107.09 A metric relation on the triangle and the circle

Introduction

In this short Note we study a metric relation between the triangle and the circle. As the Pythagorean theorem is a metric relation in a right-angled triangle, the following theorem is a metric relation between a circle and a triangle of which one of the sides is tangent to the circle and the opposite vertex belongs to the circle. We use a coordinate method to prove this relation.

Theorem

Let ABC be a triangle in the plane such that the line passing through the point A and the point C is tangent to a circle with centre O, and the point B and the point C belong to this same circle. Let $a, b, c, r \in R_+^*$ be such that $AB = c$, $AC = b$, $BC = a$ and r is the radius of the circle centre O. Then, the following relation holds:

$$r^2 = \frac{a^4b^2}{2\left(a^2b^2 + a^2c^2 + b^2c^2\right) - a^4 - b^4 - c^4}.$$

Proof

Let the cartesian coordinate system of the plane P be given by $\left(G, \vec{i}, \vec{j}\right)$. The point G is the origin of the coordinate system and the two vectors $\vec{i}, \vec{j}$ are orthogonal and unitary.

Let $d, f, p, q \in R$ be such that $d \neq 0, f \neq 0$. Let C_1 be the circle with centre O. The points B and C belong to the circle C_1. By rotation and translation, which are isometries, we may take the point O as $(0, 0)$, the point C as $(d, 0)$ and the point B as (p, q).

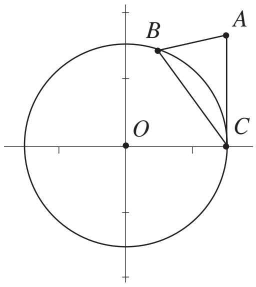

FIGURE 1

Let d_1 be the line tangent to the circle C_1 passing through the point C. The equation of the line d_1 is

$$x = d.$$

The point A belongs to the line d_1, therefore $A\,(d, f)$.

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Then

$$\begin{aligned} a^2 &= (d-p)^2 + q^2 \\ b^2 &= f^2 \\ c^2 &= (d-p)^2 + (f-q)^2 \\ r^2 &= d^2 = p^2 + q^2. \end{aligned}$$

Then, $c^2 = a^2 + b^2 - 2fq$, $\dfrac{c^2 - a^2 - b^2}{-2f} = q$, $\dfrac{(c^2 - a^2 - b^2)^2}{4b^2} = q^2$. Also $|q|$ is the height of B from OC in the triangle OBC.

Let A_1 be the area of the isosceles triangle OBC with base BC, as shown in Figure 1, so that

$$A_1 = \frac{a\sqrt{r^2 - \frac{1}{4}a^2}}{2}.$$

Let A_2 be the area of the triangle OBC with base OC and height $|q|$, so

$$A_2 = \tfrac{1}{2}r\,|q|\,.$$

Then,

$$\frac{r\,|q|}{2} = \frac{a\sqrt{r^2 - \frac{1}{4}a^2}}{2} \Rightarrow |q| = \frac{a\sqrt{4r^2 - a^2}}{2r} \Rightarrow q^2 = a^2 - \frac{a^4}{4r^2}.$$

Hence,

$$\frac{(c^2 - a^2 - b^2)^2}{4b^2} = a^2 - \frac{a^4}{4r^2}.$$

And so,

$$r^2 = \frac{a^4b^2}{2\left(a^2b^2 + a^2c^2 + b^2c^2\right) - a^4 - b^4 - c^4}.$$

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107.10 Trapezia whose side-lengths form an arithmetic progression

Everyone can easily verify that there are quadrangles with side-lengths forming an arithmetic progression. Are there trapeziums among such quadrangles? The following theorem answers this question.

Theorem

There is no trapezium for which the lengths of consecutive sides form an arithmetic progression.