Can. J. Math., Vol. XXVIII, No. 3, 1976, pp. 513–522

CONTINUOUS BOUNDARY VALUES OF HOLOMORPHIC FUNCTIONS ON KÄHLER DOMAINS

BARNET M. WEINSTOCK

1. Introduction. Let M be a complex manifold of dimension n which admits a Kähler metric, and let D be a relatively compact domain on M whose boundary B is a C^∞ submanifold of M of real codimension one. The object of this paper is to use the potential theory associated with the Laplace-Beltrami operator on M to characterize the continuous functions on B which have holomorphic extensions to D.

For the case $M = \mathbf{C}^n$ the author proved [**22**] that a necessary and sufficient condition for a continuous function f on B to have a holomorphic extension to D is that

$$(1.1) \quad \int_B f\omega = 0$$

for all C^∞ forms ω of bidegree $(n, n-1)$ on $D \cup B$ which satisfy $\bar\partial\omega = 0$ in D. The idea of the proof is to show that if (1.1) holds then the harmonic extension of f to D is given by the Bochner-Martinelli formula

$$(1.2) \quad f(y) = c_n \int_B f(x)^*\partial(|x-y|^{2-2n}).$$

Differentiation under the integral sign and a second application of (1.1) then imply that the harmonic extension is actually holomorphic. The first result of the present paper (Theorem 4.1 below) is that this argument remains valid for a domain on a Kähler manifold if the kernel $|x-y|^{2-2n}$ is replaced by the Green's function for the Laplace-Beltrami operator relative to the domain.

In the presence of certain cohomological conditions on M a somewhat weaker condition than (1.1) will imply the existence of a holomorphic extension. Andreotti and Hill [**1**] proved that if D is a domain with C^∞ boundary B on a complex manifold M such that

(1.3) $M - D$ has no compact components, and

(1.4) $H_*^{0,1}(M) = 0$ (where the subscript $*$ denotes cohomology with compact supports),

then every C^∞ solution of the tangential Cauchy-Riemann equations on B has a holomorphic extension to D.

Received April 29, 1975.

For continuous functions on B we introduce the following terminology:

Definition 1.1. A continuous function f on B is a *weak solution* of the tangential Cauchy-Riemann equations if

$$\int_B f\bar{\partial}\eta = 0 \tag{1.5}$$

for all C^∞ forms η of bidegree $(n, n-2)$ in a neighborhood of B.

Every smooth solution of the tangential Cauchy-Riemann equations on B satisfies (1.5), and every smooth function on B which satisfies (1.5) is a solution of the tangential Cauchy-Riemann equations. These are simple consequences of Stokes' theorem.

The second result of this paper (Theorem 4.2 below) is that if M admits a Kähler metric and (1.3) and (1.4) are satisfied then every weak solution of the tangential Cauchy-Riemann equations on B has a holomorphic extension to D. We recall that (1.4) is always satisfied if M is a Stein manifold by the Serre duality theorem [**20**], and that every Stein manifold admits a Kähler metric [**10**]. Thus, *if M is a Stein manifold, every weak solution of the tangential Cauchy-Riemann equations on B has a holomorphic extension to D provided only that D has no compact complementary components.* This was proved for the case $M = \mathbf{C}^n$ in [**23**].

The techniques used in this paper are potential-theoretic. The theory of the Laplace operator on Riemannian and Kähler manifolds and the associated boundary-value problems, as developed by Bidal, de Rham, Kodaira, Spencer, Duff, Garabedian and others [**3**; **6**; **7**; **9**; **18**; **21**] is used to extend to domains on Kähler manifolds the arguments used in [**22**] to study holomorphic extension from the boundary for domains in Euclidean space. The specific results which are required are collected in § 2 and § 3 below.

The extendability of smooth (C^1) solutions of the tangential Cauchy-Riemann equations for domains in $\mathbf{C}^n$ was first shown by Bochner [**4**]. (See also Martinelli [**15**].) The extendability of functions satisfying (1.1) was proved by Fichera [**8**] for domains with connected boundary in $\mathbf{C}^n$ under the additional hypothesis that f be the boundary value of a function with finite Dirichlet integral. Royden [**19**] studied the extendability of integrable functions satisfying (1.1) for the case $n = 1$, i.e., for finite Riemann surfaces. Kohn and Rossi [**13**] gave conditions for the extendability of smooth functions for more general complex manifolds. The extendability of weak solutions of the tangential Cauchy-Riemann equations has also been treated by Harvey and Lawson [**11**] by other methods, as part of their general study of boundaries of complex manifolds.

2. Differential forms on complex manifolds. We collect in this section various well-known facts about differential forms on complex manifolds. The books by Wells [**24**] and Morrow and Kodaira [**17**] are useful references for much of this material.

Let M be a complex manifold of dimension n. If D is a domain on M we denote by $A^{p,q}(D)$ the space of C^∞ forms on D of bidegree (p, q) and by $A^r(D)$ the space of complex-valued C^∞ forms on D of total degree r. If D has a C^∞ boundary B then $A^{p,q}(D \cup B)$ and $A^r(D \cup B)$ denote the forms which are "C^∞ up to the boundary", i.e., which are the restrictions to D of C^∞ forms on a neighborhood of $D \cup B$. We give $A^{p,q}(D)$ the C^∞ topology of uniform convergence on compact subsets of D of all the derivatives of the coefficients (with respect to some coordinate covering.) When D is relatively compact we give $A^{p,q}(D \cup B)$ the topology of uniform convergence on $D \cup B$ of all derivatives.

We denote by $K^{p,q}(D)$ the space of currents of bidegree (p, q) on D. (For our purposes it suffices to regard currents as differential forms whose coefficients are currents of degree 0, or distributions.) We denote by $K_*^{p,q}(D)$ the space of currents of bidegree (p, q) with compact support in D.

The dual of the Fréchet space $A^{p,q}(D)$ is the space $K_*^{n-p,n-q}(D)$. The transpose of the mapping $\bar\partial : A^{p,q} \to A^{p,q+1}$ is the mapping $(-1)^{p+q+1}\bar\partial$ acting on $K_*^{n-p,n-q-1}$ [**20**, Props. 4 and 5]. The Dolbeault cohomology groups $H^{p,q}(D)$ and $H_*^{p,q}(D)$, $*$ denoting compact supports, can be computed using either C^∞ forms of bidegree (p, q) or currents of bidegree (p, q) (see [**5**]).

The Serre duality theorem [**20**] implies that if $\bar\partial(A^{p,q-1}(D))$ and $\bar\partial(A^{p,q}(D))$ are closed then the dual of the Fréchet space $H^{p,q}(D)$ is the space $H_*^{n-p,n-q}(D)$. More generally, the argument used to prove the Serre duality theorem implies the following result:

LEMMA 2.1. *If $H_*^{p,q}(D) = 0$, then $\bar\partial(A^{n-p,n-q-1}(D))$ is dense in*

$$\{\omega \in A^{n-p,n-q}(D) : \bar\partial\omega = 0\}.$$

Suppose that M has a C^∞ hermitian metric $\{g_{\alpha\beta}\}$ whose associated differential form of bidegree (1,1) is

$$\Omega = (i/2)\sum g_{\alpha\beta}dz_\alpha \wedge d\bar z_\beta.$$

We recall that Ω is a real form (i.e. identical with its complex conjugate) because of the hermitian symmetry of $\{g_{\alpha\beta}\}$. The (n, n) form $\Omega^n = \Omega \wedge \ldots \wedge \Omega$ is a real form which is non-vanishing on M and which induces the natural orientation of M as a complex manifold.

The hermitian metric on M induces a hermitian inner product on the space of convectors of bidegree (p, q) at each point of M. If this inner product at $X \in M$ is denoted by $(\varphi, \psi)_x$ then in particular $(\Omega^n, \Omega^n)_x \equiv 1$ on M.

The Hodge adjoint $* : A^{p,q} \to A^{n-q,n-p}$ is defined by the identity

$$\varphi_x \wedge * \bar\psi_x = (\varphi, \psi)_x\Omega_x^n \quad \text{for } \varphi, \psi \in A^{p,q}.$$

https://doi.org/10.4153/CJM-1976-051-1 Published online by Cambridge University Press

We extend $*$ to A^r by linearity. The adjoint operator satisfies

$$\begin{aligned} \overline{*\varphi} &= *\bar{\varphi} \\ \text{(2.1)} \quad \varphi \wedge *\bar{\psi} &= \bar{\psi} \wedge *\varphi \\ **\varphi &= (-1)^r \varphi \quad \text{if } \varphi \in A^r. \end{aligned}$$

If $\varphi, \psi \in A^r(D)$ we define an inner product $(\varphi, \psi)_D$ by the formula

$$(\varphi, \psi)_D = \int_D \varphi \wedge *\bar{\psi}$$

whenever the integral on the right converges. If $(\varphi, \varphi)_D$ is finite we define $||\varphi||$ to be $(\varphi, \varphi)_D{}^{1/2}$. We note that if $p + q = s + t$ and $\varphi \in A^{p,q}(D)$, $\psi \in A^{s,t}(D)$ then $(\varphi, \psi)_D = 0$ if $(p - s)^2 + (q - t)^2 > 0$ since $\varphi \wedge *\bar{\psi}$ is of bidegree $(n + p - s, n + q - t)$ and one of these degrees must exceed n.

The formal adjoints of the exterior differential operators d, ∂ and $\bar{\partial}$ are the operators d', ∂' and $\bar{\partial}'$ defined by the equations $d' = -*d*$, $\partial' = -*\bar{\partial}*$, and $\bar{\partial}' = -*\partial*$. Thus, for example, $\bar{\partial}'(A^{p,q}) \subset A^{p,q-1}$. If D is a relatively compact domain on M with nice boundary B, and if φ and ψ are forms of the appropriate degrees then Stokes' theorem implies that

$$\text{(2.2)} \quad \int_B \varphi \wedge *\bar{\psi} = (\bar{\partial}\varphi, \psi)_D - (\varphi, \bar{\partial}'\psi)_D$$

$$\text{(2.3)} \quad \int_B \varphi \wedge *\bar{\psi} = (\partial\varphi, \psi)_D - (\varphi, \partial'\psi)_D$$

$$\text{(2.4)} \quad \int_B \varphi \wedge *\bar{\psi} = (d\varphi, \psi)_D - (\varphi, d'\psi)_D$$

The Laplacian on M is the operator $\Delta = dd' + d'd$. The complex Laplacians $\square$ and $\bar{\square}$ are defined by $\square = \partial\partial' + \partial'\partial$ and $\bar{\square} = \bar{\partial}\bar{\partial}' + \bar{\partial}'\bar{\partial}$. It is easily seen that $\Delta(A^r) \subset A^r$, $\square(A^{p,q}) \subset A^{p,q}$ and $\bar{\square}(A^{p,q}) \subset A^{p,q}$. If $\varphi \in A^r(M)$ then φ is called harmonic on a domain $D \subset M$ if $\Delta\varphi = 0$ in D.

A complex manifold M is called a Kähler manifold if there exists a Hermitian metric on M whose associated (1,1) form Ω satisfies $d\Omega = 0$ on M. The basic property of Kähler manifolds which we utilize in this paper is the following:

THEOREM 2.2 ([**24**, V, Theorem 3.7]). *If M is a Kähler manifold then $\square = \bar{\square} = \frac{1}{2}\Delta$. In particular, $\Delta(A^{p,q}) \subset A^{p,q}$ and Δ commutes with ∂, $\bar{\partial}$ and $*$, so that every function which is holomorphic in a domain $D \subset M$ is harmonic in D.*

If we consider the Laplacian Δ acting on functions (this operator is often referred to as the Laplace-Beltrami operator) then in local coordinates $\Delta = d'd$ has the form

$$\sum h_{\alpha\beta} \frac{\partial^2}{\partial x_\alpha \partial x_\beta} - \sum h_t \frac{\partial}{\partial x_t}$$

where $\{h_{\alpha\beta}\}$ is positive-definite (cf. [**18**, § 26]). In particular, Δ is elliptic and annihilates constants, so that the maximum principle of E. Hopf for second-order elliptic operators ([**16**, Theorem 2.8.1; **25**, Theorem 2.1]) can be used to deduce the following result:

THEOREM 2.2. *If D is a relatively compact domain on M and if a complex-valued function f is harmonic in D and continuous on $D \cup B$ then $|f|$ takes its maximum on B.*

3. Potential theory on manifolds. The purpose of this section is to recall some results from the potential theory associated with the Laplacian Δ on a Kähler manifold. Some of these facts are valid, more generally, on any Riemannian manifold, and so we state them in that context.

Let D be a relatively compact domain with C^∞ boundary B on a Riemannian manifold M. If $x \in B$ let $\{x_1, \ldots, x_m\}$ be a local coordinate system in a neighborhood of x on M such that $\{x_1, \ldots, x_{m-1}\}$ is a coordinate system for a neighborhood of x on B and such that x_m is a defining function for B near x. If $\varphi = \sum a_I \, dx_I$, $I = (i_1, \ldots, i_r)$, is an r-form defined near B then $\varphi = t\varphi + n\varphi$ where $t\varphi$, the tangential component of φ, is the sum of those monomials $a_I \, dx_I$ for which $m \notin I$. The summand $n\varphi$ is called the normal component of φ. This decomposition of φ is independent of the choice of coordinates $\{x_1, \ldots, x_m\}$ having the above properties. If φ is a function then by definition $t\varphi = \varphi$ and $n\varphi = 0$. Finally, it is easily shown that $*t = n*$.

The Dirichlet problem for r-forms on D is to find a harmonic r-form ω on D with given tangential and normal components on B.

Garabedian and Spencer [**9**] showed that if the only r-form ω of class C^1 on $D \cup B$ satisfying $\Delta\omega = 0$ in D and $t\omega = n\omega = 0$ on B is the form $\omega \equiv 0$ then the Dirichlet problem has a unique solution. Indeed, for a domain D satisfying this uniqueness condition they constructed a Green's form $G_r(x, y)$, a double form of degree r in x and y, such that

(i) $\Delta_x G_r(x, y) = 0$ if $x \neq y$,
(ii) $t_x G_r(x, y) = n_x G_r(x, y) = 0$ on B,
(iii) $G_r(x, y) = G_r(y, x)$,
(iv) $G_r(x, y) = O(s^{2-m})$ for x in a small neighborhood of y, where $s(x, y)$ is the geodesic distance between x and y.

In particular, for fixed $y \in D$, $G_r(x, y)$ is of class C^∞ on $D \cup B - \{y\}$.

In terms of this Green's form the solution to the Dirichlet problem with smooth boundary data $\varphi(x)$ is given by (cf. [**9**, § 8])

$$\omega(y) = -\int_B \{\varphi(x) \wedge * d_x G_r(x, y) - d' G_r(x, y) \wedge * \varphi(x)\}. \tag{3.1}$$

Soon thereafter Spencer [**21**] established the uniqueness property for forms of bidegree $(p, 0)$, $(0, q)$, (n, q) or (p, n) on an n-dimensional Kähler manifold, and thus proved the existence of the Green's form of those bidegrees. Although this result of Spencer's suffices for the applications in the present paper we note for the sake of completeness that the unique continuation theorem of Aronszajn, Krzywicki and Szerski [**2**] implies that the above uniqueness property holds for forms of any degree on any Riemannian manifold (cf. [**16**, Theorem 7.8.3]).

Using the classical technique of integral equations on the boundary Duff [**6**] showed that the Dirichlet problem for r-forms on a relatively compact domain D with C^∞ boundary B has a solution for arbitrary continuous boundary data on B. In particular every continuous function on B is the boundary value of a harmonic function in D. Taking Theorem 2.1 and (3.1) into account we can thus state the following result:

THEOREM 3.1. *Let D be a relatively compact domain with C^∞ boundary B on a Riemannian manifold M. Let f be a continuous function on B. Then there is a unique function F which is continuous on $D \cup B$, harmonic in D, and which satisfies $F = f$ on B. Moreover, F is given by*

$$(3.2) \quad F(y) = \int_B f(x) * dG_0(x, y).$$

The Green's forms G_r enjoy several other properties which generalize those of the familiar Green's function for a domain in Euclidean space.

From (2.4) one derives easily the following Green's identity:

$$(3.3) \quad \begin{aligned}(\Delta\varphi, \psi) - (\varphi, \Delta\psi) = \int_B \{\varphi \wedge * d\bar{\psi} - \psi \wedge * d\bar{\varphi} + d'\varphi \wedge * \bar{\psi} \\ - d'\psi \wedge * \bar{\varphi}\}.\end{aligned}$$

Also, the Green's form G_r satisfies

$$(3.4) \quad \lim_{\rho \to 0} \int_{\partial\Delta(y,\rho)} \varphi \wedge * dG_r - d'G_r \wedge * \bar{\varphi} = -\varphi(y)$$

and

$$(3.5) \quad \lim_{\rho \to 0} \int_{\partial\Delta(y,\rho)} G_r \wedge * d\bar{\varphi} - d'\varphi \wedge * G_r = 0.$$

where $\Delta(y, \rho)$ is the geodesic ball of radius $\frac{1}{2}$ about y. (Formula (3.5) is a simple consequence of the fact that $G_r = O(s^{2-n})$, while (3.4) follows from [**3**, pp. 27–31]).

If we apply (3.3) to the domain $D - \Delta(y. \rho)$ with ρ small, with $\psi = G_r$, and use (3.4) and (3.5), we obtain

THEOREM 3.3. *If φ is a smooth form of degree r on $D \cup B$ such that $\varphi = 0$ on B then*

$$(3.6) \quad \varphi(y) = \int_D (\Delta\varphi)(x) * G_r(x, y).$$

If M is a Kähler manifold then $\Delta(A^{p,q}) \subset A^{p,q}$. It follows that the Green's

https://doi.org/10.4153/CJM-1976-051-1 Published online by Cambridge University Press

form G_r for a domain $D \subset M$ can be written as a sum

$$G_r = \sum_{p+q=r} G_{p,q}$$

where $G_{p,q}(x, y)$ is of bidegree (p, q) in x, and of bidegree (q, p) in y, so that if $\varphi \in A^{p,q}$, and $\varphi = 0$ on B, (3.6) takes the form $\varphi(y) = (\Delta_x \varphi, G_{p,q}(x, y))_D$.

From the fact that Δ commutes with $\bar{\partial}$ and $\bar{\partial}'$ when M is Kähler we obtain the following identity:

THEOREM 3.4. *If D is a relatively compact domain with C^∞ boundary on a Kähler manifold then for each $y \in D$ there is a C^∞ function $H_y(x)$ on $D \cup B$ such that*

$$(3.7) \quad \partial_x' G_{1,0}(x, y) - \bar{\partial}_y G_0(x, y) = H_y(x)$$

and

$$(3.8) \quad \Delta_x H_y(x) = 0 \quad \text{in } D.$$

Proof. Since Δ commutes with ∂', (3.8) follows from (3.7). To prove (3.7) let f be a smooth function in a neighborhood of $D \cup B$. By Theorem 3.3,

$$\Delta_y (f, G_0)_D = f(y)$$

so that

$$\partial_y f = \partial_y \Delta_y (f, G_0)_D = \Delta_y \partial_y (f, G_0)_D = \Delta_y (f, \bar{\partial}_y G_0)_D.$$

But the same theorem also implies that

$$(3.9) \quad \partial_y f = \Delta_y (\partial_x f, G_{1,0})_D$$

and, since $G_{1,0} = 0$ on B, we conclude from (2.2) that

$$\partial_y f = \Delta_y (f, \partial_x' G_{1,0})_D,$$

the integration by parts being justified by the nature of the singularity of the Green's form.

Now $(f, \partial_x' G_{1,0})_D - (f, \bar{\partial}_y G_0)_D$ is a harmonic $(1, 0)$ form on D, and from (3.9) we see that $(f, \bar{\partial}_x' G_{1,0}) = 0$ for $y \in B$. Using (3.1) and interchanging the order of integration we obtain

$$(f(x), [\partial_x' G_{1,0} - \bar{\partial}_y G](x, y))_D$$

$$= \left(f(x), \int_B \{d_z' G_1(z, y) \wedge * \bar{\partial}_z G_0(x, z) - \bar{\partial}_z G_0(x, z) \wedge * d_z G_1(z, y)\}\right)_D.$$

This proves (3.7) if we define $H_y(x)$ to be the boundary integral in the previous line, since $H_y(x)$ is clearly C^∞ in D for fixed $y \in D$, and since the left side of (3.7) is smooth up to the boundary for fixed y.

4. Boundary values of holomorphic functions.

THEOREM 4.1. *Let M be a Kähler manifold of dimension n and let D be a relatively compact domain on M whose boundary B is a C^∞ submanifold of real codimension one. A continuous function f on B has a continuous extension F to $D \cup B$ which satisfies $\bar\partial F = 0$ in D if and only if*

$$\int_B f\omega = 0$$

for all $\omega \in A^{n,n-1}(D \cup B)$ such that $\bar\partial\omega = 0$ in D.

Proof. The necessity follows immediately from Stokes' theorem since if F is continuous on $D \cup B$, $\bar\partial F = 0$ in D, and $\partial\omega = 0$ in D, then

$$\int_B F\omega = \int_D d(F\omega) = \int_D \bar\partial(F\omega) = 0.$$

To prove the converse, let F be the harmonic extension of f to D. By Theorem 3.2,

$$F(y) = \int_B f(x) * d_x G_0(x, y).$$

But $*dG_0$ and $2*\partial G_0$ define the same $2n - 1$ form on B. This follows from the fact that $*dG_0 - 2*\partial G_0 = *\bar\partial G_0 - *\partial G_0$ and the identity

$$dG_0 \wedge (*\bar\partial G_0 - *\partial G_0) = \partial G_0 \wedge *\bar\partial G_0 - \bar\partial G_0 \wedge *\partial G_0$$

together with (2.1) since G_0 is real and is constant on B. Thus

$$F(y) = 2\int_B f(x) * \partial_x G_0(x, y)$$

(cf. Royden [**19**] and Weinstock [**22**]).

Now by Theorem 3.4,

$$\partial_y F(y) = 2\int_B f(x) * \partial_x \bar\partial_y G_0(x, y)$$
$$= 2\int_B f(x) * \partial_x(\partial_x{}' G_{0,1}(x, y) + H(x, y))$$

But for fixed $y \in D$, $\bar\partial_x *\partial_x H = 0$ since H is harmonic, i.e., $*\partial_x H$ is a $\bar\partial$-closed $(n, n-1)$ form on $D \cup B$, so $\int_B f*\partial H = 0$ by hypothesis. Also, $G_{0,1}(x, y)$ is harmonic so

$$0 = \partial\partial' G_{0,1} + \partial'\partial G_{0,1}.$$

Thus,

$$\bar\partial_y F(y) = 2\int_B f(x) * \partial_x \partial_x{}' G_{0,1}(x, y) = -2\int_B f(x) * \partial_x{}' \partial_x G_{0,1}(x, y)$$
$$= -2\int_B f(x)\bar\partial_x * \partial_x G_{0,1}(x, y),$$

since $\partial' = -*\bar\partial*$. Now for fixed $y \in D$, choose $\Omega_y(x) \in A^{n,n-2}(D \cup B)$ which

https://doi.org/10.4153/CJM-1976-051-1 Published online by Cambridge University Press

agrees with $*\partial_x G_{0,1}(x, y)$ in a neighborhood of B. Then

$$\bar{\partial}_y F(y) = -2 \int f(x) \bar{\partial} \Omega_y(x) = 0,$$

since $\bar{\partial}^2 = 0$, i.e. F is holomorphic.

THEOREM 4.2. *Let M be a Kähler manifold of dimension M and let D be a relatively compact domain on M whose boundary B is a C^∞ submanifold of real codimension one. Suppose that conditions* (1.3) *and* (1.4) *are satisfied. Then a continuous function f on B has a holomorphic extension to D if and only if f is a weak solution of the tangential Cauchy-Riemann equations on B.*

In view of Definition 1.1, it is clear that Theorem 4.2 is an immediate consequence of Theorem 4.1, Lemma 2.1, and the following approximation theorem (cf. [**23**]).

THEOREM 4.3. *Let D be a relatively compact domain on a complex manifold M of dimension n whose boundary B is a C^1 submanifold of real codimension one. Suppose that* (1.3) *and* (1.4) *are satisfied. Then every $\omega \in A^{n,n-1}(D \cup B)$ such that $\bar{\partial}\omega = 0$ in D can be approximated in $A^{n,n-1}(D \cup B)$ by a sequence $\{\Omega_\nu\}$ where $\Omega_\nu \in A^{n,n-1}(M)$, and $\bar{\partial}\Omega_\nu = 0$ in M.*

Proof. Let $E = \{\Omega \in A^{n,n-1}(M) : \bar{\partial}\Omega = 0\}$. We must show that $E|D \cup B$ is dense in $F = \{\omega \in A^{n,n-1}(D \cup B) : \bar{\partial}\omega = 0 \text{ in } D\}$.

Now every continuous linear functional on $A^{n,n-1}(D \cup B)$ can be regarded as a current on M with support in $D \cup B$, i.e., as an element of $K_*^{0,1}(M)$. If $T \in K_*^{0,1}(M)$ and $T(\Omega) = 0$ for $\Omega \in E$ then in particular T is orthogonal to the kernel of the mapping $\bar{\partial} : A^{n,n-1}(M) \to A^{n,n}(M)$, hence T is in the weak* closure of the image of the transpose ${}^t\bar{\partial}$, which is the mapping $\bar{\partial} : K_*^{0,0}(M) \to K_*^{0,1}(M)$. But this mapping has closed image since $H_*{}^{0,1}(M) = 0$, thus $T = \bar{\partial}S$, $S \in K_*^{0,0}(M)$. To see that S is supported in $D \cup B$, observe that $\bar{\partial}S = 0$ in $M - (D \cup B)$ hence S is an analytic function in $M - (D \cup B)$. But S has compact support, and $M - D$ has no compact components. Thus $S \equiv 0$ in $M - D$.

To complete the proof, use a partition of unity to write $S = S_0 + \sum S_i$ where S_0 has compact support in D and each S_i is supported in a small coordinate neighborhood of some boundary point. Clearly $S_0(\omega) = 0$ for all $\omega \in F$, and the same holds for each S_i since each S_i can be approximated by a translate in the direction of a suitable outward normal.

We remark that Theorem 4.3 is valid under weaker hypotheses. In particular, it suffices to assume that $H^{n,n}(M) = 0$, since this implies that the mapping $\bar{\partial} : A^{n,n-1}(M) \to A^{n,n}(M)$ has closed range, which by a well-known property of Fréchet spaces (cf. [**12**]) guarantees that its adjoint $\bar{\partial} : K_*^{0,0}(M) \to K_*^{0,1}(M)$ has closed range. Moreover, by a theorem of Malgrange [**14**] the condition $H^{n,n}(M) = 0$ is satisfied by every non-compact manifold which admits a real-analytic Kähler metric, so in particular by every Stein manifold [**10**].

References

1. A. Andreotti and C. Denson Hill, *E. E. Levi convexity and the Hans Lewy problem, Part I: Reduction to vanishing theorems*, Ann. Scuola Norm. Sup. Pisa *26* (1972), 325–363.
2. N. Aronszajn, A. Krzywicki and J. Szarski, *A unique continuation theorem for exterior differential forms on Riemannian manifolds*, Arkiv Mat. *4* (1962), 417–453.
3. P. Bidal and Georges de Rham, *Les formes différentielles harmoniques*, Comm. Mat. Helv. *19* (1946), 1–49.
4. S. Bochner, *Analytic and meromorphic continuation by means of Green's formula*, Ann. Math. *44* (1943), 652–673.
5. P. Dolbeault, *Formes différentielles et cohomologie sur une variété analytique complexe, I*, Ann. Math. *64* (1956), 83–130.
6. G. F. D. Duff, *Boundary value problems associated with the tensor Laplace equation*, Can. J. Math. *5* (1953), 196–210.
7. G. F. D. Duff and D. C. Spencer, *Harmonic tensors on Riemannian manifolds with boundary*, Ann. Math. *56* (1952), 128–156.
8. G. Fichera, *Caratterizzazione della traccia sulla frontiera di un campo di una funzione analitica di più variabili complesse*, Atti Accad. Nat. Lincei Rend. (8) *22* (1957), 706–715.
9. P. R. Garabedian and D. C. Spencer, *A complex tensor calculus for Kähler manifolds*, Acta Math. *89* (1953), 279–331.
10. H. Grauert, *Charakterisierung der Holomorphiegebiete durch die vollständige Kählersche metrik*, Math. Ann. *131* (1956), 38–75.
11. F. R. Harvey and H. Blaine Lawson, Jr., *Boundaries of complex analytic varieties*, Bull. Amer. Math. Soc. *80* (1974), 180–183.
12. J. L. Kelley and Isaac Namioka *et al.*, *Linear topological spaces* (D. Van Nostrand, Princeton, N. J., 1963).
13. J. J. Kohn and Hugo Rossi, *On the extension of holomorphic functions from the boundary of a complex manifold*, Ann. Math. *81* (1965), 451–472.
14. B. Malgrange, *Existence et approximation des solutions des équations aux dérivées partielles et des équations de convolution*, Ann. Inst. Fourier Grenoble *6* (1955–56), 271–355.
15. E. Martinelli, *Sopra una dimonstrazione di R. Fueter per un teorema di Hartogs*, Comm. Mat. Helv. *15* (1942), 340–349.
16. C. B. Morrey, Jr., *Multiple integrals in the calculus of variations* (Springer-Verlag, New York, 1966).
17. J. Morrow and K. Kodaira, *Complex manifolds* (Holt, Rinehart and Winston, New York, 1971).
18. G. de Rham, *Variétés différentiables* (Hermann, Paris, 1960).
19. H. Royden, *The boundary values of analytic and harmonic functions*, Math. Z. *78* (1962), 1–24.
20. J.-P. Serre, *Une théorème de dualité*, Comm. Mat. Helv. *29* (1955), 9–26.
21. D. C. Spencer, *Cauchy's formula on Kähler manifolds*, Proc. Nat. Acad. Sci. *38* (1952), 76–80.
22. B. Weinstock, *Continuous boundary values of analytic functions of several complex variables*, Proc. Amer. Math. Soc. *21* (1969), 463–466.
23. ——— *An approximation theorem for ∂-closed forms of type $(n, n-1)$*, Proc. Amer. Math. Soc. *26* (1970), 625–628.
24. R. O. Wells, Jr., *Differential analysis on complex manifolds*. (Prentice-Hall, Englewood Cliffs, N. J., 1973).
25. K. Yano and Salomon Bochner, *Curvature and Betti numbers*, Annals of Mathematics Studies *32* (Princeton University Press, Princeton, N. J., 1953).

University of Kentucky,
Lexington, Kentucky

https://doi.org/10.4153/CJM-1976-051-1 Published online by Cambridge University Press