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# GENERALIZED RADON TRANSFORM AND LÉVY'S BROWNIAN MOTION, II*)

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## § 1. Introduction

As a continuation of the author's paper [19], we shall investigate the null spaces of a dual Radon transform $R^*$, in connection with a Lévy's Brownian motion $X$ with parameter space $(R^n, d)$. We shall follow the notation used in (I), [19].

We begin with a brief review of the general framework behind the representation of Chentsov type:

$$X(x) = \int_{B_x} W(dh) = W(B_x)\,, \tag{1}$$

with $B_x := \{h \in H;\ x \in h\}$. It consists of the following:

(i) A Lévy's Brownian motion $X = \{X(x);\ x \in M\}$ with mean 0 and variance $d(x, y) = E[(X(x) - X(y))^2]$, where $d(x, y)$ is an $L^1$-embeddable (semi-)metric on $M$;

(ii) A Gaussian random measure $W = \{W(dh);\ h \in H\}$ based on a measure space $(H, \nu)$ such that $H \subset 2^M$ and $\nu$ is a positive measure on $H$ satisfying $\nu(B_x) < \infty$ and

$$d(x, y) = \nu(B_x \triangle B_y) = \int_H \pi_h(x, y)\nu(dh) \qquad \text{for all } x, y \in M\,, \tag{2}$$

where

$$\pi_h(x, y) := |\chi_h(x) - \chi_h(y)| = |\chi_{B_x}(h) - \chi_{B_y}(h)|\,.$$

As a bridge connecting the metric space $(M, d)$ and the measure space $(H, \nu)$, the equation (2) guarantees the existence of a representation of the form (1) for a Lévy's Brownian motion $X$ with parameter space $(M, d)$.

The representation (1) of Chentsov type played in (I) (and will play

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also in the present (II)) an important role, and led us to introduce a pair of integral transformations; one is the *generalized Radon transform*,

$$(3) \qquad (Rf)(h) := \int_h f(x)m(dx)\,, \qquad f \in L^1(M, m)\,,$$

and the other is the *dual Radon transform*

$$(4) \qquad (R^*g)(x) := \int_{B_x} g(h)\nu(dh), \qquad g \in L^2(H, \nu)\,.$$

DEFINITION 1. For each subset $A \subset M$, we define

$$(5) \qquad N_1(A) := \{g \in L^2(H, \nu);\ (R^*g)(x) \equiv 0 \text{ on } A\} = [\chi_{B_x}(h);\ x \in A]^{\perp}\,.$$

This closed subspace of $L^2(H, \nu)$ is called the *null space of $R^*$ relative to the subset $A$.*

The study of such null spaces $N_1(A)$ is of great importance for the following reason. For each Lévy's Brownian motion $X$ with parameter space $(M, d)$, we have a representation of the form (1). Consider an increasing family of closed linear spans $[X(x); x \in A_\rho]$ corresponding to each increasing family of subsets $A_\rho$ with $\cup_{0<\rho<\infty} A_\rho = M$. Just as in the well-known theory of canonical representations of Gaussian processes, we wish to give a description of these $[X(x); x \in A_\rho]$ in terms of a Gaussian random measure $W$; they are all contained in the big closed subspace

$$\left\{\int_H g(h)W(dh);\ g \in L^2(H, \nu)\right\}$$

of $L^2(\Omega, P)$. Since one can easily see that

$$(6) \qquad [X(x);\ x \in A] = \left\{\int_H g(h)W(dh);\ g \in N_1(A)^{\perp}\right\}$$

for every $A \subset M$, our problem is to determine completely the null space $N_1(A_\rho)$ of the dual Radon transform $R^*$.

So the main purpose of this paper is to investigate the null spaces $N_1(A_\rho)$ for a certain increasing family of closed subsets $A_\rho$ of $M$, such as $A_\rho = V_\rho$ in the case $M = R^n$, where $V_\rho$ denotes the closed ball of radius $\rho$ about the origin $O$, $0 < \rho < \infty$. Examples of $L^1$-embeddable metrics $d$ on $R^n$ in which we have succeeded in finding a complete description of $N_1(V_\rho)$ as well as of $[X(x); x \in V_\rho]$ will be explained below.

Sections 3 and 4 concern rotation-invariant distances $d$ on $M = R^n$ which are derived, via the equation (2), from the following choice of $H$:

$H = \{h_{t,\omega};\ t > 0,\ \omega \in S^{n-1}\}$ is the set of all half-spaces
$h_{t,\omega} := \{x \in R^n;\ (x, \omega) > t\}$ not containing the origin $O$.

The Euclidean distance $|x - y|$ is a familiar example of such a distance.

The generalized Radon transform $(Rf)(h_{t,\omega})$ is then given by the integral of $f$ over the half-space $h_{t,\omega}$ and hence closely related to the classical Radon transform. This observation leads us to apply the fruitful theory of the classical Radon transform (see, for example, [9], [12] and [16]) and solve the problem concerning the null spaces of $R^*$. In fact, by using the theorem of Ludwig [16] (cf. [20] and [21]), we are able to find a complete description of $N_1(V_\rho)$ (Theorem 7) as well as that of $[X(x);\ x \in V_\rho]$ (Theorem 8).

Our result on the structure of $[X(x);\ x \in V_\rho]$ can be restated in terms of mutually independent Gaussian processes $M_{m,k}(t)$ introduced by McKean [17]:

$$(7) \qquad M_{m,k}(t) := \int_{S^{n-1}} X(t\omega) S_{m,k}(\omega) \sigma(d\omega)\,, \qquad t > 0\,,$$

where $\sigma$ denotes the uniform probability measure on the unit sphere $S^{n-1}$ and $\{S_{m,k}(\omega);\ (m, k) \in \Delta\}$, $\Delta := \{(m, k);\ m \geq 0 \text{ and } 1 \leq k \leq h(m)\}$, is taken to be a CONS in $L^2(S^{n-1}, \sigma)$ consisting of spherical harmonics. The basic representation (1) of $X$ yields

$$(8) \qquad M_{m,k}(t) = \int_0^t \lambda_m(u/t) dB_{m,k}(u)\,,$$

where the kernel $\lambda_m(t)$ is expressed in terms of the Gegenbauer polynomial $C_m^q(u)$ of degree $m$ with $q := (n-2)/2$:

$$\lambda_m(t) = (\text{const.}) \int_t^1 C_m^q(u)(1 - u^2)^{q-1/2} du\,.$$

It turns out that the representation (8) of $M_{m,k}(t)$ is canonical only for $m \leq 2$ (Theorem 10). Moreover, for $m \geq 3$, we determine the dimension of $[B_{m,k}(t);\ t \leq \rho] \ominus [M_{m,k}(t);\ t \leq \rho]$ (orthogonal complement in $L^2(\Omega, P)$) which can be regarded as the degree of non-canonicality of (8). In this way, our Theorems 8 and 10 might be viewed as a development (or refinement) of the result in [17] proved for a Brownian motion with $n$-dimensional parameter.

In Section 2 we shall give various kinds of $L^1$-embeddable metrics $d$ on $R^n$. Some of them should be mentioned here.

The first kind of $d$ depends on the choice of a bounded subset $K \subset R^n$ such that $|K| > 0$ and $O \in K$. Take the following measure space $(H_K, \nu_K)$:

$$H_K: = \{h_{\alpha,p}: = \{x \in R^n;\ \alpha(x-p) \in K\} = \alpha^{-1}K + p;\ \alpha \in SO(n)/\Sigma_K, p \in R^n\}$$

and

$$d\nu_K(h_{\alpha,p}): = c\,d\alpha\,dp\,, \qquad c > 0\,,$$

where $\Sigma_K: = \{\alpha \in SO(n);\ \alpha K = K\}$ and $d\alpha$ denotes the normalized Haar measure on $SO(n)/\Sigma_K$. Then, the equation (2) gives us an $L^1$-embeddable metric $d_K$ invariant under every rigid motion on $R^n$:

$$\begin{aligned} d_K(x, y) &= c\int_{SO(n)/\Sigma_K} |(\alpha^{-1}K + x - y) \triangle \alpha^{-1}K|\,d\alpha \\ &= c\int_{SO(n)} |(K + \alpha(x-y)) \triangle K|\,d\alpha = r_K(|x-y|)\,. \end{aligned}$$

The typical choice of $K = V_\rho$ allows us to compute the explicit form of $r_{V_\rho}$ and get a large class of invariant distances by forming a superposition of the family $\{d_{V_\rho};\ 0 < \rho < \infty\}$ (cf. Section 2, 2–1). This idea of superposition is due to Takenaka [24] who gave a nice account of representations of self-similar Gaussian random fields.

It deserves mention that the generalized Radon transform

$$(Rf)(h_{\alpha,p}) = \int_K f(\alpha^{-1}x + p)dx\,, \qquad h_{\alpha,p} \in H_K\,,$$

was discussed in connection with the Pompeiu problem (cf. [26]).

The next kind of $d$ is of the form $\|x-y\|$, where $\|x\|$ is a norm of negative type ([6] and [8]). Such a norm is characterized as the support function of a special convex body in $R^n$ called a *zonoid* ([5]), and therefore admits of the following expression in terms of a bounded symmetric positive measure $\tau$ on $S^{n-1}$:

$$\|x\| = \int_{S^{n-1}} |(x, \omega)|\,\tau(d\omega)\,. \tag{9}$$

With the help of this well-known expression, the measure space $(H, \nu)$ combined with $\|x-y\|$ via (2) is naturally taken to be

$$\nu(dh_{t,\omega}) = dt\,\tau(d\omega) \qquad \text{on the set } H \text{ of half-spaces } h_{t,\omega}\,.$$

Note that rotation-invariance of $\tau$ yields the Euclidean distance $|x-y|$ up to a constant multiple.

It is worthwhile to remark that every Lévy's Brownian motion $X$ with parameter space $(R^n, ||x - y||)$ possesses a notable property: For each line $L$ in $R^n$, restrict the whole parameter space $R^n$ to the one-dimensional set $L$; then the Gaussian process $X_{|L} = \{X(x);\ x \in L\}$ coincides with a standard Brownian motion. In order to get at his definition of Brownian motion with $n$-dimensional parameter, Lévy [15] added one more simple condition that the probability law of $X(x) - X(O)$ is invariant under every rotation $\in SO(n)$. The class of Lévy's Brownian motions corresponding to norms of negative type is thus thought of as a nice extension of Lévy's original one.

## § 2. $L^1$-embeddable metrics on Euclidean space

This section is devoted to the study of the equation (2) connecting a metric space $(R^n, d)$ with a measure space $(H, \nu)$. Indeed we describe a variety of $L^1$-embeddable metrics $d$ on $R^n$ and corresponding measures $\nu$ on $H \subset 2^{R^n}$. Among them, we should like to mention the following class of rotation-invariant distances:

$$d(x, y) = c|x - y| + \int_0^\infty \mu(dt) \int_{S^{n-1}} |e^{t(x,\omega)} - e^{t(y,\omega)}| \sigma(d\omega)\,, \tag{10}$$

where $c \geq 0$ and $\mu$ is a non-negative measure on $(0, \infty)$ such that

$$\int_0^\infty t e^{at} \mu(dt) < \infty \qquad \text{for any } a > 0\,.$$

This class will be further discussed in Sections 3 and 4.

**2-1.** The first type of an $L^1$-embeddable metric $d$ on $R^n$ is derived from the $d_K$ in Section 1 with the choice of $K = V_{u/2}$. For each $u > 0$, we set

$$H_u := \{k_p := V_{u/2} + p;\ p \in R^n\} \quad \text{and} \quad \nu_u(dk_p) := dp/2|S^{n-1}|(u/2)^{n-1},$$

to get the desired distance

$$d_u(x, y) := \int_{H_u} \pi_{k_p}(x, y) \nu_u(dk_p) = r_u(|x - y|)\,,$$

where

$$\begin{aligned} r_u(t) &= |(V_{u/2} + te_1) \triangle V_{u/2}|/2|S^{n-1}|(u/2)^{n-1} \\ &= u\int_0^{\min(t/u,1)} (1 - v^2)^{(n-1)/2} dv\,, \qquad e_1 = (1, 0, \cdots, 0) \in R^n\,. \end{aligned} \tag{11}$$

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Observe that $\lim_{u\to\infty} r_u(t) = r_\infty(t) = t$ for each $t > 0$. Hence we put $d_\infty(x, y): = |x - y|$.

Having found the family $\{d_u;\ 0 < u \leq \infty\}$, we now form its superposition by means of a positive measure $G(du)$ on $(0, \infty]$:

$$d(x, y): = \int_{(0,\infty]} d_u(x, y)G(du)\,. \tag{12}$$

The corresponding measure space $(H, \nu)$ is obviously taken as follows:

$$H = \{k_{u,p}: = V_{u/2} + p;\ 0 < u < \infty,\ p \in R^n\} \cup \{h_{t,\omega};\ t > 0,\ \omega \in S^{n-1}\}$$

(disjoint union) and

$$\nu(dk_{u,p}) = \frac{2^{n-2}}{|S^{n-1}|}\, u^{-n+1}G(du)dp, \qquad \nu(dh_{t,\omega}) = G(\{\infty\})\, \frac{(n-1)|S^{n-1}|}{|S^{n-2}|}\, dt\sigma(d\omega)\,.$$

Here is a brief comment on the choice of $(H, \nu)$. Even if $\nu(B_x) = \infty$ for some $x \in R^n$, the equation (2) still has a meaning under the condition that $\nu(B_x \triangle B_y) < \infty$ for all $x, y \in R^n$. We therefore impose the condition $\int_{(0,\infty]} \min(u, 1)G(du) < \infty$ on the measure $G$. In order to get at the stronger conclusion that $\nu(B_x) < \infty$ for all $x \in R^n$, it suffices to change every element $h \in H$ containing the origin $O$ for its complement $h^c$, so that $B_O$ is empty and $\nu(B_x) = \nu(B_x \triangle B_O) < \infty$. This manipulation was explained in (I), Section 2.

The above distance (12) is invariant under every rigid motion on $R^n$ and takes the form $r(|x - y|)$ with

$$r(t) = \int_{(0,\infty]} r_u(t)G(du)\,. \tag{12'}$$

It follows that

$$r'(t) = \int_{(t,\infty]} (1 - t^2/u^2)^{(n-1)/2}G(du)\,. \tag{13}$$

In the one-dimensional case, this expression (13) immediately shows the following

Proposition 1. *Suppose $r(t)$ is a continuous function on $[0, \infty)$, $r(0) = 0$ and has the right derive $r'_+(t) \geq 0$ which is non-increasing on $(0, \infty)$ and satisfies $\left|\int_0^1 t\,dr'_+(t)\right| < \infty$. Then the distance $d(x, y): = r(|x - y|)$ on $R^1$ is $L^1$-embeddable.*

For $n \geq 2$, we devote our attention to the case where $G(du)$ is absolutely continuous on $(0, \infty)$ with density $g(u)$ and $G(\{\infty\}) = 0$. The equality (13) becomes

$$r'(t) = \int_t^\infty (1 - t^2/u^2)^{(n-1)/2} g(u) du \,, \tag{13'}$$

which coincides with the classical Radon transform $\hat{f}(\delta h_{t,\omega})$ applied to the radial function $f(y) := g(|y|)/|S^{n+1}||y|^n$ on $R^{n+2}$, i.e., the integral of $f$ over the hyperplane $\delta h_{t,\omega} := \{y \in R^{n+2};\ (y, \omega) = t\}$ in $R^{n+2}$ ([9], p. 103). By appeal to the inversion formula ([9], p. 120), we get

$$g(u) = d_n \int_u^\infty \left\{\left(-\frac{d}{dt}\right)^{n+1} r'(t)\right\}(t^2 - u^2)^{(n-1)/2} dt \,, \tag{14}$$

with

$$d_n := \frac{2^{n-1}}{\pi}\left\{\frac{\Gamma(n/2)}{\Gamma(n)}\right\}^2 .$$

We consider the functions $\psi_\lambda(t) := (1 - e^{-\lambda t})/\lambda$, $\lambda > 0$; every $\psi_\lambda(t)$ satisfies $(-d/dt)^{n+1}\psi'_\lambda(t) \geq 0$ for all $n \geq 2$. By (14), the $L^1$-embeddable metric $\psi_\lambda(|x - y|)$ on $R^n$ is of the form (12) with the corresponding density

$$g_\lambda(u) = d_n \lambda^{n+1} \int_u^\infty e^{-\lambda t}(t^2 - u^2)^{(n-1)/2} dt \,.$$

Thus, the method of superposition gives us the following

PROPOSITION 2 (cf. [2] and [3]). *Suppose a function $r(t)$ on $[0, \infty)$ is expressed in the form*

$$r(t) = ct + \int_0^\infty \psi_\lambda(t) \gamma(d\lambda) \,, \tag{15}$$

*where $c \geq 0$ and $\gamma$ is a non-negative measure on $(0, \infty)$ such that*

$$\int_0^\infty \min(1, \lambda^{-1}) \gamma(d\lambda) < \infty \,.$$

*Then the distance $d(x, y) := r(|x - y|)$ on $R^n$ is $L^1$-embeddable.*

**2-2.** The second type of $d$ is an extension of the norm $||x - y||$ admitting of the expression (9).

PROPOSITION 3. *Suppose $r(t)$ is a function described in Proposition 1. Then the distance*

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(16) $$d(x, y) := \int_{S^{n-1}} r(|(x - y, \omega)|)\tau(d\omega) \qquad \text{on } R^n$$

*is $L^1$-embeddable.*

*Proof.* The proof is carried out by constructing a measure space $(H, \nu)$ combined with (16) via the equation (2). Since $r(t)$ is of the form

(17) $$r(t) = ct + \int_0^\infty \min(t, u)G(du), \qquad c := G(\{\infty\}),$$

it is convenient to divide $d$ into two parts:

$$d_1(x, y) := c\int_{S^{n-1}} |(x - y, \omega)|\tau(d\omega) = c||x - y||,$$

and

$$d_2(x, y) := \int_0^\infty G(du)\int_{S^{n-1}} \min(|(x - y, \omega)|, u)\tau(d\omega).$$

We have already described a measure space $(H_1, \nu_1)$ for the first part $d_1$ in Section 1. On the other hand, a measure space $(H_2, \nu_2)$ combined with $d_2$ is easily found; it is

$H_2 := \{k_{u,t,\omega} := \{x \in R^n;\ |(x, \omega) - t| < u/2\};\ 0 < u < \infty,\ t \in R^1 \text{ and } \omega \in S^{n-1}\}$ equipped with $\nu_2(dk_{u,t,\omega}) := G(du)dt\tau(d\omega)/2$.

The proof is thus completed.

For a given norm $||x||$ of negative type, we consider the distance $||x - y||^\alpha$, $0 < \alpha < 1$. It is known ([8] and [14]) that $||x - y||^\alpha$ can be expressed in the form (16) with $r(t) = t^\alpha$. Hence the method of superposition again shows that the distance $d(x, y) := \psi(||x - y||)$ on $R^n$ is $L^1$-embeddable if $\psi(t) = \int_{(0,1]} t^\alpha m(d\alpha)$, where $m$ is a bounded positive measure on $(0, 1]$.

**2-3.** In connection with the theory of continuous functions $\phi(x)$ of negative type on the semigroup $(R^n, +)$ ([4]), we proceed to discuss a new class of $L^1$-embeddable metrics on $R^n$.

First recall the known expression of $\phi$ ([4], p. 220):

$$\phi(x) = a + (b, x) - Q(x) + \int_{R^n\setminus\{0\}} \left(1 - e^{(x,\xi)} + \frac{(x, \xi)}{1 + |\xi|^2}\right)\gamma(d\xi),$$

where $a \in R^1$, $b \in R^n$, $Q$ is a non-negative quadratic form on $R^n$ and $\gamma$ is a non-negative measure on $R^n\setminus\{0\}$ such that

$$\int_{0<|\xi|<1} |\xi|^2 \gamma(d\xi) < \infty \quad \text{and} \quad \int_{|\xi|>1} e^{(x,\xi)} \gamma(d\xi) < \infty \quad \text{for all } x \in R^n .$$

Set $d(x, y) := 2\phi(x + y) - \phi(2x) - \phi(2y)$, to get

$$d(x, y) = Q(x - y) + \int_{R^n \setminus \{0\}} (e^{(x,\xi)} - e^{(y,\xi)})^2 \gamma(d\xi) .$$

This form of $d$ gaurantees the existence of a centered Gaussian random field $X = \{X(x);\ x \in R^n\}$ such that $d(x, y) = E[(X(x) - X(y))^2]$.

We are ready to state the following

PROPOSITION 4. *Suppose $r(t)$ is a function described in Proposition 1, and define a distance on $R^n$ by*

$$d(x, y) := \int_{R^n \setminus \{0\}} r(|e^{(x,\xi)} - e^{(y,\xi)}|) \gamma(d\xi) , \tag{18}$$

*where $\gamma$ is a positive measure on $R^n \setminus \{0\}$ such that*

$$\int_{0<|\xi|<1} r(|\xi|) \gamma(d\xi) < \infty \quad \text{and} \quad \int_{|\xi|>1} e^{(x,\xi)} \gamma(d\xi) < \infty$$

*for all $x \in R^n$. Then $d$ is $L^1$-embeddable.*

*Proof.* In view of the general form (17) of $r$, it suffices to treat the two special cases: (i) $r(t) = t$ and (ii) $r(t) = \min(t, u)$, $0 < u < \infty$.

(i) The case $r(t) = t$. A measure space corresponding to (18) is given by

$$\nu(dh_{t,\omega}) = \int_{R^n \setminus \{0\}} \gamma(d\xi) \{|\xi| e^{|\xi| t} dt \delta_{\xi/|\xi|}(d\omega)\}$$

on the set $H$ of half-spaces $h_{t,\omega}$, $t > 0$ and $\omega \in S^{n-1}$, where $\delta_a$ denotes the Dirac measure at the point $a \in S^{n-1}$.

(ii) The case $r(t) = \min(t, u)$. Consider the following subset parametrized by $(t, \xi) \in R^1 \times R^n$:

$$\tilde{k}_{t,\xi} := \{x \in R^n ; |e^{(x,\xi)} - t| < u/2\} .$$

Then it is easy to verify that the measure

$$\nu_u(d\tilde{k}_{t,\xi}) := dt \gamma(d\xi)/2 \qquad \text{on } H_u := \{\tilde{k}_{t,\xi};\ t \in R^1,\ \xi \in R^n\}$$

yields the desired distance (18) in this second case, which completes the proof.

If a rotation-invariant distance of the form (18) is requested, we must take a rotation-invariant measure $\gamma$, which is of the form

$$\gamma(d\xi) = d\mu(|\xi|)d\sigma(\xi/|\xi|) \qquad \text{with a positive measure } \mu \text{ on } (0, \infty)$$

such that $\int_0^\infty r(t)e^{at}d\mu(t) < \infty$ for all $a > 0$. It also deserves mention that one can derive the distance $\|x - y\|^\alpha$ in Section 2–2 as the limit of distances of the form (18) with $r(t) = t^\alpha$, $0 < \alpha \leq 1$. Indeed, for each $\rho > 0$, take the measure $\gamma_\rho(d\xi) := d\tau(\xi/\rho)/\rho^\alpha$ concentrated on the sphere $\delta V_\rho$ of radius $\rho$; then one can see that

$$\lim_{\rho \downarrow 0} \int_{\delta V_\rho} |e^{(x,\xi)} - e^{(y,\xi)}|^\alpha \gamma_\rho(d\xi) = \int_{S^{n-1}} |(x - y, \omega)|^\alpha \tau(d\omega) = \|x - y\|^\alpha .$$

**2-4.** Let $X$ be a centered Gaussian random field with homogeneous increments ([25]). Then the variance $d(x, y) := E[(X(x) - X(y))^2]$ takes the analogous form

$$d(x, y) = Q(x - y) + \int_{R^n\setminus\{0\}} |e^{i(x,\xi)} - e^{i(y,\xi)}|^2 \gamma(d\xi) ,$$

where $\gamma$ is a spectral measure on $R^n\setminus\{0\}$ satisfying $\int \min(|\xi|^2, 1)\gamma(d\xi) < \infty$. On the lines of Proposition 4, we can prove the following

PROPOSITION 5. *Suppose $r(t)$ is a function described in* [19], *Proposition* 2. *Set*

$$d(x, y) := \int_{R^n\setminus\{0\}} r(d_G(e^{i(x,\xi)}, e^{i(y,\xi)}))\gamma(d\xi) , \tag{19}$$

*where $d_G$ denotes the geodesic distance on the unit circle $S^1 = \{z \in C : |z| = 1\}$ and $\gamma$ is a symmetric positive measure on $R^n\setminus\{0\}$ such that*

$$\int_{R^n\setminus\{0\}} \min(r(|\xi|), 1)\gamma(d\xi) < \infty .$$

*Then the distance $d$ on $R^n$ is $L^1$-embeddable.*

## § 3. Null spaces of dual Radon transforms

In this section we are concerned with every rotation-invariant distance $d$ on $R^n$ of the form (10). The corresponding measure space is then taken to be the set $H$ of half-spaces $h_{t,\omega}$ equipped with the rotation-invariant measure

(20) $$\nu(dh_{t,\omega}) = \left\{c + \int_0^\infty ue^{tu}\mu(du)\right\}dt\sigma(d\omega).$$

Our aim is to determine the null space $N_1(V_\rho)$ of the dual Radon transform $R^*$ on this $L^2(H, \nu)$ (see (5)). In view of the relation (6) for a Lévy's Brownian motion $X$ with parameter space $(R^n, d)$, our result on $N_1(V_\rho)$ will show a gap between the two closed subspaces $[X(x); x \in V_\rho]$ and $[W(dh_{t,\omega}); h_{t,\omega} \in H(\rho)]$ in $L^2(\Omega, P)$, where $H(\rho) := \{h_{t,\omega} \in H; 0 < t \leq \rho, \omega \in S^{n-1}\}$ is the set of all half-spaces intersecting $V_\rho$.

**3-1.** We shall start with a brief discussion of the restriction $X_{|V_\rho}$ of the whole parameter space $R^n$ to the closed ball $V_\rho$. Since $B_x \subset H(\rho)$ for every $x \in V_\rho$, the complement of $H(\rho)$ is of no importance. That is, a measure space combined with the distance $d_{|V_\rho}$ on $V_\rho$ via (2) is given by

$$\tilde{H}_\rho := \{\tilde{h}_{t,\omega} := h_{t,\omega} \cap V_\rho;\ h_{t,\omega} \in H(\rho)\} \qquad \text{and } d\tilde{\nu}(\tilde{h}_{t,\omega}) = d\nu(h_{t,\omega}),$$

which is isomorphic to the original $(H(\rho), \nu)$.

The relevant dual Radon transform $R_\rho^*$ is, therefore, considered to be a Hilbert-Schmidt operator from $L^2(H(\rho), \nu)$ to $L^2(V_\rho, dx)$, although both $R_\rho^*$ and $R^*$ take the same form

$$\int_{B_x} g(h_{t,\omega})\nu(dh_{t,\omega}), \qquad g \in L^2(H(\rho), \nu).$$

As was shown in (I), Theorem 5, the singular value decomposition of $R_\rho^*$ is expressed by means of $\lambda_{\rho,i} > 0$, $f_{\rho,i}(x)$ and $g_{\rho,i}(h_{t,\omega})$, $i \in I_\rho$:

$$(R_\rho^* g)(x) = \sum_{i \in I_\rho} \lambda_{\rho,i}(g, g_{\rho,i})_{L^2(H(\rho),\nu)} f_{\rho,i}(x),$$

where $\{f_{\rho,i};\ i \in I_\rho\}$ (resp. $\{g_{\rho,i};\ i \in I_\rho\}$) forms an ONS in $L^2(V_\rho, dx)$ (resp. $L^2(H(\rho), \nu)$).

The Gaussian system $X_{|V_\rho}$ now admits of the Karhunen-Loève expansion

(21) $$X(x) = \sum_{i \in I_\rho} \lambda_{\rho,i}\xi_{\rho,i} f_{\rho,i}(x), \qquad x \in V_\rho,$$

where the system

$$\xi_\rho = \left\{\xi_{\rho,i} := \int_{H(\rho)} g_{\rho,i}(h_{t,\omega})W(dh_{t,\omega});\ i \in I_\rho\right\}$$

is an i.i.d. sequence of standard Gaussian random variables. Moreover we have

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$$(22)\qquad [X(x);\, x \in V_\rho] = [\xi_{\rho,i};\ i \in I_\rho] = \left\{ \int_{H(\rho)} g(h_{t,\omega}) W(dh_{t,\omega});\ g \in N_\rho^\perp \right\},$$

with the null space $N_\rho$ of $R_\rho^*$:

$$N_\rho : = \{g \in L^2(H(\rho), \nu);\ (R_\rho^* g)(x) \equiv 0 \text{ on } V_\rho\}.$$

Note that $N_1(V_\rho) = N_\rho \oplus L^2(H(\rho)^c, \nu)$, which implies that (22) coincides with (6) for $A = V_\rho$.

**3-2.** We are now going to determine the null space $N_\rho$ of $R_\rho^*$, $0 < \rho < \infty$.

For that purpose we need

LEMMA 6. *We have an expansion*

$$(23)\qquad \begin{aligned} \chi_{B_x}(h_{t,\omega}) &= \sum_{m=0}^{\infty} \lambda_m(t/|x|) \sum_{k=1}^{h(m)} S_{m,k}(x/|x|) S_{m,k}(\omega) \\ &= \sum_{m=0}^{\infty} \lambda_m(t/|x|) h(m) \Phi_m^q((x, \omega)/|x|), \end{aligned}$$

*where* $\Phi_m^q(t) : = C_m^q(t)/C_m^q(1)$ *with* $q : = (n-2)/2$, *and*

$$(24)\qquad \lambda_m(t) = \frac{|S^{n-2}|}{|S^{n-1}|} \chi_{(0,1]}(t) \int_t^1 \Phi_m^q(u)(1-u^2)^{q-1/2} du.$$

*Furthermore we have*

$$(25)\qquad \lambda_m(t) = \frac{|S^{n-2}|}{|S^{n-1}|(n-1)} \Phi_{m-1}^{q+1}(t)(1-t^2)^{q+1/2}$$

*for* $m \geq 1$ *and* $0 < t < 1$.

*Proof.* Since $\chi_{B_x}(h_{t,\omega}) = \chi_{h_{t,\omega}}(x) = \chi_{(t/|x|,1]}((x', \omega))$, $x' : = x/|x|$, the above assertions for the variables $\omega$, $x' \in S^{n-1}$ coincide with (I), Lemma 7 stated in terms of the variables $x, y \in S^n$.

Now, take an arbitrary function $g$ from $L^2(H(\rho), \nu)$. Such a function is written in the form

$$g(h_{t,\omega}) = \sum_{(m,k)\in \Delta} g_{m,k}(t) S_{m,k}(\omega),$$

where

$$g_{m,k}(t) : = \int_{S^{n-1}} g(h_{t,\omega}) S_{m,k}(\omega) \sigma(d\omega), \qquad 0 < t \leq \rho.$$

The density in the expression (20) of $\nu$ is simply denoted by

(20′) $$v(t) := c + \int_0^\infty u e^{ut}\mu(du)\,.$$

Then all functions $g_{m,k}(t)$ belong to $L^2((0,\rho], v(t)dt)$, because

$$\sum_{(m,k)\in\Delta}\int_0^\rho g_{m,k}^2(t)v(t)dt = \|g\|^2_{L^2(H(\rho),\nu)} < \infty\,.$$

Lemma 6 implies that

$$(R_\rho^* g)(x) = \sum_{(m,k)\in\Delta} S_{m,k}(x/|x|)\int_0^{|x|}\lambda_m(t/|x|)g_{m,k}(t)v(t)dt\,.$$

We now assume that $g \in N_\rho$. Then we have

(26) $$\int_0^u \lambda_m(t/u)g_{m,k}(t)v(t)dt \equiv 0\,, \qquad 0 < u \le \rho\,,$$

for every $(m, k) \in \Delta$.

In case $m = 0$, we make use of (24) to get

$(26)_0$ $$\int_0^u (1 - t^2/u^2)^{(n-3)/2} G_{0,1}(t)dt \equiv 0, \qquad 0 < u \le \rho\,,$$

where we have put

$$G_{0,1}(t) := \int_0^t g_{0,1}(s)v(s)ds\,.$$

As is well known ([12]), p. 14), the integral equation $(26)_0$ yields the unique solution $G_{0,1}(t) \equiv 0$, i.e., $g_{0,1}(t) \equiv 0$ on $(0, \rho]$.

The equation (26) for $m \ge 1$ takes a different form: By (25), we have

$(26)_m$ $$\int_0^1 C_{m-1}^{q+1}(t)(1-t^2)^{q+1/2}G_{m,k}(ut)dt = \int_{-1}^1 C_{m-1}^{q+1}(t)(1-t^2)^{q+1/2}G_{m,k}(ut)dt/2 \equiv 0\,, \quad 0<u\le\rho,$$

where

$$G_{m,k}(t) = g_{m,k}(t)v(t) \quad \text{for} \quad 0 < t \le \rho \quad \text{and} \quad G_{m,k}(t) = (-1)^{m-1}g_{m,k}(-t)v(-t)$$

for $-\rho \le t < 0$. The theorem in Ludwig [16] for the Gegenbauer transform (see also [20] and [21]) now concludes that $G_{m,k}(t)$, $t > 0$, is a polynomial of the form $\sum_{j=1}^{[(m-1)/2]} a_{m,k,j}t^{m-1-2j}$ with some coefficients $a_{m,k,j} \in R^1$, We have thus proved that $g$ in $N_\rho$ is necessarily of the form

$$\sum_{(m,k,j)\in J} a_{m,k,j}p_{m,k,j}, \qquad \text{where } p_{m,k,j}(h_{t,\omega}) := S_{m,k}(\omega)t^{m-1-2j}/v(t)$$

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and $J: = \{(m, k, j) \in Z^3;\ m \geq 3,\ 1 \leq k \leq h(m) \text{ and } 1 \leq j \leq [(m-1)/2]\}$.

Conversely, the functions $p_{m,k,j}(h_{t,\omega})$, $(m, k, j) \in J$, form an orthogonal system in $L^2(H(\rho), \nu)$ and we can check that every $p_{m,k,j}$ belongs to the null space $N_\rho$.

What we have proved is summarized below.

THEOREM 7. *Let $R_\rho^*$ be the dual Radon transform on $L^2(H(\rho), \nu)$, where $\nu$ is a measure of the form* (20). *Then we have*

$$N_\rho = [p_{m,k,j}(h_{t,\omega});\ (m, k, j) \in J]\,.$$

*In other words, a function $g$ belongs to $N_\rho$ if and only if $g$ is expressed in the form*

$$g(h_{t,\omega}) = \sum_{(m,k,j) \in J} a_{m,k,j} S_{m,k}(\omega) t^{m-1-2j} / v(t)\,. \tag{27}$$

Let $\rho$ go to infinity in the above theorem. Then we obtain, as a by-product of Theorem 7, a complete description of the full null space of $R^*$:

$$N_\infty: = \{g \in L^2(H, \nu);\ (R^*g)(x) \equiv 0,\ x \in R^n\}\,.$$

THEOREM 7′. *If the measure $\mu$ in* (20′) *is equal to* 0 (*in other words, if $d(x, y) = c|x - y|$, $c > 0$*), *then $N_\infty = \{0\}$, i.e., $R^*$ is injective on $L^2(H, \nu)$. While, if $\mu$ is positive we have $N_\infty = [p_{m,k,j}(h_{t,\omega});\ (m, k, j) \in J]$.*

**3-3.** We are now in a position to state noteworthy consequences of the preceding results. By virtue of the relation (22), our conclusion follows from Theorems 7 and 7′.

THEOREM 8. *Let $X$ be a Lévy's Brownian motion with parameter space $(R^n, d)$, where $d$ is of the form* (10). *Then we have, for $0 < \rho < \infty$,*

$$[X(x)\colon x \in V_\rho] = \Big\{\int_{H(\rho)} g(h) W(dh);\ g \in L^2(H(\rho), \nu) \text{ satisfying } \int_0^\rho t^{m-1-2j} dt \int_{S^{n-1}} S_{m,k}(\omega) g(h_{t,\omega}) \sigma(d\omega) = 0 \text{ for all } (m, k, j) \in J\Big\}.$$

*For the case $\rho = \infty$, we have*

$$[X(x);\ x \in R^n] = \Big\{\int_H g(h) W(dh);\ g \in L^2(H, \nu)\Big\}, \quad \text{if } d(x, y) = c|x - y|,\quad c > 0\,,$$

*and*

$$[X(x);\ x \in R^n] = \left\{ \int_H g(h) W(dh);\ g \in L^2(H, \nu) \ \textit{satisfying} \right.$$
$$\int_0^\infty t^{m-1-2j} dt \int_{S^{n-1}} S_{m,k}(\omega) g(h_{t,\omega}) \sigma(d\omega) = 0$$
$$\left. \textit{for all}\ (m, k, j) \in J \right\},$$

*if* $d$ *is given by* (10) *with positive* $\mu$.

**3-4.** With a suitable choice of $\alpha(x) > 0$ satisfying $\int_{R^n} \nu(B_x)\alpha(x)dx < \infty$, the Hilbert-Schmidt operator $R \circ T_\alpha$ from $L^2(R^n, \alpha(x)dx)$ to $L^2(H, \nu)$ was discussed in connection with a factorization of the covariance operator of $X$ ((I), Theorem 3). As a counterpart of the exterior Radon transform (cf. [21] and [22]), it would be interesting to study the exterior halfspace transform

(28) $$(R \circ T_\alpha f)(h_{t,\omega}) := \int_{h_{t,\omega}} f(x)\alpha(x)dx\,, \qquad f \in L^2(V_\rho^c, \alpha(x)dx)\,,$$

where the resultant function $R \circ T_\alpha f$ is considered to be in $L^2(H(\rho)^c, \nu)$.

Under the assumption that $\alpha$ is a radial function, $\alpha(x) = \alpha(|x|)$ on $V_\rho^c$, we can determine the null space of $R \circ T_\alpha$:

$$N_\rho(\alpha) := \{f \in L^2(V_\rho^c, \alpha(|x|)dx);\ (R \circ T_\alpha f)(h_{t,\omega}) \equiv 0 \ \text{on}\ H(\rho)^c\}\,.$$

First observe that, for a given $f \in L^2(V_\rho^c, \alpha(|x|)dx)$,

(29) $$(R \circ T_\alpha f)(h_{t,\omega}) = (R_\alpha^* \tilde{f})(\omega/t)\,, \qquad t > \rho \quad \text{and} \quad \omega \in S^{n-1}\,,$$

where $\tilde{f}(h_{t,\omega}) := f(\omega/t) \in L^2(H(\rho^{-1}), \nu_\alpha)$, $\nu_\alpha$ being a measure on $H(\rho^{-1})$ defined by

$$\nu_\alpha(dh_{t,\omega}) := |S^{n-1}| t^{-n-1} \alpha(1/t) dt \sigma(d\omega)\,,$$

and $R_\alpha^*$ is the dual Radon transform defined on $L^2(H(\rho^{-1}), \nu_\alpha)$. On the lines of Theorem 7, we can prove the following

PROPOSITION 9. *For* $0 < \rho < \infty$, *we have*

$$N_\rho(\alpha) = [f_{m,k,j}(x);\ (m, k, j) \in J(\alpha)]\,,$$

*where*

$$f_{m,k,j}(x) := S_{m,k}(x/|x|)|x|^{-m-n+2j}/\alpha(|x|), \qquad |x| \geq \rho\,,$$

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*and*

$$J(\alpha) := \left\{(m, k, j) \in J; \int_\rho^\infty t^{-2m-n+4j-1}(\alpha(t))^{-1}dt < \infty\right\}.$$

## § 4. The McKean processes

As in Section 3, we shall assume that $X = \{X(x);\ x \in R^n\}$ is a Lévy's Brownian motion with parameter space $(R^n, d)$, where $d$ is a rotation-invariant distance of the form (10). Since the representations (8) of the McKean processes $M_{m,k}(t)$, $(m, k) \in \varDelta$, follow from the original representation (1) of $X$, we can answer, as a byproduct of Theorem 8, the basic question concerning the canonical property of (8).

We being by applying Lemma 6 to the representation (1) of $X$; we get

$$(30) \qquad X(x) = \sum_{(m,k)\in\varDelta} S_{m,k}(x/|x|) \int_{H(|x|)} \lambda_m(t/|x|) S_{m,k}(\omega)\, W(dh_{t,\omega})$$
$$= \sum_{(m,k)\in\varDelta} S_{m,k}(x/|x|) \int_0^{|x|} \lambda_m(t/|x|) dB_{m,k}(t)\,,$$

where the Gaussian processes $B_{m,k}(t)$, $(m, k) \in \varDelta$, are defined by

$$(31) \qquad B_{m,k}(t) := \int_{H(t)} S_{m,k}(\omega) W(dh_{u,\omega})\,, \qquad t > 0\,.$$

Observe that

$$E[B_{m,k}(t)B_{m',k'}(t')] = \int_{H(t)\cap H(t')} S_{m,k}(\omega)S_{m',k'}(\omega)\nu(dh_{u,\omega})$$
$$= \delta_{(m,k),\,(m',k')} \int_0^{\min(t,t')} v(u)du\,,$$

where $v(u)$ was given by (20′). This shows that the processes $B_{m,k}(t)$ are mutually independent Gaussian additive processes with common spectral density $v(t) = E[(B_{m,k}(dt))^2]/dt$.

In view of the expression (30) of $X$, we are naturally led to the following

DEFINITION 2 (cf. [17]). The Gaussian process

$$(7) \qquad M_{m,k}(t) := \int_{S^{n-1}} X(t\omega) S_{m,k}(\omega) \sigma(d\omega)\,, \qquad t > 0\,,$$

is called the *McKean process with index* $(m, k)$, $(m, k) \in \varDelta$. In the case $m = 0$, $M_{0,1}(t)$ has a more familiar name, the *$M(t)$-process* (cf. [15]).

With this definition, the expression (30) is rewritten as follows:

$$X(x) = \sum_{(m,k)\in\Delta} S_{m,k}(x/|x|)M_{m,k}(|x|)\,,$$

and

(8) $$M_{m,k}(t) = \int_0^t \lambda_m(u/t)dB_{m,k}(u)\,,$$

where the kernel $\lambda_m(t)$ was computed in Lemma 6.

Now, Theorem 8 is rephrased in terms of these Gaussian processes $M_{m,k}(t)$ and $B_{m,k}(t)$, $(m, k) \in \Delta$.

THEOREM 10. (i) *In the case* $m \leq 2$, *the representation* (8) *of* $M_{m,k}(t)$ *is canonical, i.e.,*

$$[M_{m,k}(t); t \leq \rho] = [B_{m,k}(t);\ t \leq \rho] \qquad \textit{for every } \rho > 0.$$

(ii) *In the case* $m \geq 3$, *the representation* (8) *of* $M_{m,k}(t)$ *is not canonical. Furthermore we have*

$$\begin{aligned}&[B_{m,k}(t);\ t \leq \rho] \ominus [M_{m,k}(t);\ t \leq \rho]\\ &\quad = \Big[\int_0^\rho t^{m-1-2j}(v(t))^{-1}dB_{m,k}(t);\ 1 \leq j \leq [(m-1)/2]\Big]\end{aligned}$$

*for every* $0 < \rho < \infty$, *and*

$$\begin{aligned}&[B_{m,k}(t);\ t > 0] \ominus [M_{m,k}(t);\ t > 0]\\ &\quad = \begin{cases}\{0\} & \textit{if } d(x, y) = c|x-y|\,, \quad c > 0\,,\\ \Big[\int_0^\infty t^{m-1-2j}(v(t))^{-1}dB_{m,k}(t);\ 1 \leq j \leq [(m-1)/2]\Big]\,, & \end{cases}\end{aligned}$$

*otherwise.*

*Concluding remarks.* (i) Our discussions in Sections 3 and 4 can be extended to the case with other parameter spaces such as $M = S^n$ ($n$-sphere) or $H^n$ ($n$-dimensional real hyperbolic space). In particular, consider a familiar Lévy's Brownian motion $X$ with parameter space $(M, d_G)$, $d_G$ being the usual geodesic distance on $M = S^n$ or $H^n$ (cf. [18] and [23]). Such an $X$ admits of a nice representation ([23]) analogous to (1) for a Brownian motion with $n$-dimensional parameter. By making use of this known representation of $X$, we can show that Theorems 8 and 10 have respective counterparts in these two cases of $(S^n, d_G)$ and $(H^n, d_G)$. The details are omitted.

(ii) In their study of conformal invariance of white noise, Hida, Lee and Lee [13] introduced a generalized Gaussian random field $Y = \{Y(x); x \in R^n,\ 0 < |x| < 1\}$ defined by

(32) $$Y(x) = \int_{B_x} F(x, h_{t,\omega}) W(dh_{t,\omega}),$$

where the kernel $F$ is given by

(33) $$F(x, h_{t,\omega}) = a(x) t^{-n+1} / \{(x, \omega) - t|x|\},$$

and $W = \{W(dh_{t,\omega}); h_{t,\omega} \in H(1)\}$ is a Gaussian random measure (white noise) with variance $\nu(dh_{t,\omega}) := t^{n-1} dt \sigma(d\omega)$, $\nu$ being a measure on the set $H(1)$ of half-spaces $h_{t,\omega}$, $0 < t < 1$ and $\omega \in S^{n-1}$.

This representation (32) of $Y$ might be thought of as a multi-dimensional version of canonical representations of Gaussian processes, and takes a more general form than the representation (1) of Chentsov type (which corresponds to the choice of $F(x, h_{t,\omega}) \equiv 1$). This generality would cause us many difficulties in investigating the integral transformation $R_F^*$ associated with (32):

(34) $$(R_F^* g)(x) := \int_{B_x} F(x, h_{t,\omega}) g(h_{t,\omega}) \nu(dh_{t,\omega}),$$

$g$ being in a suitable class of functions on $H(1)$. But in the persent situation where the kernel $F$ is specified by (33) with the additional condition that $a(x) > 0$, we can prove analogous results on the null spaces $N_\rho(F)$ of $R_F^*$, $0 < \rho < 1$:

$$N_\rho(F) := \{g(h_{t,\omega}); \operatorname{supp} g \subset H(\rho) \quad \text{and } (R_F^* g)(x) \equiv 0,\ 0 < |x| \leq \rho\}.$$

Indeed, similar arguments to Section 3–2 lead us to the following conclusion:

$$N_\rho(F) = [g_{m,k,j}(h_{t,\omega});\ m \geq 2,\ 1 \leq k \leq h(m) \text{ and } 1 \leq j \leq [m/2]],$$

where we put

$$g_{m,k,j}(h_{t,\omega}) := t^{m-2j} \chi_{(0,\rho]}(t) S_{m,k}(\omega).$$

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