal{K}$ is a *variety* (or *equational class*) if it can be axiomatized using a set of identities; equivalently, by Birkhoff's theorem, $\mathcal{K}$ is a variety if and only if it is closed under H, S, and P. The smallest variety containing $\mathcal{K}$ is $\mathsf{HSP}(\mathcal{K})$ and is denoted by $\mathsf{V}(\mathcal{K})$. A *quasivariety* is a class of algebras that can be axiomatized by a set of quasi-identities. By Mal'cev's theorem, the class $\mathcal{K}$ is a quasivariety if and only if $\mathcal{K}$ is closed under I, S, P, and $\mathsf{P_U}$; the smallest quasivariety containing $\mathcal{K}$ is $\mathsf{ISPP_U}(\mathcal{K})$, also denoted by $\mathsf{Q}(\mathcal{K})$. Finally, recall that $\mathcal{K}$ is *universal* if it can be axiomatized by a set of universal sentences, which is equivalent to $\mathcal{K}$ being closed under I, S, and $\mathsf{P_U}$. Moreover, the smallest universal class containing $\mathcal{K}$ is given by $\mathsf{ISP_U}(\mathcal{K})$.

Given a class $\mathcal{K}$ and two sentences φ, ψ, we say that φ and ψ are *equivalent* in $\mathcal{K}$, and write $\varphi \sim \psi$ in $\mathcal{K}$, if for every $\mathbf{A} \in \mathcal{K}$ we have that $\mathbf{A} \vDash \varphi$ if and only if $\mathbf{A} \vDash \psi$.

2.2. Algebraically expandable classes. In order to define algebraically expandable classes [10], one of the fundamental notions in this article, we need to introduce the special type of sentences that axiomatize them. An *equational function definition sentence* (EFD-sentence for short) in the language τ is a sentence of the form

$$\forall x_1 \dots x_n \exists! z_1 \dots z_m \bigwedge_{i=1}^{k} s_i(\bar{x}, \bar{z}) = t_i(\bar{x}, \bar{z}), \tag{1}$$

where s_i, t_i are τ-terms, $n \geq 0$, and $m \geq 1$. Suppose φ is the EFD-sentence in (1). Observe that φ is valid in a structure $\mathbf{A}$ if and only if the system of equations $\bigwedge_{i=1}^{k} s_i(\bar{x}, \bar{z}) = t_i(\bar{x}, \bar{z})$ defines a (total) function $F\colon A^n \to A^m$. If $\pi_j\colon A^m \to A$ is the j-th projection function, we write $[\varphi]_j^{\mathbf{A}} := \pi_j \circ F$ for $j \in \{1, \dots, m\}$.

[1]We write the first-order connectives $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$ in bold font to distinguish them from algebraic operations and connectives in sentential logics.

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Let φ be as in (1). We define:

- $E(\varphi) := \forall \bar{x} \exists \bar{z} \bigwedge_{i=1}^{k} s_i(\bar{x}, \bar{z}) = t_i(\bar{x}, \bar{z})$,
- $U(\varphi) := \forall \bar{x} \bar{y} \bar{z} \bigwedge_{i=1}^{k} s_i(\bar{x}, \bar{y}) = t_i(\bar{x}, \bar{y}) \wedge \bigwedge_{i=1}^{k} s_i(\bar{x}, \bar{z}) = t_i(\bar{x}, \bar{z}) \rightarrow \bar{y} = \bar{z}$.

The following basic facts are used without explicit reference throughout the article.

- φ is equivalent to $E(\varphi) \wedge U(\varphi)$.
- $U(\varphi)$ is (equivalent to) a conjunction of quasi-identities.
- $E(\varphi)$ is preserved by homomorphic images, that is, for any surjective homomorphism $f : \mathbf{A} \to \mathbf{B}$, if $\mathbf{A} \vDash E(\varphi)$, then $\mathbf{B} \vDash E(\varphi)$.

A class of algebras $\mathcal{K}$ is an *algebraically expandable class* (AE-class for short) if there is a set of EFD-sentences Σ such that $\mathcal{K} = \mathrm{Mod}(\Sigma)$. Let $\mathcal{K}$ and $\mathcal{C}$ be classes of algebras, $\mathcal{K} \subseteq \mathcal{C}$. We say that $\mathcal{K}$ is an *AE-subclass* of $\mathcal{C}$ if $\mathcal{K}$ is axiomatizable by EFD-sentences relative to $\mathcal{C}$, that is, $\mathcal{K} = \mathcal{C} \cap \mathrm{Mod}(\Sigma)$ for some set Σ of EFD-sentences. The reader should be aware that $\mathcal{K}$ may be an AE-subclass of $\mathcal{C}$, but fail to be an AE-class itself.

Let $\mathcal{Q}$ be a quasivariety in the language τ and let Σ be a set of EFD-sentences. There is an obvious expansion of the AE-subclass $\mathcal{K} := \mathcal{Q} \cap \mathrm{Mod}(\Sigma)$ of $\mathcal{Q}$ obtained by skolemizing the existential quantifiers in Σ. More precisely, for each $\varphi \in \Sigma$ of the form $\forall x_1 \dots x_n \exists! z_1 \dots z_m \bigwedge_{i=1}^{k} s_i(\bar{x}, \bar{z}) = t_i(\bar{x}, \bar{z})$ consider new n-ary function symbols $f_1^{\varphi}, \dots, f_m^{\varphi}$ and the set of identities

$$E_{\varphi} := \{\forall \bar{x}\, s_i(\bar{x}, f_1^{\varphi}(\bar{x}), \dots, f_m^{\varphi}(\bar{x})) = t_i(\bar{x}, f_1^{\varphi}(\bar{x}), \dots, f_m^{\varphi}(\bar{x})) : 1 \leq i \leq k\}.$$

Let τ_{Σ} be the expansion of τ obtained by adding the f_j^{φ}'s for each $\varphi \in \Sigma$, and put

$$E_{\Sigma} := \bigcup_{\varphi \in \Sigma} E_{\varphi}, \qquad U_{\Sigma} := \{U(\varphi) : \varphi \in \Sigma\}.$$

Define

$$\mathcal{Q}^{\Sigma} := \mathrm{Mod}(\Gamma \cup E_{\Sigma} \cup U_{\Sigma}),$$

where Γ is a set of quasi-identities axiomatizing $\mathcal{Q}$. We call $\mathcal{Q}^{\Sigma}$ an *algebraic expansion of* $\mathcal{Q}$. Note that $\mathcal{Q}^{\Sigma}$ is a quasivariety over the language τ_{Σ} whose members are precisely the expansions of the members of $\mathcal{K}$. A fact worth mentioning is that if Σ and Σ' are two sets of EFD-sentences with the same models in $\mathcal{Q}$, then the quasivarieties $\mathcal{Q}^{\Sigma}$ and $\mathcal{Q}^{\Sigma'}$ are term-equivalent (see [11, Theorem 5]).

We conclude this section with a preservation result for EFD-sentences needed in the sequel. Recall that a structure $\mathbf{A}$ is *finitely subdirectly irreducible* if its diagonal congruence is meet-irreducible in the congruence lattice of $\mathbf{A}$. We write $\mathcal{K}_{\mathrm{fsi}}$ for the class of finitely subdirectly irreducible members of $\mathcal{K}$. A variety is *arithmetical* provided that it is both congruence distributive and congruence permutable.

Lemma 2.1. *Let $\mathcal{V}$ be an arithmetical variety such that $\mathcal{V}_{\mathrm{fsi}} \cup \{$trivial algebras$\}$ is a universal class, and let $\mathbf{A} \in \mathcal{V}$. If φ is an EFD-sentence such that $\mathsf{H}(\mathbf{A})_{\mathrm{fsi}} \vDash \varphi$, then $\mathbf{A} \vDash \varphi$.*

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PROOF. By [18, Theorem 5.1] $\mathbf{A}$ has a global representation with factors in $\mathsf{H}(\mathbf{A})_{\mathrm{fsi}}$, and by [25] global representations preserve EFD-sentences. ⊣

§3. The algebraic expansions of a logic. Following [15, Definition 1.5] we take a (sentential) *logic* to be a pair $L := \langle \tau, \vdash_L \rangle$ where τ is an algebraic language and $\vdash_L$ is a substitution-invariant consequence operator on the set of τ-formulas. We refer the reader to [15] for definitions and results about abstract algebraic logic not explicitly mentioned in this article. Important disclaimer: *all logics considered in the sequel are assumed to be finitary.*

Let $L := \langle \tau_L, \vdash_L \rangle$ and $L' := \langle \tau_{L'}, \vdash_{L'} \rangle$ be logics. Recall that L' is an *expansion* of L if $\tau \subseteq \tau'$ and $\vdash_L \subseteq \vdash_{L'}$.

Suppose L is algebraizable and let $\Delta(x, y)$ be a set of equivalence formulas for L. Given a finite set of τ-formulas $\Phi(\bar{x}, \bar{z})$ in variables $x_1, \dots, x_n, z_1, \dots, z_m$, $n, m \in \omega$, let τ_Φ be the language obtained by expanding τ with new n-ary function symbols $f_1^\Phi, \dots, f_m^\Phi$. Next, define $L^\Phi := \langle \tau_\Phi, \vdash_{L^\Phi} \rangle$, where $\vdash_{L^\Phi}$ is the least substitution-invariant consequence operator containing $\vdash_L$ such that

$$\vdash_{L^\Phi} \Phi(\bar{x}, f_1^\Phi(\bar{x}), \dots, f_m^\Phi(\bar{x})), \tag{E_Φ}$$

$$\Phi(\bar{x}, \bar{y}), \Phi(\bar{x}, \bar{z}) \vdash_{L^\Phi} \Delta(\bar{y}, \bar{z}). \tag{U_Φ}$$

($\Delta(\bar{y}, \bar{z})$ is shorthand for $\bigcup_{j=1}^{m} \Delta(y_j, z_j)$.) We say that L^Φ is the *algebraic expansion* of L by Φ. Recall that if $\Delta'(x, y)$ is another set of equivalence formulas for L, then $\Delta(x, y) \dashv\vdash_L \Delta'(x, y)$. Thus, the expansion L^Φ does not depend on the choice of the set of equivalence formulas.

Given a set Σ of finite sets of τ-formulas, define $L^\Sigma := \langle \tau_\Sigma, \vdash_{L^\Sigma} \rangle$, where $\tau_\Sigma := \bigcup\{\tau_\Phi : \Phi \in \Sigma\}$ and $\vdash_{L^\Sigma}$ is the least substitution-invariant consequence operator containing $\vdash_{L^\Phi}$ for every $\Phi \in \Sigma$. (Of course, we assume that the new symbols for each L^Φ are different.) The logic L^Σ is called the *algebraic expansion* of L by Σ.

Observe that, in the definition of L^Φ, for the case $m = 0$ condition U_Φ holds vacuously, so L^Φ is just the axiomatic extension of L by Φ. Hence, axiomatic extensions of L are algebraic expansions of L.

As mentioned in the introduction, in [5] Caicedo studies expansions of finitely algebraizable logics where the behaviour of the new connectives is determined by the added axioms and rules. More precisely, let $L := \langle \tau, \vdash \rangle$ be an algebraizable logic with equivalence formulas $\Delta(x, y)$, and let F be a set, disjoint from τ, of function symbols. Following [5], an expansion $L(F)$ is said to be an *expansion of L by implicit connectives* provided that

$$\vdash_{L(F) \cup L(F')} \Delta(f(\bar{x}), f'(\bar{x})) \text{ for } f \in F,$$

where F' is a copy of F disjoint from τ and F, the logic $L(F')$ is the copy of $L(F)$ in the language $\tau \cup F'$, and $L(F) \cup L(F')$ is the logic in the language $\tau \cup F \cup F'$ whose consequence operator is the least substitution invariant consequence operator containing $\vdash_{L(F)}$ and $\vdash_{L(F')}$.

It is easy to see that the expansion L^Σ defined above is in fact an expansion of L by implicit connectives (where (E_Φ) and (U_Φ) correspond to new axioms and rules, respectively). As an immediate consequence of this fact we have that L^Σ is

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algebraizable with the same equivalence formulas and defining equations as L [5, Theorem 1]. Furthermore, the equivalent algebraic semantics of L^Σ is the expected one [5, Corollary 2], which in this case turns out to be an algebraic expansion of the equivalent algebraic semantics of L. The details are worked out next.

Let $\mathcal{Q}$ be the equivalent algebraic semantics of L via the set of equivalence formulas $\Delta(x, y)$ and the set of defining equations $\varepsilon(x)$. Given a finite set $\Phi(\bar{x}, \bar{z})$ of τ-formulas, let $e(\Phi)$ be the EFD-sentence $\forall \bar{x} \exists! \bar{z} \bigwedge \varepsilon(\Phi(\bar{x}, \bar{z}))$. For Σ a set of finite sets of τ-formulas define $e(\Sigma) := \{e(\Phi) : \Phi \in \Sigma\}$. Now, Corollary 2 of [5] says that the algebraic expansion $\mathcal{Q}^{e(\Sigma)}$ is the equivalent algebraic semantics of L^Σ. Thus, for each algebraic expansion of L we have a corresponding algebraic expansion of $\mathcal{Q}$. Of course, we can also go in the other direction. Given an EFD-sentence $\varphi := \forall \bar{x} \exists! \bar{z}\, \alpha(\bar{x}, \bar{z})$, put $d(\varphi) := \Delta(\alpha(\bar{x}, \bar{z}))$. Here and in the sequel $\Delta(\alpha(\bar{x}, \bar{z}))$ abbreviates $\bigcup_{i=1}^{k} \Delta(s_i(\bar{x}, \bar{z}), t_i(\bar{x}, \bar{z}))$ if $\alpha(\bar{x}, \bar{z})$ is the conjunction of equations $\bigwedge\limits_{i=1}^{k} s_i(\bar{x}, \bar{z}) = t_i(\bar{x}, \bar{z})$. For a set Σ of EFD-sentences, we write $d(\Sigma)$ for the set $\{d(\varphi) : \varphi \in \Sigma\}$. Again, it is straightforward to check that $\mathcal{Q}^\Sigma$ is the equivalent algebraic semantics of $L^{d(\Sigma)}$. Furthermore,

- $L^\Sigma = L^{d(e(\Sigma))}$,
- $\mathcal{Q}^\Sigma = \mathcal{Q}^{e(d(\Sigma))}$

for suitable Σ's. This establishes a direct correspondence between algebraic expansions of a logic and those of its equivalent algebraic semantics. Theorem 3.2 explores this connection in greater detail. In the sequel, to avoid cumbersome notation, given a logic L and a set Σ of EFD-sentences we write L^Σ instead of $L^{d(\Sigma)}$.

For future reference, the facts above are summarized in the following:

THEOREM 3.1. *Let L be a finitely algebraizable logic with equivalent algebraic semantics $\mathcal{Q}$. Let Σ be a set of EFD-sentences in the language of $\mathcal{Q}$. Then the algebraic expansion L^Σ is algebraizable with the same equivalence formulas and defining equations as L, and its equivalent algebraic semantics is the quasivariety $\mathcal{Q}^\Sigma$. Moreover, there is a one-to-one correspondence between the algebraic expansions of L and the algebraic expansions of $\mathcal{Q}$.*

We conclude this section with an example of a logic that has an expansion by implicit connectives that is not algebraic. Let L_{int} be the Intuitionistic Logic and let L^S_{int} be the extension of L_{int} by the implicit connective S defined in [7, Example 5.2]. The equivalent algebraic semantics of L^S_{int} is the variety $\mathcal{H}^S$ of Heyting algebras with successor. It is not hard to show that the class of Heyting-reducts of algebras in $\mathcal{H}^S$ is not an AE-subclass of $\mathcal{H}$. Thus, by Theorem 3.1, L^S_{int} cannot be an algebraic expansion of L_{int}.

3.1. The lattice of algebraic expansions. Let L be a finitely algebraizable logic with equivalent algebraic semantics $\mathcal{Q}$. The AE-subclasses of the quasivariety $\mathcal{Q}$ are naturally (lattice-)ordered by inclusion. In the current section we show how this ordering translates to the algebraic expansions of $\mathcal{Q}$, and thus to the algebraic expansions of L. For this we need to look into interpretations between logics and between classes of algebras.

Fix a countably infinite set of variables $X := \{x_1, x_2, ...\}$; given a language τ we write $Tm(\tau)$ for the set of τ-terms over the variables in X. Let τ_1 and τ_2 be two

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expansions of a language τ. A τ-*translation* from τ_1 into τ_2 is a function $T : \tau_1 \to Tm(\tau_2)$ such that T maps each symbol of arity n to a term in the variables $x_1, \dots, x_n$, and $T(f) = f(x_1, \dots, x_n)$ for every n-ary symbol $f \in \tau$.

Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be two classes of algebras over τ_1 and τ_2, respectively. A τ-*interpretation* of $\mathcal{K}_1$ in $\mathcal{K}_2$ is a τ-translation $T : \tau_1 \to Tm(\tau_2)$ such that for every member $\mathbf{A} := (A, \{g^{\mathbf{A}} : g \in \tau_2\})$ in $\mathcal{K}_2$, the algebra $\mathbf{A}^T := (A, \{T(f)^{\mathbf{A}} : f \in \tau_1\})$ belongs to $\mathcal{K}_1$. If T and S are τ-interpretations of $\mathcal{K}_1$ in $\mathcal{K}_2$ and $\mathcal{K}_2$ in $\mathcal{K}_1$, respectively, such that the maps $\mathbf{A} \mapsto \mathbf{A}^T$ and $\mathbf{A} \mapsto \mathbf{A}^S$ are mutually inverse, we say that $\mathcal{K}_1$ and $\mathcal{K}_2$ are τ-*term-equivalent*.

We turn now to maps between logics. A τ-translation T from τ_1 into τ_2 extends in a natural way to a mapping from $Tm(\tau_1)$ to $Tm(\tau_2)$:

- $T(x) = x$ for every variable $x \in X$;
- $T(f(\varphi_1, \dots, \varphi_n)) = T(f)(T(\varphi_1), \dots, T(\varphi_n))$ for f in τ_1 of arity n and $\varphi_1, \dots \varphi_n$ in $Tm(\tau_1)$.

Given a set Γ of τ_1-terms we write $T(\Gamma)$ for $\{T(\varphi) : \varphi \in \Gamma\}$.

Let τ_1 and τ_2 be expansions of a language τ, and suppose L_1 and L_2 are logics in τ_1 and τ_2, respectively. A τ-*morphism* from L_1 to L_2 is a τ-translation from τ_1 into τ_2 such that

$$\Gamma \vdash_{L_1} \varphi \text{ implies } T(\Gamma) \vdash_{L_2} T(\varphi)$$

for $\Gamma \cup \{\varphi\} \subseteq Tm(\tau_1)$. We say that L_1 and L_2 are τ-bimorphic, and write $L_1 \leftrightarrows_\tau L_2$, provided there exist a τ-morphism from L_1 to L_2 and a τ-morphism from L_2 to L_1.

The following result shows the connection between the above defined relations.

Theorem 3.2. *Let L be a finitely algebraizable logic in the language τ with equivalent algebraic semantics $\mathcal{Q}$. Let Σ and Σ' be two sets of EFD-sentences in τ.*

1. *The following are equivalent*:
   (i) *There is a τ-morphism from $L^{\Sigma'}$ to L^{Σ}.*
   (ii) *There is a τ-interpretation of $\mathcal{Q}^{\Sigma'}$ in $\mathcal{Q}^{\Sigma}$.*
   (iii) $\mathcal{Q} \cap \mathrm{Mod}(\Sigma) \subseteq \mathcal{Q} \cap \mathrm{Mod}(\Sigma')$.
2. *The following are equivalent*:
   (i) *$L^{\Sigma'}$ and L^{Σ} are τ-bimorphic.*
   (ii) *$\mathcal{Q}^{\Sigma'}$ and $\mathcal{Q}^{\Sigma}$ are τ-term-equivalent.*
   (iii) $\mathcal{Q} \cap \mathrm{Mod}(\Sigma) = \mathcal{Q} \cap \mathrm{Mod}(\Sigma')$.

Proof. The proofs of all equivalences are routine with the exception of the implication $(iii) \Rightarrow (ii)$, which follows from the proof of [11, Theorem 5]. ⊣

A word of caution: it is not true in general that two bimorphic algebraizable logics have term-equivalent algebraic semantics. For example, let $C := \langle \tau, \vdash \rangle$ be the Classical Propositional Logic and let $C_e = \langle \tau \cup \{e\}, \vdash_e \rangle$ where e is a new constant symbol and $\vdash_e$ is the least substitution invariant consequence operator containing $\vdash$. Clearly C_e is algebraizable with equivalent algebraic semantics $\mathcal{B}_e$, the class of pointed Boolean algebras. It is straightforward to check that $C_e \leftrightarrows_\tau C$; however, $\mathcal{B}_e$ is obviously not τ-term-equivalent to the class of Boolean algebras.

Let L be a logic algebraized by a quasivariety $\mathcal{Q}$. As is the case for any quasivariety, the AE subclasses of $\mathcal{Q}$ form a lattice Λ under inclusion. In the light of Theorem 3.2,

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the algebraic expansions of L modulo $\leftrightarrows$, ordered by morphisms, form a lattice as well, dually isomorphic to Λ. Thus, classifying the algebraically expandable classes of $\mathcal{Q}$ yields a classification of all algebraic expansions of L up to $\leftrightarrows$.

3.2. Some examples. When the AE-subclasses of a quasivariety are known, Theorem 3.2 immediately gives a description of the algebraic expansions of the corresponding logic. We present here three examples.

3.2.1. The primal case. An algebra **A** is called *primal* if it is finite and every function $f : A^n \to A$ for $n \geq 1$ is a term-operation of **A**. It is proved in [10, Theorem 13] that the only AE-subclasses of $\mathsf{V}(\mathbf{A})$ for a primal **A** are $\mathsf{V}(\mathbf{A})$ and the class of trivial algebras. Thus, the only (modulo $\leftrightarrows$) algebraic expansions of a logic L algebraized by such a variety are L itself and the inconsistent logic. This applies, e.g., to Classical Propositional Logic and m-valued Post's logic.

3.2.2. Gödel logic. Recall that Gödel Logic L_G is the extension of Intuitionistic Logic by the prelinearity axiom $(x \to y) \vee (y \to x)$. It is known that the equivalent algebraic semantics of L_G is the variety $\mathcal{H}_G$ of Gödel algebras, also known as prelinear Heyting algebras. The only AE-subclasses of $\mathcal{H}_G$ are its subvarieties [8]. Thus, the algebraic expansions of L_G agree with its axiomatic extensions.

3.2.3. The implicative fragment of classical logic. Let $L_\to$ be the implicative fragment of classical propositional logic. The equivalent algebraic semantics of $L_\to$ is the variety $\mathcal{I}$ of implication algebras. Recall that disjunction is expressible in terms of $\to$, and thus for $n \geq 2$ and $1 \leq i \leq n$

$$s_i^n(x_1, \dots, x_n) := \bigvee_{j=1, j \neq i}^{n} x_j$$

is an $\{\to\}$-term. For each $n \geq 2$ let

$$\Phi_n := \{z \to s_i^n(\bar{x}) : i \in \{1, \dots, n\}\} \cup \left\{ \bigvee_{i=1}^{n} (s_i^n(\bar{x}) \to z) \right\}.$$

By definition, $L_\to^{\Phi_n}$ is the least expansion of $L_\to$ that satisfies (E_{Φ_n}) and (U_{Φ_n}). However, condition (U_{Φ_n}) is already true for $L_\to$. Thus $L_\to^{\Phi_n}$ is the expansion of $L_\to$ by the following axioms:

$\mu_n(\bar{x}) \to s_i^n(\bar{x})$ for $i \in \{1, \dots, n\}$,

$\bigvee_{i=1}^n (s_i^n(\bar{x}) \to \mu_n(\bar{x}))$,

where μ_n is a new n-ary symbol.

By the characterization of the AE-subclasses of $\mathcal{I}$ given in [9, Theorem 13] it follows from Theorem 3.2 that, up to $\leftrightarrows$, the consistent algebraic expansions of $L_\to$ are

$$L_\to < \cdots < L_\to^{\Phi_3} < L_\to^{\Phi_2},$$

where $L < L'$ means that there is an $\{\to\}$-morphism from L to L' but there is no $\{\to\}$-morphism from L' to L'. Observe that μ_2 is the classical conjunction and, more generally, we have that $\mu_n(\bar{x}) = \bigwedge_{i=1}^n s_i^n(\bar{x})$.

Example 3 of [5] shows classical negation is implicitly definable in $L_{\rightarrow}$. Since none of the algebraic expansions of $L_{\rightarrow}$ has classical negation as a term, we have another example of an expansion by implicit connectives that is not algebraic.

§4. Algebraic expansions of abelian ℓ-groups and the Logic of Equilibrium. In this section we give a complete description of the AE-classes of abelian ℓ-groups. In particular, we show that they form a lattice isomorphic to $\mathbf{1} \oplus \mathbf{2}^{\omega}$ (here, and in the sequel, $\mathbf{A} \oplus \mathbf{B}$ denotes the ordinal sum of $\mathbf{A}$ and $\mathbf{B}$ where $\mathbf{B}$ goes on top of $\mathbf{A}$). In view of Theorem 3.2 this produces a complete characterization of the algebraic expansions of the Logic of Equilibrium [16, 21].

Recall that an *abelian* ℓ-group is a structure in the language $\tau_{\mathcal{G}} := \{+, -, 0, \vee, \wedge\}$ such that:

- $(A, +, -, 0)$ is an abelian group,
- $(A, \vee, \wedge)$ is a lattice,
- $a + (b \vee c) = (a + b) \vee (a + c)$ for every $a, b, c \in A$.

Clearly abelian ℓ-groups form a variety, which we denote by $\mathcal{G}$. We write $\mathcal{G}_{\text{to}}$ to denote its subclass of totally ordered members. Since all ℓ-groups in this article are abelian, we sometimes omit the word abelian. In the following lemma we collect some well-known properties that are needed in the sequel (see, e.g., [17, 26]).

Lemma 4.1.

1. *The variety $\mathcal{G}$ is arithmetical, that is, every member of $\mathcal{G}$ has permutable and distributive congruences.*
2. *For every nontrivial $\mathbf{A} \in \mathcal{G}_{\text{to}}$ we have $\mathsf{ISP_U}(\mathbf{A}) = \mathcal{G}_{\text{to}}$.*
3. *An abelian ℓ-group is finitely subdirectly irreducible if and only if it is nontrivial and totally ordered.*
4. *For every nontrivial $\mathbf{A} \in \mathcal{G}$ we have $\mathsf{Q}(\mathbf{A}) = \mathsf{V}(\mathbf{A}) = \mathcal{G}$.*

4.1. AE-classes of abelian ℓ-groups. We proceed to characterize EFD-sentences modulo equivalence in $\mathcal{G}$. We first reduce the problem to totally ordered abelian ℓ-groups.

Lemma 4.2. *Given EFD-sentences φ, ψ, if $\varphi \sim \psi$ in $\mathcal{G}_{\text{to}}$, then $\varphi \sim \psi$ in $\mathcal{G}$.*

Proof. Suppose $\varphi \sim \psi$ in $\mathcal{G}_{\text{to}}$; take a nontrivial $\mathbf{A}$ in $\mathcal{G}$, and assume $\mathbf{A} \vDash \varphi$. On the one hand, since $U(\varphi)$ is a quasi-identity, Lemma 4.1(4) implies that $\mathsf{H}(\mathbf{A}) \vDash U(\varphi)$. On the other hand, $\mathsf{H}(\mathbf{A}) \vDash E(\varphi)$ because $E(\varphi)$ is preserved by homomorphic images. Hence $\mathsf{H}(\mathbf{A}) \vDash \varphi$ and, in particular, $\mathsf{H}(\mathbf{A})_{\text{fsi}} \vDash \varphi$. As, by Lemma 4.1(3), every member in $\mathsf{H}(\mathbf{A})_{\text{fsi}}$ is totally ordered, we have $\mathsf{H}(\mathbf{A})_{\text{fsi}} \vDash \psi$. So, using Lemma 2.1, we are done. ⊣

For each positive integer k define

$$\delta_k := \forall x \exists! z \; kz = x.$$

Our next step is to show that every EFD-sentence is equivalent to a δ_k in $\mathcal{G}$, which is accomplished in Theorem 4.12.

Recall that an ℓ-group $\mathbf{G}$ is *divisible* if for every $g \in G$ and every positive integer n, there exists $h \in G$ such that $g = nh$. Given a divisible ℓ-group $\mathbf{D}$, since ℓ-groups

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are torsion-free, we have that δ_k holds in $\mathbf{D}$ for all k; thus, we can define the expansion

$$\overline{\mathbf{D}} := (\mathbf{D}, ([\delta_k]^{\mathbf{D}})_{k\geq 1}). \tag{2}$$

The next result shows that the only functions defined by EFD-sentences in these expansions are term-operations.

THEOREM 4.3. *Let* $\mathbf{D}$ *be a totally ordered divisible* ℓ*-group and let* φ *be an EFD-sentence that holds in* $\mathbf{D}$*. Then, the functions* $[\varphi]_1^{\mathbf{D}}, \dots, [\varphi]_m^{\mathbf{D}}$ *defined by* φ *on* $\mathbf{D}$ *are term-functions on* $\overline{\mathbf{D}}$.

The above theorem can be derived from [6, Theorem 20]. We provide a different proof that relies on the characterization of existentially closed algebras in $\mathcal{G}_{\text{to}}$.

Given a class $\mathcal{K}$ of algebras closed under isomorphisms and $\mathbf{A} \in \mathcal{K}$, we say that $\mathbf{A}$ is *existentially closed* in $\mathcal{K}$ if for every $\mathbf{B} \in \mathcal{K}$ such that $\mathbf{A} \subseteq \mathbf{B}$, every existential formula $\varphi(\bar{x})$, and every $\bar{a} \in A^n$

$$\mathbf{B} \vDash \varphi(\bar{a}) \text{ implies } \mathbf{A} \vDash \varphi(\bar{a}).$$

The next proposition characterizes the existentially closed members of the class of totally ordered ℓ-groups.

PROPOSITION 4.4. *Given a totally ordered* ℓ*-group* $\mathbf{G}$*, we have that* $\mathbf{G}$ *is existentially closed in* $\mathcal{G}_{\text{to}}$ *if and only if* $\mathbf{G}$ *is divisible.*

PROOF. The result follows from [24, Theorem 3.1.13] when considering totally ordered ℓ-groups as structures of the language $\tau := \{S, \leq\}$, where S is a ternary relation symbol interpreted as the graph of the addition operation and $\leq$ is a binary relation symbol interpreted as the ordering relation. Now, since the operations $+, -, \vee, \wedge$ are definable by quantifier-free τ-formulas, the statement follows. ⊣

COROLLARY 4.5. *Let* $\mathbf{D} \subseteq \mathbf{G}$ *be totally ordered* ℓ*-groups and assume* $\mathbf{D}$ *is divisible. Then, for every EFD-sentence* φ *we have that* $\mathbf{G} \vDash \varphi$ *implies* $\mathbf{D} \vDash \varphi$.

PROOF. Suppose $\mathbf{G}$ satisfies the EFD-sentence φ. Since $U(\varphi)$ is universal, we have $\mathbf{D} \vDash \varphi$, and the fact that $\mathbf{D}$ is existentially closed implies $\mathbf{D} \vDash E(\varphi)$. ⊣

COROLLARY 4.6. *If* φ *is an EFD-sentence with a nontrivial model in* $\mathcal{G}$*, then every totally ordered divisible* ℓ*-group satisfies* φ.

PROOF. Assume $\mathbf{H}$ is a nontrivial model of φ and let $\mathbf{H}'$ be a nontrivial totally ordered homomorphic image of $\mathbf{H}$. Clearly $\mathbf{H}' \vDash E(\varphi)$ and, since $\mathsf{Q}(\mathbf{H})$ is the class of all ℓ-groups, we have $\mathbf{H}' \vDash U(\varphi)$. Hence $\mathbf{H}' \vDash \varphi$. By Lemma 4.1, we know that $\mathsf{ISP_U}(\mathbf{H}') = \mathcal{G}_{\text{to}}$. Thus, if $\mathbf{D}$ is a totally ordered divisible ℓ-group, there is $\mathbf{G} \in \mathsf{P_U}(\mathbf{H}')$ such that $\mathbf{D} \subseteq \mathbf{G}$. Finally, Corollary 4.5 yields $\mathbf{D} \vDash \varphi$. ⊣

After this sequence of results we are ready to present:

PROOF OF THEOREM 4.3. Assume $\mathbf{D} \vDash \varphi$ for some EFD-sentence φ, $\mathbf{D}$ nontrivial. Let $\overline{\mathbf{D}}$ be as in (2). Observe that $\mathsf{V}(\overline{\mathbf{D}})$ is arithmetical since arithmeticity is witnessed by a Pixley term (see [4, Theorem 12.5]).

We prove first that $\mathsf{V}(\overline{\mathbf{D}})_{\text{fsi}} \vDash \varphi$. Since all divisions are basic operations of $\overline{\mathbf{D}}$, we have that the algebras in $\mathsf{SP_U}(\overline{\mathbf{D}})$ are totally ordered divisible ℓ-groups, and Corollary

4.6 produces $\mathsf{SP_U}(\overline{\mathbf{D}}) \vDash \varphi$. Clearly $\mathsf{HSP_U}(\overline{\mathbf{D}}) \vDash E(\varphi)$ and, since $\mathcal{G} = \mathsf{Q}(\mathbf{D}) \vDash U(\varphi)$, it follows that $\mathsf{HSP_U}(\overline{\mathbf{D}})$ satisfies $U(\varphi)$ as well. Thus, $\mathsf{HSP_U}(\overline{\mathbf{D}}) \vDash \varphi$, and we are done since $\mathsf{V}(\overline{\mathbf{D}})_{\text{fsi}} \subseteq \mathsf{HSP_U}(\overline{\mathbf{D}})$ by Jónsson's lemma (see [20]).

Since ℓ-group congruences are compatible with division operations, we have that the congruences of algebras in $\mathsf{V}(\overline{\mathbf{D}})$ agree with the congruences of their ℓ-group reducts. Now, an algebra $\mathbf{A} \in \mathsf{V}(\overline{\mathbf{D}})$ is finitely subdirectly irreducible if and only if its diagonal congruence is meet irreducible in its congruence lattice. Thus, $\mathbf{A}$ is finitely subdirectly irreducible if and only if $\mathbf{A}|_{\tau_\mathcal{G}}$ is finitely subdirectly irreducible, which in turn is equivalent to $\mathbf{A}|_{\tau_\mathcal{G}}$ being totally ordered. Therefore, the class $\mathsf{V}(\overline{\mathbf{D}})_{\text{fsi}}$ is universal, and by Lemma 2.1 we have $\mathsf{V}(\overline{\mathbf{D}}) \vDash \varphi$. The proof concludes applying [11, Lemma 3], which states that if an EFD-sentence φ holds in a variety $\mathcal{V}$, then there are terms that agree with the functions defined by φ on each member of $\mathcal{V}$. ⊣

Given a positive integer k and a term $t(\bar{x})$ in $\tau_\mathcal{G}$, let

$$\delta_{k,t} := \forall \bar{x} \exists! z\, kz = t(\bar{x}).$$

Observe that $U(\delta_{k,t})$ is valid in $\mathcal{G}$ because abelian ℓ-groups are torsion-free.

We denote by $\mathcal{D}$ the class of expansions $\overline{\mathbf{D}}$ of divisible ℓ-groups $\mathbf{D} \in \mathcal{G}$. We write $\tau_\mathcal{D}$ for the language of the algebras in the class $\mathcal{D}$.

LEMMA 4.7. *Given a term $s(\bar{x})$ in $\tau_\mathcal{D}$, there is a term $t(\bar{x})$ in $\tau_\mathcal{G}$ and a positive integer k such that $k\, s(\bar{x}) = t(\bar{x})$ is valid in $\mathcal{D}$. Hence, for any divisible $\mathbf{D} \in \mathcal{G}$ the term-function $s^{\overline{\mathbf{D}}}$ agrees with the function $[\delta_{k,t}]^{\mathbf{D}}$.*

PROOF. It follows by induction on the structure of $s(\bar{x})$. ⊣

LEMMA 4.8. *Let φ be an EFD-sentence with a nontrivial model in $\mathcal{G}$. Then there are positive integers $k_1, \dots, k_m$ and terms $t_1, \dots, t_m$ in $\tau_\mathcal{G}$ such that $\varphi \sim \bigwedge_{j=1}^{m} \delta_{k_j,t_j}$ in $\mathcal{G}$.*

PROOF. Fix $\varphi := \forall x_1 \dots x_n \exists! z_1 \dots z_m\, \alpha(\bar{x}, \bar{z})$. Note that $\mathcal{G} \vDash U(\varphi)$ since φ has a nontrivial model in $\mathcal{G}$ and $\mathcal{G}$ has no proper subquasivarieties. Let $\mathbf{D}$ be a nontrivial totally ordered divisible ℓ-group. By Corollary 4.6, we have that $\mathbf{D} \vDash \varphi$. So Theorem 4.3 provides terms $s_1, \dots, s_m$ in $\tau_\mathcal{D}$ such that $[\varphi]_j^{\mathbf{D}} = s_j^{\overline{\mathbf{D}}}$ for $j \in \{1, \dots, m\}$. Moreover, by Lemma 4.7, there are positive integers $k_1, \dots, k_m$ and terms $t_1, \dots, t_m$ in $\tau_\mathcal{G}$ such that $s_j^{\overline{\mathbf{D}}} = [\delta_{k_j,t_j}]^{\mathbf{D}}$. This shows that $\mathbf{D} \vDash \forall \bar{x} \bar{z}\, (\alpha(\bar{x}, \bar{z}) \leftrightarrow \bigwedge_{j=1}^{m} k_j z_j = t_j(\bar{x}))$, and again using that $\mathcal{G}$ has no proper subquasivarieties, we have $\mathcal{G} \vDash \forall \bar{x} \bar{z}\, (\alpha(\bar{x}, \bar{z}) \leftrightarrow \bigwedge_{j=1}^{m} k_j z_j = t_j(\bar{x}))$. Finally, since $\mathcal{G}$ satisfies $U(\varphi)$ and $U(\delta_{k_j,t_j})$ for $j \in \{1, \dots, m\}$, it follows that $\varphi \sim \bigwedge_{j=1}^{m} \delta_{k_j,t_j}$ in $\mathcal{G}$. ⊣

In the following, by a *system of linear inequalities* we mean a finite conjunction of inequalities of the form $a_1 x_1 + \dots + a_n x_n \geq 0$ where $a_1, \dots, a_n$ are integers. (Note that such a system can be written as a conjunction of equations in $\tau_\mathcal{G}$.)

We say that a system of linear inequalities $\alpha(\bar{x})$ is *full*-dimensional on an abelian ℓ-group $\mathbf{G}$ if there is no $(a_1, \dots, a_n) \in \mathbb{Z}^n \setminus \{\bar{0}\}$ such that $\mathbf{G} \models \forall \bar{x}\,(\alpha(\bar{x}) \rightarrow \sum a_i x_i = 0)$. That is, the system $\alpha(\bar{x})$ imposes no linear dependencies on its solutions in $\mathbf{G}$. Observe that Lemma 4.1(4) implies that $\alpha(\bar{x})$ is full-dimensional on some nontrivial ℓ-group $\mathbf{G}$ if and only if it is full-dimensional on every nontrivial ℓ-group. Hence, we say that $\alpha(\bar{x})$ is *full-dimensional* provided it is full-dimensional on some nontrivial ℓ-group.

LEMMA 4.9. *A system of linear inequalities $\alpha(\bar{x})$ is full-dimensional if and only if for every totally ordered ℓ-group $\mathbf{G}$ the set $\{\bar{g} \in G^n : \mathbf{G} \models \alpha(\bar{g})\}$ generates $\mathbf{G}^n$ as an abelian group.*

PROOF. Assume $\alpha(\bar{x})$ is a full-dimensional system of linear inequalities and let $S_{\mathbf{G}} := \{\bar{g} \in G^n : \mathbf{G} \models \alpha(\bar{g})\}$ for any totally ordered ℓ-group $\mathbf{G}$. Let $\mathbf{Q}$ and $\mathbf{Z}$ denote the ℓ-groups of rational and integer numbers, respectively. First observe that $S_{\mathbf{Z}} = S_{\mathbf{Q}} \cap Z^n$. Note also that $S_{\mathbf{G}}$ is closed under linear combinations whose coefficients are non-negative integers, and $S_{\mathbf{Q}}$ is closed under non-negative rational linear combinations.

We start by proving that $S_{\mathbf{Z}}$ generated $\mathbf{Z}^n$ as an abelian group. Let V be the $\mathbf{Q}$-vector subspace of $\mathbf{Q}^n$ generated by $S_{\mathbf{Q}}$. Observe that $V = Q^n$; otherwise, there would exist integers $a_1, \dots, a_n$, not all zero, such that $V \subseteq \{\bar{x} \in Q^n : \sum a_i x_i = 0\}$, contradicting the fact that $\alpha(\bar{x})$ is full-dimensional. Since $V = Q^n$, the solution set $S_{\mathbf{Q}}$ contains a $\mathbf{Q}$-basis of $\mathbf{Q}^n$, which, multiplied by a suitable positive integer, yields a $\mathbf{Q}$-basis $\{\bar{b}_1, \dots, \bar{b}_n\} \subseteq S_{\mathbf{Z}}$. Since $S_{\mathbf{Z}}$ is closed under positive integer linear combinations, $\bar{b} := \sum \bar{b}_i \in S_{\mathbf{Z}}$. Now, let $\bar{c} \in Z^n$ be arbitrary and write $\bar{c} = \sum r_i \bar{b}_i$ for suitable rational numbers r_i. Let k be a positive integer such that $k \geq -r_i$ for all i. Then $k\bar{b} + \bar{c} = \sum_i (k + r_i)\bar{b}_i \in S_{\mathbf{Q}}$, since it is a positive linear combination of elements in $S_{\mathbf{Q}}$. Thus $k\bar{b} + \bar{c} \in S_{\mathbf{Q}} \cap Z^n = S_{\mathbf{Z}}$, and so $\bar{c} = (k\bar{b} + \bar{c}) - k\bar{b}$ belongs to the abelian group generated by $S_{\mathbf{Z}}$.

We prove now that $S_{\mathbf{G}}$ generates $\mathbf{G}^n$ as an abelian group for any totally ordered group $\mathbf{G}$. For any $\bar{a} \in Z^n$ and $g \in G$ we write $\bar{a}g := (a_1 g, \dots, a_n g)$. Note that if $\bar{a} \in S_{\mathbf{Z}}$ and g is a non-negative member of G, then $\bar{a}g \in S_{\mathbf{G}}$. Fix $j \in \{1, \dots, n\}$, and let $\bar{e}_j \in Z^n$ be such that $e_{ji} = 1$ if $i = j$ and $e_{ji} = 0$ otherwise. We write $\bar{e}_j = \sum k_l \bar{a}_l$ for integers k_l and $\bar{a}_l \in S_{\mathbf{Z}}$. Hence, if $g \in G, g \geq 0$, then $\bar{e}_j g = \sum k_l \bar{a}_l g$ is an integer linear combination of solutions $\bar{a}_l g \in S_{\mathbf{G}}$. This proves that $S_{\mathbf{G}}$ generates $\bar{e}_j g$ for all j and all $g \in G, g \geq 0$. Now it follows easily that any $\bar{g} \in G^n$ is generated by elements in $S_{\mathbf{G}}$.

The converse implication is straightforward. ⊣

LEMMA 4.10. *Let $t(\bar{x})$ be a term in τ_G. There are full-dimensional systems of linear inequalities $\alpha_1(\bar{x}), \dots, \alpha_m(\bar{x})$ and terms $t_1(\bar{x}), \dots, t_m(\bar{x})$, which are integer linear combinations of the variables $x_1, \dots, x_n$, such that for all $\mathbf{G} \in \mathcal{G}_{\text{to}}$ and all $\bar{g} \in G^n$ we have*

$$t^{\mathbf{G}}(\bar{g}) = \begin{cases} t_1^{\mathbf{G}}(\bar{g}), & \text{if } \alpha_1(\bar{g}) \text{ holds in } \mathbf{G}, \\ \vdots & \\ t_m^{\mathbf{G}}(\bar{g}), & \text{if } \alpha_m(\bar{g}) \text{ holds in } \mathbf{G}. \end{cases} \tag{3}$$

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PROOF. Fix a $\tau_{\mathcal{G}}$-term $t(\bar{x})$. We show first that there are full-dimensional systems $\alpha_1(\bar{x}), \dots, \alpha_m(\bar{x})$ and abelian group terms $t_1(\bar{x}), \dots, t_m(\bar{x})$ such that (3) holds for $\mathbf{G} = \mathbf{R}$, the ℓ-group of real numbers.

Using the way the lattice and group operations interact, we may assume $t(\bar{x}) = s(u_1(\bar{x}), \dots, u_p(\bar{x}))$ where $u_1(\bar{x}), \dots, u_p(\bar{x})$ are abelian group terms (i.e., linear combinations of variables with integer coefficients) and $s(\bar{y})$ is a lattice term. For each permutation σ of $\{1, \dots, p\}$ let $\alpha_\sigma(\bar{x})$ be the system of linear inequalities expressing that $u_{\sigma(1)}(\bar{x}) \leq \dots \leq u_{\sigma(p)}(\bar{x})$. Since $\mathbf{R}$ is totally ordered, for each σ there is $j_\sigma \in \{1, \dots, p\}$ such that $t^{\mathbf{R}}(\bar{r}) = u^{\mathbf{R}}_{j_\sigma}(\bar{r})$ for all $\bar{r}$ such that $\alpha_\sigma(\bar{r})$.

Next, for each σ let $S_\sigma := \{\bar{r} \in R^n : \alpha_\sigma(\bar{r})\}$. As each $\bar{r} \in R^n$ satisfies at least one $\alpha_\sigma(\bar{x})$, we have that $R^n = \bigcup_\sigma S_\sigma$. Let $\{\sigma_1, \dots, \sigma_m\}$ be the set of permutations σ such that S_σ has nonempty interior. Note that $\alpha_{\sigma_j}(\bar{x})$ is full-dimensional on $\mathbf{R}$ for all $j \in \{1, \dots, m\}$ (and thus on every ℓ-groups). Since each S_σ is a closed subset of R^n, by a simple topological argument, we have $R^n = S_{\sigma_1} \cup \dots \cup S_{\sigma_m}$. So, defining $\alpha_j(\bar{x}) := \alpha_{\sigma_j}(\bar{x})$ and $t_j(\bar{x}) := u_{\sigma_j}(\bar{x})$ for $j \in \{1, \dots, m\}$, we have established (3) in the case $\mathbf{G} = \mathbf{R}$. To conclude we show that the same α_j's and t_j's work for any $\mathbf{G} \in \mathcal{G}_{\text{to}}$. In fact, note that (3) holds if and only if $\mathbf{G}$ satisfies the following universal formulas:

- $\forall \bar{x}\ (\alpha_j(\bar{x}) \to t(\bar{x}) = t_j(\bar{x}))$ for $j \in \{1, \dots, m\}$,
- $\forall \bar{x}\ (\alpha_1(\bar{x}) \vee \dots \vee \alpha_m(\bar{x}))$.

Since these formulas hold in $\mathbf{R}$, Lemma 4.1(2) says that they must hold in $\mathbf{G}$. ⊣

LEMMA 4.11. *Given a positive integer k and a $\tau_{\mathcal{G}}$-term t, there is a positive integer k' such that $\delta_{k,t} \sim \delta_{k'}$ in $\mathcal{G}$.*

PROOF. Fix a positive integer k and a $\tau_{\mathcal{G}}$-term t; let $\alpha_j(\bar{x})$ and $t_j(\bar{x})$ for $j \in \{1, \dots, m\}$ be as in Lemma 4.10. Suppose $t_j(\bar{x}) = a_{j1}x_1 + \dots + a_{jn}x_n$, and let d be the greatest common divisor of the set $\{k\} \cup \{a_{ji} : i \in \{1, \dots, n\}, j \in \{1, \dots, m\}\}$. Define k' by $k = dk'$; we prove that $\delta_{k,t} \sim \delta_{k'}$ in $\mathcal{G}$. Observe that, due to Lemma 4.2, it suffices to show that $\delta_{k,t} \sim \delta_{k'}$ in $\mathcal{G}_{\text{to}}$.

Take $\mathbf{G} \in \mathcal{G}_{\text{to}}$ and assume $\mathbf{G} \vDash \delta_{k,t}$. We claim that $t_j(\bar{g})$ is divisible by k for every $\bar{g} \in G^n$ and $j \in \{1, \dots, m\}$. Indeed, given $\bar{g} \in G^n$ and $j \in \{1, \dots, m\}$, by Lemma 4.9, we can write $\bar{g} = \sum b_l \bar{g}_l$ for some integers b_l and some $\bar{g}_l \in G^n$ such that $\mathbf{G} \vDash \alpha_j(\bar{g}_l)$. Note that $t(\bar{g}_l) = t_j(\bar{g}_l)$ for each l. Since $\mathbf{G} \vDash \delta_{k,t}$, for each l there is $h_l \in G$ such that $kh_l = t(\bar{g}_l) = t_j(\bar{g}_l)$. Thus

$$t_j(\bar{g}) = t_j\left(\sum b_l \bar{g}_l\right) = \sum b_l t_j(\bar{g}_l) = \sum b_l k h_l = k \sum b_l h_l,$$

which proves the claim.

Now write $d = kc + \sum_{i,j} a_{ji} c_{ji}$ for suitable integers c and c_{ji}. Then, for any $g \in G$,

$$dg = kcg + \sum_j \sum_i a_{ji} c_{ji} g = kcg + \sum_j t_j(\bar{g}_j),$$

where $\bar{g}_j := (c_{j1}g, \dots, c_{jn}g)$. Since each $t_j(\bar{g}_j)$ is divisible by k, it follows that there is $g' \in G$ such that $dg = kg'$. Thus $dg = dk'g'$, so $d(g - k'g') = 0$ and, since $\mathbf{G}$ is torsion-free, $g = k'g'$. This proves that $\mathbf{G} \vDash \delta_{k'}$.

Conversely, assume any element in $\mathbf{G}$ is divisible by k', and fix $\bar{g} := (g_1, \dots, g_n) \in G^n$. We prove that $t(\bar{g})$ is divisible by k. Let $j \in \{1, \dots, m\}$ be such that $t(\bar{g}) = t_j(\bar{g})$. Since each a_{ji} is divisible by d, there is $g' \in G$ such that $t_j(\bar{g}) = dg'$. Now, since g' is divisible by k', there is $g'' \in G$ such that $g' = k'g''$. Putting all together we obtain $t(\bar{g}) = t_j(\bar{g}) = dg' = dk'g'' = kg''$. ⊣

We are now ready to present our characterization of EFD-sentences in $\mathcal{G}$.

THEOREM 4.12. *Given an EFD-sentence φ with a nontrivial model in $\mathcal{G}$ there is a positive integer k such that $\varphi \sim \delta_k$ in $\mathcal{G}$.*

PROOF. Given φ, combining Lemmas 4.8 and 4.11, we have that there are positive integers $k_1, \dots, k_m$ such that $\varphi \sim \bigwedge_{j=1}^{m} \delta_{k_j}$ in $\mathcal{G}$. Now take $k := k_1 \cdots k_m$, and note that $\bigwedge_{j=1}^{m} \delta_{k_j}$ is equivalent to δ_k in $\mathcal{G}$. ⊣

Given a set S of prime numbers, let $\Sigma_S := \{\delta_p : p \in S\}$. Since for an ℓ-group divisibility by k is equivalent to divisibility by the prime factors of k, we have the following:

THEOREM 4.13. *Every set of EFD-sentences either has only trivial models or is equivalent over $\mathcal{G}$ to Σ_S for some set S of prime numbers. Furthermore, the map $S \mapsto \Sigma_S$ is one-to-one, and thus, the lattice of AE-subclasses of $\mathcal{G}$ is isomorphic to $\mathbf{1} \oplus \mathbf{2}^{\omega}$.*

4.2. The algebraic expansions of the Logic of Equilibrium. As shown in [16, Theorem 17] the variety $\mathcal{G}$ of abelian ℓ-groups is the equivalent algebraic semantics of the Logic of Equilibrium *Bal* in the language $\{\rightarrow, ^+\}$ whose consequence operator is defined in [16, Section 2]. The derived connectives

$$0 := x \rightarrow x, \qquad x \vee y := (x \rightarrow y)^+ + x,$$
$$-x := x \rightarrow 0, \qquad x \wedge y := -(-x \vee -y),$$
$$x + y := -x \rightarrow y,$$

form a complete set for *Bal* since $\varphi(x \rightarrow y, \bar{z}) \dashv\vdash \varphi(-x + y, \bar{z})$ and $\varphi(x^+, \bar{z}) \dashv\vdash \varphi(x \vee 0, \bar{z})$ for any formula φ. This allows us to say that $\mathcal{G}$ is the equivalent algebraic semantics of *Bal* via equivalence formulas $\Delta(x, y) = \{x \rightarrow y\}$ and defining equations $\varepsilon(x) = \{x = 0\}$.

Given a prime number p, the algebraic expansion of *Bal* corresponding to the EFD-sentence δ_p is, by definition, obtained from *Bal* by adding a unary function symbol d_p to the langauge of *Bal*, the rule $\mathrm{U}_{\{x \rightarrow pz\}}$, and the axiom:

$$x \rightarrow p\mathsf{d}_p(x). \tag{$\mathrm{A_p}$}$$

Since the rule $\mathrm{U}_{\{x \rightarrow pz\}}$ is derivable in *Bal*, the expansion is obtained simply by adding A_p. For a set S of prime numbers define Bal^S as the expansion of *Bal* by the axioms $\{A_p : p \in S\}$. Note that, since Bal^S is an axiomatic expansion of *Bal*, its equivalent algebraic semantics is a variety. These expansions were also considered in [6, Section 7] where it is proved that every implicit connective in the logic Bal^{Primes} is explicit.

Recall that an expansion $L' := \langle \tau', \vdash_{L'} \rangle$ of a logic $L := \langle \tau, \vdash_L \rangle$ is called *conservative* provided that for each set of τ-formulas $\Gamma \cup \{\varphi\}$ we have that $\Gamma \vdash_{L'} \varphi$ implies $\Gamma \vdash_L \varphi$.

THEOREM 4.14.

1. *Every algebraic expansion of Bal is $\tau_{\mathcal{G}}$-bimorphic to exactly one of the following*:
   - *Inconsistent Logic*,
   - *Bal^S for some set S of prime numbers.*
2. *The algebraic expansions of Bal form a lattice isomorphic to $\mathbf{2}^{\omega} \oplus \mathbf{1}$ when ordered by $\tau_{\mathcal{G}}$-morphisms.*
3. *Given sets S, S' of prime numbers with $S \subseteq S'$, the expansion $Bal^{S'}$ is conservative over Bal^S.*

PROOF. Items 1. and 2. follow from Theorems 3.2 and 4.13. We prove 3.

Fix sets of prime numbers $S \subseteq S'$, and let $\mathcal{V}$ and $\mathcal{V}'$ be the equivalent algebraic semantics of Bal^S and $Bal^{S'}$, respectively. Since $Bal^{S'}$ is finitary, to prove 3. it is enough to show that any quasi-identity in the language of $\mathcal{V}$ valid in $\mathcal{V}'$ is also valid in $\mathcal{V}$. Let $\mathbf{Q}_S$ be the ℓ-group of rational numbers expanded with the divisions by the primes in S. It is not hard to show that $\mathbf{Q}_S$ generates $\mathcal{V}$ as a quasivariety, that is, $\mathsf{Q}(\mathbf{Q}_S) = \mathcal{V}$. Now let φ be a quasi-identity in the language of $\mathcal{V}$ that is valid in $\mathcal{V}'$. Then, we have that $\mathbf{Q}_{S'} \vDash \varphi$, and thus, $\mathbf{Q}_S \vDash \varphi$. Since $\mathsf{Q}(\mathbf{Q}_S) = \mathcal{V}$, the proof is finished. ⊣

§5. Algebraic expansions of perfect MV-algebras and their logic. The class of MV-algebras is the equivalent algebraic semantics of Łukasiewicz infinite-valued logic and has been extensively studied [12]. In this section we characterize the AE-subclasses of the variety generated by perfect MV-algebras, and thus, by Theorem 3.2, we also obtain a full description of lattice of algebraic expansions of $L_{\mathcal{P}}$, the Logic of Perfect MV-Algebras (see, e.g., [1]). Our approach is to export the results for abelian ℓ-groups to perfect MV-algebras, exploiting the connection between these two classes (see [13, 22, 23]).

Concerning notation and basic facts of MV-algebras we follow [12]; in particular, we consider MV-algebras in the language $\tau_{\mathcal{MV}} := \{+, \neg, 0\}$. Let $\mathbf{A}$ be an MV-algebra; the *radical* of $\mathbf{A}$ is the intersection of all maximal ideals of $\mathbf{A}$, which is denoted by $\operatorname{rad}\mathbf{A}$. We say that $\mathbf{A}$ is *perfect* if it is nontrivial and $A = \operatorname{rad}\mathbf{A} \cup \neg \operatorname{rad}\mathbf{A}$, where $\neg \operatorname{rad}\mathbf{A} := \{\neg a : a \in \operatorname{rad}\mathbf{A}\}$. The class of perfect MV-algebras is denoted by $\mathcal{P}$; we write $\mathcal{P}_{\text{to}}$ for its subclass of totally ordered members.

5.1. EFD-sentences in perfect MV-algebras. As shown by Theorem 4.13, every EFD-sentence of ℓ-groups is equivalent to some δ_k. Of course, each of these sentences induces the inverse function of multiplication by some positive integer. Next, for each k we introduce a term whose interpretation plays the role of multiplication by k in $\mathsf{V}(\mathcal{P})$, namely

$$\mathsf{t}_k(z) := (kz \wedge \neg 2z^2) \vee z^k.$$

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Thus, division by k in $\mathsf{V}(\mathcal{P})$ is embodied by the function induced by the EFD-sentence

$$\varepsilon_k := \forall x \exists! \, z \, \mathsf{t}_k(z) = x.$$

Both of these operations enjoy some natural algebraic properties whose proofs we omit as they are easy exercises.

LEMMA 5.1. *Let* $\mathbf{A} \in \mathcal{P}$.

1. *For every* $a \in A$ *we have*

$$\mathsf{t}_k^{\mathbf{A}}(a) = \begin{cases} a^k, & \text{if } a \in \neg \operatorname{rad} \mathbf{A}, \\ ka, & \text{if } a \in \operatorname{rad} \mathbf{A}. \end{cases}$$

2. *The term-function* $\mathsf{t}_k^{\mathbf{A}}$ *is a one-to-one endomorphism of* $\mathbf{A}$.
3. *The following are equivalent*:
   (i) $\mathsf{t}_k^{\mathbf{A}}$ *is surjective.*
   (ii) $\mathbf{A} \vDash \varepsilon_k$.
   (iii) *For every* $a \in \operatorname{rad} \mathbf{A}$, *there is* $b \in \operatorname{rad} \mathbf{A}$ *such that* $kb = a$.
   (iv) *For every* $a \in \neg \operatorname{rad} \mathbf{A}$, *there is* $b \in \neg \operatorname{rad} \mathbf{A}$ *such that* $b^k = a$.
4. *If* $\mathbf{A} \vDash \varepsilon_k$, *then* $[\varepsilon_k]^{\mathbf{A}}$ *is an automorphism, which is the inverse of* $\mathsf{t}_k^{\mathbf{A}}$.

Given the close connection between ℓ-groups and perfect MV-algebras, it is hardly surprising that a version of Theorem 4.13 holds for $\mathsf{V}(\mathcal{P})$ when we take ε_k in place of δ_k. Even though the proof we discovered is nontrivial, for reasons of brevity we only outline its key steps.

THEOREM 5.2. *For every EFD-sentence* φ *in* $\tau_{\mathcal{MV}}$ *with a model in* $\mathcal{P}$ *either* $\varphi \sim \forall x \, 2x = x$ *in* $\mathsf{V}(\mathcal{P})$ *or there is a positive integer* k *such that* $\varphi \sim \varepsilon_k$ *in* $\mathsf{V}(\mathcal{P})$.

SKETCH OF PROOF. The key to translate our classification of EFD-sentences for ℓ-groups to perfect MV-algebras is that the positive cone of an ℓ-group and the radical of a perfect MV-algebra are, essentially, the same thing.

Given an abelian ℓ-group $\mathbf{G}$, its *positive cone* is the subset $G^+ := \{x \in G : x \geq 0\}$. We define the algebraic structure $\mathbf{G}^+ := (G^+, +, \dot{-}, 0)$ where $x \dot{-} y := (x - y) \vee 0$. We write $\mathcal{C}$ for the class of positive cones of abelian ℓ-groups considered as algebras in the language $\tau_{\mathcal{C}} := \{+, \dot{-}, 0\}$. The members of $\mathcal{C}$ are known as *cancellative hoops*; see, e.g., [3, 14]. Given a cancellative hoop $\mathbf{A}$, there is (up to isomorphism) a unique abelian ℓ-group whose positive cone is isomorphic to $\mathbf{A}$ (see [2, Chapter XIV]); we write $\mathbf{A}^*$ for this ℓ-group.

By means of a syntactical translation argument we can apply Theorem 4.12 to obtain the following:

(1) For every EFD-sentence ψ in $\tau_{\mathcal{C}}$ with a nontrivial model there is a positive integer k such that $\psi \sim \delta_k$ in $\mathcal{C}$.[2]

In what follows let $\varphi := \forall \bar{x} \exists! \, \bar{z} \, \alpha(\bar{x}, \bar{z})$ denote a fixed but arbitrary EFD-sentence in $\tau_{\mathcal{MV}}$ with a model in $\mathcal{P}$. If the two-element MV-algebra is the only model of φ in $\mathcal{P}$, it is easy to see that $\varphi \sim \forall x \, 2x = x$ in $\mathsf{V}(\mathcal{P})$. So, assume φ has a non-Boolean model in $\mathcal{P}$. By an argument analogous to the one in the proof of Lemma 4.2 it

[2]Note that δ_k is also a sentence in $\tau_{\mathcal{C}}$.

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suffices to show that φ is equivalent to some ε_k in $\mathcal{P}_{\text{to}}$. We begin by showing that φ can be rewritten in a way such that it is evaluated just in the radical. Again, this is proved by a syntactical manipulation, and it turns out to be the most technically challenging part of the whole argument.

(2) There is an EFD-sentence $\varphi^{\mathrm{R}} := \forall \bar{x} \exists! \, \bar{z} \, \alpha^{\mathrm{R}}(\bar{x}, \bar{z})$ in $\tau_{\mathcal{MV}}$ such that $\varphi \sim \varphi^{\mathrm{R}}$ in $\mathcal{P}_{\text{to}}$ and for every $\mathbf{A} \in \mathcal{P}_{\text{to}}$ and every $\bar{a} \in A^n$ the following holds:
   (i) if $\bar{a} \in (\operatorname{rad} \mathbf{A})^n$ and $\mathbf{A} \vDash \alpha(\bar{a}, \bar{b})$ for some $\bar{b} \in A^m$, then $\bar{b} \in (\operatorname{rad} \mathbf{A})^m$,
   (ii) if $\bar{a} \notin (\operatorname{rad} \mathbf{A})^n$ we have that $\mathbf{A} \vDash \alpha(\bar{a}, \bar{0})$ and $\bar{z} = \bar{0}$ is the unique solution to $\alpha(\bar{a}, \bar{z})$ in $\mathbf{A}$.

Given $\mathbf{A} \in \mathcal{MV}$, we define $x \dot{-} y := \neg(\neg x + y)$. Via this shorthand we can interpret $\tau_{\mathcal{C}}$-terms in MV-algebras. The radical of $\mathbf{A}$ is closed under $+$ and $\dot{-}$. Moreover, $\mathbf{rad}\,\mathbf{A} := (\operatorname{rad} \mathbf{A}, +, \dot{-}, 0)$ is a cancellative hoop (see [13, Lemma 3.2]). The special properties of φ^{R} allow us to exchange it for a sentence to be evaluated on $\mathbf{rad}\,\mathbf{A}$.

(3) There is an EFD-sentence ψ in the language $\tau_{\mathcal{C}}$ such that: $\mathbf{A} \vDash \varphi^{\mathrm{R}} \Leftrightarrow \mathbf{rad}\,\mathbf{A} \vDash \psi$ for all $\mathbf{A} \in \mathcal{P}_{\text{to}}$.

Combining (1)–(3) yields

(4) There is a positive integer k such that $\mathbf{A} \vDash \varphi \Leftrightarrow \mathbf{rad}\,\mathbf{A} \vDash \delta_k$ for all $\mathbf{A} \in \mathcal{P}_{\text{to}}$.

Finally, the equivalence between (ii) and (iii) in Lemma 5.1.3 gives us

(5) $\mathbf{rad}\,\mathbf{A} \vDash \delta_k \Leftrightarrow \mathbf{A} \vDash \varepsilon_k$ for all $\mathbf{A} \in \mathcal{P}_{\text{to}}$. ⊣

5.2. The algebraic expansions of $L_{\mathcal{P}}$. The Logic $L_{\mathcal{P}}$ of Perfect MV-Algebras [13] is the extension of Łukasiewicz Logic by the axiom $2x^2 \leftrightarrow (2x)^2$ (recall that $x \leftrightarrow y := (\neg x + y) \wedge (\neg y + x)$). As the name suggests, the equivalent algebraic semantics of $L_{\mathcal{P}}$ is the variety $\mathsf{V}(\mathcal{P})$.

Given a prime number p, the algebraic expansion of $L_{\mathcal{P}}$ corresponding to the EFD-sentence ε_p is, by definition, obtained from $L_{\mathcal{P}}$ by adding a unary function symbol d_p to the language of $L_{\mathcal{P}}$, the axiom:

$$((p\mathsf{d}_p(x) \wedge \neg 2\mathsf{d}_p(x)^2) \vee \mathsf{d}_p(x)^p) \leftrightarrow x, \tag{$\mathrm{D_p}$}$$

and the rule $\mathrm{U}_{\{((kz \wedge \neg 2z^2) \vee z^k) \leftrightarrow x\}}$. However, since this rule is derivable in $L_{\mathcal{P}}$, the expansion is obtained simply by adding D_p.

For a set S of prime numbers define $L_{\mathcal{P}}^S$ as the expansion of $L_{\mathcal{P}}$ by the axioms $\{D_p : p \in S\}$. Note that, by the comment above, $L_{\mathcal{P}}^S$ is the algebraic expansion of $L_{\mathcal{P}}$ corresponding to the AE-class axiomatized by $\Sigma_S := \{\varepsilon_p : p \in S\}$. Thus, the equivalent algebraic semantics $\mathsf{V}(\mathcal{P})^{\Sigma_S}$ of $L_{\mathcal{P}}^S$ is a variety.

THEOREM 5.3.

1. *Every algebraic expansion of $L_{\mathcal{P}}$ is $\tau_{\mathcal{MV}}$-bimorphic to exactly one of the following*:
   - *Inconsistent Logic*,
   - *Classical Propositional Logic*,
   - *$L_{\mathcal{P}}^S$ for some set S of prime numbers.*

2. *The algebraic expansions of $L_{\mathcal{P}}$ form a lattice isomorphic to* $\mathbf{2}^{\omega} \oplus \mathbf{2}$ *when ordered by $\tau_{\mathcal{MV}}$-morphisms.*
3. *Given sets S, S' of prime numbers with $S \subseteq S'$, the expansion $L_{\mathcal{P}}^{S'}$ is conservative over $L_{\mathcal{P}}^{S}$.*

PROOF. Items 1. and 2. follow from Theorems 3.2 and 5.2. To prove 3. let $\mathbf{D}_S$ be the expansion of $\Gamma(\mathbf{Z} \vec{\times} \mathbf{Q}, (1, 0))$ by the operations d_p for $p \in S$. (Where Γ is Mundici's functor and $\vec{\times}$ is the lexicographical product; see [12].) Now the proof follows the argument of that of 3. in Theorem 4.14 with $\mathbf{D}_S$ in place of $\mathbf{Q}_S$. ⊣

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