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LINEAR HAHN–BANACH EXTENSION OPERATORS

by BRAILEY SIMS and DAVID YOST

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Given any subspace N of a Banach space X, there is a subspace M containing N and of the same density character as N, for which there exists a linear Hahn–Banach extension operator from M^* to X^*. This result was first proved by Heinrich and Mankiewicz [**4**, Proposition 3.4] using some of the deeper results of Model Theory. More precisely, they used the Banach space version of the *Löwenheim–Skolem theorem* due to Stern [**11**], which in turn relies on the Löwenheim–Skolem and Keisler–Shelah theorems from Model Theory. Previously Lindenstrauss [**7**], using a finite dimensional lemma and a compactness argument, obtained a version of this for reflexive spaces. We shall show that the same finite dimensional lemma leads directly to the general result, without any appeal to Model Theory.

Using Model Theoretic methods, Heinrich and Mankiewicz [**4**] developed a substantial theory for Lipschitz and uniform homeomorphisms of Banach spaces. A careful reading of their work shows that their results on Lipschitz homeomorphisms, and certain of their results on uniform homeomorphisms [**4**, Proposition 4.1 and Theorem 5.1], follow from the above result (on linear extension operators), without any further need for Model Theory. Thus our proof provides a purely analytic approach to these aspects of their theory.

Let X be a Banach space and let M be a closed subspace of X. For each bounded linear functional $f: M \to \mathbf{R}$; that is, for f an element of the dual space M^*, we define

$$H_M(f) := \{\tilde{f} \in X^* : \|\tilde{f}\| = \|f\| \text{ and } R_M \tilde{f} = f\},$$

where $R_M: X^* \to M^*$ is the natural restriction operator $\tilde{f} \mapsto \tilde{f}|_M$. Thus $H_M(f)$ is the set of *Hahn–Banach extensions* of f to X. It is **nonempty**, courtesy of the Hahn–Banach Theorem, w^*—**compact** and **convex**.

A selector $T: M^* \to X^*$ with $Tf \in H_M(f)$ for all $f \in M^*$ is a *Hahn–Banach extension operator* for M. Clearly such a T is norm preserving.

It is natural to consider the question of when T can be chosen to be linear. Clearly this is always the case when X is a Hilbert space. That the converse is also true is demonstrated in the proposition below.

We begin with the following easily verified observations.

Observation (1). If $T: M^* \to X^*$ is a linear Hahn–Banach extension operator, then $P := TR_M: X^* \to X^*$ is a norm-1 projection with range $T(M^*)$ and $Ker\,P = M^\perp$.

Observation (2). (i) If M is the range of a norm-1 projection P on X, then P^* is a linear Hahn–Banach extension operator from M^* to X^*.

Dually,

(ii) If $T: M^* \to X^*$ is a linear Hahn–Banach extension operator, then T^* is a norm-1 projection of X^{**} onto M^{**}.

Proposition. *Every (2-dimensional) subspace of X admits a linear Hahn–Banach extension operator if (and only if) X is a Hilbert space.*

Proof. Let N be a 3-dimensional subspace of X and let M be a 2-dimensional subspace of N. Then by (2) (ii) there exists a norm-1 projection from X^{**} onto $M^{**} = M$, and so by restriction from N onto M. Hence by Kakutani [5] X is a Hilbert space.

An isomorphic version of this is given by Fakhoury [1, Théorème 3.7].

The above proposition suggests that subspaces which admit a linear Hahn–Banach extension operator may not be very common. On the other hand, the main theorem shows that, in some sense, subspaces with this property are plentiful. As previously noted, our proof needs the following lemma due to Lindenstrauss [7].

Lemma. *Let F be a finite dimensional subspace of X and let $k \in N$ and $\varepsilon > 0$ be given. Then there exists a finite dimensional subspace $Z > F$ such that for all subspaces $E \geqq F$ with $\dim E/F \leqq k$ there is a linear mapping $T: E \to Z$ with $\|T\| \leqq 1 + \varepsilon$ and $T|_F = Id$.*

Theorem. *Let N be a subspace of a Banach space X. Then there exists a subspace $M \geqq N$ with dens $M =$ dens N and a linear Hahn–Banach extension operator $T: M^* \to X^*$.*

Proof. We first prove the result for N separable. Let $(x_n)_{n=1}^\infty$ be a dense sequence in N. Starting with $M_0 := \{0\}$ we inductively define subspaces M_n by: M_n is the subspace Z given by the above lemma with $F := \langle M_{n-1}, x_n \rangle$, $k := n$ and $\varepsilon := 1/n$. Put $M := \overline{\bigcup_n M_n}$. Clearly M is separable and contains N.

Now for each n define

$$I_n := \{E \leqq X : E \geqq M_n \quad \text{and} \quad \dim E/M_n \leqq n\},$$

and let

$$I := \bigcup_n I_n.$$

Since $E_n \in I_n$ and $E_m \in I_m$ implies $E_n + E_m + M_{\dim E_n + \dim E_m} \in I_{\dim E_n + \dim E_m}$ we have that I,

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ordered by inclusion, is a directed set. Hence the family of sets of the form $\{E\in I: E\supseteq E_0\}$, with $E_0\in I$, is a subbase for a filter. Let U be any extension of this filter to an ultra-filter on I. Further we note that for $x\in X$ we have that $I_x:=\{E\in I: x\in E\}=\{E\in I: E\supseteq\langle x_1,x\rangle\}\in U$.

For each $E\in I$ let $n(E):=max\,\{n: E\in I_n\}$, which exists since the dimension of E is finite. Then by the lemma there exists $T_E: E\to M_{n(E)+1}<M$ with $T_E|M_{n(E)}=Id$ and $\|T_E\|\leqq 1+1/n(E)$.

Extend T_E (non-linearly) to X by setting

$$\tilde{T}_E(x):=\begin{cases} T_E(x), & \text{if } x\in E;\\ 0, & \text{otherwise.}\end{cases}$$

Regarding $\tilde{T}_E(x)$ as an element of M^{**} we define T on M^* by

$$T(f)(x):=\lim_U(\tilde{T}_Ex)(f).$$

For the definition and existence of limits over ultra-filters in compact Hausdorff spaces see, for example, [**10**].

It is now routine to verify that T is the required linear Hahn–Banach extension operator for M. For example, to establish that $T(f)\in X^*$, we need only note that given $x,y\in X$ we have

$$\tilde{T}_E(x+y)=T_E(x+y)=T_Ex+T_Ey=\tilde{T}_Ex+\tilde{T}_Ey,$$

for all $E\in I_x\cap I_y\in U$.

The general result we now establish by transfinite induction on *dens* N.

Suppose $dens\,N>\aleph_0$. Let η be the first ordinal of cardinality $dens\,N$, and let $\{x_\alpha:\alpha<\eta\}$ be dense in N. The argument above yields a separable subspace $M_{\aleph_0}$ containing $\{x_\alpha:\alpha<\aleph_0\}$ and a linear Hahn–Banach extension operator $T_{\aleph_0}: M^*_{\aleph_0}\to X^*$ By the induction hypothesis, if $\aleph_0<\alpha<\eta$ we can find a subspace M_α with $\bigcup_{\beta<\alpha}M_\beta\cup\{x_\alpha\}\subset M_\alpha$, $dens\,M_\alpha\leqq card\;\alpha$ and a linear Hahn–Banach extension operator $T_\alpha: M^*_\alpha\to X^*$.

Put $M:=\overline{\bigcup_{\alpha<\eta}M_\alpha}$. Clearly $N\leqq M$ and $dens\,M=dens\,N$. Now define $T: M^*\to X^*$ by

$$T(f):=w^*-\lim_U\,T_\alpha R_{M_\alpha}(f),$$

where U is any non-trivial ultra-filter on $\{\alpha:\alpha<\eta\}$. It is readily verified that T is a linear Hahn–Banach extension operator for M, thereby establishing the theorem.

Combining this result with Observation (1) we have the following.

Corollary [4, p. 227]. *If X is the dual of a non-separable Banach space then X contains uncountably many proper norm-one complemented subspaces.*

Similarly from Observation 2(ii) we have

Corollary [7]. *If X is reflexive then, every subspace of X is contained in a norm-one complemented subspace with the same density character.*

Various other results on Hahn–Banach extension operators are scattered throughout the literature. Questions concerning uniqueness of extensions, existence of linear selections and continuity properties of the mapping $f \mapsto H_M(f)$ arise naturally. We shall conclude with a few observations and a brief survey of known results.

Let us note that **the mapping $f \mapsto H_M(f)$ is norm to w^*-upper semicontinuous**. To see this let $\|f_n - f\| \to 0$ in M^* and let N be a w^*-neighbourhood of $H_M(f)$. If $H_M(f_n)$ is not eventually in N we may, by passing to a subsequence if necessary, assume that $H_M(f_n) \subseteq N$ for any n. But, then for each n there exists $g_n \in H_M(f_n) \backslash N$. Let (g_{n_α}) be a subnet converging w^* to g, then $g \notin N$. Now

$$\|g\| \leqq \liminf_\alpha \|g_{n_\alpha}\| = \liminf_\alpha \|f_{n_\alpha}\| = \|f\|$$

and for $m \in M$, $g(m)$ is a cluster point of the sequence $(g_n(m)) = (f_n(m))$ which converges to $f(m)$. Thus $g \in H_M(f)$, contradicting $g \notin N$.

The question of when T is unique (that is, when $H_M(f)$ is a singleton set for all $f \in M^*$) has been considered by Taylor [**13**] and Foguel [**2**]. Their results show that there is a unique Hahn–Banach extension operator for every $M \leqq X$ if and only if X^* is strictly convex. Phelps [**9**] proved that a given subspace M has a unique Hahn–Banach extension operator if and only if $M^\perp$ contains a unique closest point to each element of X^*.

When T is unique it is by the above result norm to w^* continuous. We ask, is the converse also true? That is, if there is a norm to w^* continuous Hahn–Banach extension operator for M, is $H_M(f)$ necessarily a singleton set for each $f \in M^*$? The analogy with the Duality map, see for example [**3**], should be noted.

We finally summarize sufficient conditions for a fixed subspace to admit a linear Hahn–Banach extension operator.

Fakhoury [**1**, Corollaire 2.16] shows that a subspace M admits a linear Hahn–Banach extension operator to $\langle M, x \rangle$ for each x in X if and only if for every n the following condition $\mathscr{P}_n$ is satisfied: If $m_1, m_2, \ldots, m_n \in M$ and $r_1, r_2, \ldots, r_n > 0$ are such that $\bigcap_1^n B_{r_i}(m_i) \neq \emptyset$, then $M \cap \bigcap_1^n B_{r_i}(m_i) \neq \emptyset$. Lima [**6**, Proposition 3.2] gave a different proof of this result, via a consideration of the following question: When, given $n > 2$, and any $f_1, f_2, \ldots, f_n \in M^*$ with $\sum_1^n f_i = 0$, can we find $\tilde{f}_i \in H_M(f_i)$ with $\sum_1^n \tilde{f}_i = 0$? He characterized such M as those for which $\mathscr{P}_n$ holds for the prescribed n [**6**, Theorem 3.1]. He also showed [**6**, Theorem 4.8] that if $\mathscr{P}_n$ holds for all n and in addition M is weakly Hahn–Banach smooth in X [**12**], then M admits a linear Hahn–Banach extension operator to all of X. In particular this last condition is satisfied when the extension operator is unique.

An alternative condition sufficient for a linear Hahn–Banach extension operator for M follows directly from Pełczyński [**8**, pp. 61–62]. Namely: M admits a linear Hahn–Banach extension operator if there exists a *retract* $R: X \to M$ such that for some $r > 0$ and all $x, y \in X$ we have $\|Rx - Ry\| \leqq r$ whenever $\|x - y\| \leqq r$. This is also proved in [1, Corollaire 2.12].

We remark that if $M \leqq X$ and M is a Lindenstrauss space (that is, an L_1-predual) then there is a linear Hahn–Banach extension operator from M^* to X^*. This follows directly from the injectivity of M^{**}. See Fakhoury [1, Corollaire 3.3].

Fakhoury [1, Théorème 3.1] shows that a subspace M of X admits a linear Hahn–Banach extension operator if and only if every finite rank (compact/weakly compact) linear operator from M into another Banach space has a finite rank (compact/weakly compact) norm preserving extension to all of X.

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Department of Mathematics, Statistics and Computing Science
University of New England
Armidale
N.S.W. 2351
Australia

Department of Mathematics
Institute of Advanced Studies
Australian National University
G.P.O. Box 4
A.C.T. 2601
Australia

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