A FAMILY OF DISTRIBUTIONS WITH THE SAME RATIO PROPERTY AS NORMAL DISTRIBUTION

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1. Introduction. If U and V are independent random variables, both drawn from the Normal distribution, then it is known that the distribution of U/V follows the Cauchy law, i.e. has frequency function $1/\pi(1+x^2)$. Conversely if U and V are independently drawn from the same distribution and U/V is known to follow the Cauchy law of distribution must U and V be necessarily drawn from a Normal distribution? This question has been considered by several authors ([3], [4], [5], [7]) who have obtained several examples to show that the ratio property above is not confined to the Normal distribution.

In this paper I shall discuss a family of distributions each of which possesses the ratio property described above. Denoting one member of the family by $D_\nu(x)$ and its associated frequency function by $N_\nu(x)$, where $\nu \geq 0$, we have the following definitions:

$$(1) \qquad x \geq 0 \ ; \ N_\nu(x) = \frac{x^\nu e^{-x^2/2}}{(2\pi)^{1/2} 2^{\nu/2}} \frac{\Gamma(\frac{1}{2}\nu+1)}{\Gamma(\nu+1)} \ {}_1F_1(\tfrac{1}{2}\nu \ ; \nu+1 \ ; \tfrac{1}{2}x^2) \ ,$$

and

$$(2) \qquad x < 0 \ ; \ N_\nu(x) = N_\nu(-x) \ ,$$

where the Hypergeometric function ${}_1F_1$ is defined by

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$$(3) \quad \frac{\Gamma(\frac{1}{2}\nu)}{\Gamma(\nu+1)} \; {}_1F_1(\frac{1}{2}\nu\,;\;\nu+1\,;\;\frac{1}{2}x^2) = \sum_{m=0}^{\infty} \frac{\Gamma(\frac{1}{2}\nu+m)}{m!\,\Gamma(\nu+1+m)} \frac{x^{2m}}{2^m} .$$

The case $\nu < 0$ is excluded. Evidently the Hypergeometric function ${}_1F_1$ converges for all values of x, $N_\nu(x) \geq 0$ for all real values of x and $N_0(x)$ is the frequency function of the Normal distribution.

We shall now prove that if U and V are two independent random variables, both drawn from the same distribution $D_\nu(x)$, with frequency function $N_\nu(x)$ given in (1) and (2) above, then the distribution of U/V is the Cauchy distribution with frequency function $1/\pi(1+x^2)$.

This result is proved if we can establish the following two equations

$$(4) \qquad 2\int_0^\infty N_\nu(x)\,dx = 1$$

and

$$(5) \qquad 2\int_0^\infty u\,N_\nu(ux)\,N_\nu(u)\,du = \frac{1}{\pi(1+x^2)} .$$

If (4) is established then, by virtue of (2) and $N_\nu(x) \geq 0$, it follows that $N_\nu(x)$ is the frequency function of some distribution.

To discuss (5) we first note that if U and V are independent random variables both drawn from the same distribution, with frequency function $f(x)$ where $f(x) = f(-x)$, then it is known that the distribution of U/V has $2\int_0^\infty u\,f(ux)\,f(u)\,d$ for its frequency function.

We can prove this statement as follows. Since U and V are independent the joint distribution of (U, V) is f(y) f(z). Now let $x = y/z$ and $u = z$, i.e. $y = ux$ and $z = u$; which is a transformation whose Jacobian is $|u|$. Then the joint distribution of (U/V, V) is

$$|u| \, f(ux) \, f(u) , \qquad -\infty < u, x < \infty .$$

From the theory of marginal distributions, [10] § 21.1, it then follows that the frequency function for the distribution of U/V is

$$\int_{-\infty}^{\infty} |u| \, f(ux) \, f(u) \, du = 2 \int_{0}^{\infty} u f(ux) \, f(u) \, du ,$$

where the equality arises from the assumption $f(u) = f(-u)$.

Now let U and V be random variables independently drawn from the distribution whose frequency function is $N_\nu(x)$, as given by (1) and (2). If (5) can be proved it is then evident from the remarks above that the distribution of U/V will follow the Cauchy law, i.e. have the frequency function $1/\pi(1+x^2)$.

2. <u>Proof of (4).</u> From (1) substitute for $N_\nu(x)$ in (7) below and change the order of integration and summation, assuming for the moment that such a change can be justified. On using

$$(6) \qquad \int_0^\infty e^{-x^2/2} \left(\frac{1}{2}x^2\right)^{\frac{1}{2}\nu+m} dx = 2^{-1/2} \, \Gamma\left(\frac{1}{2}\nu + \frac{1}{2} + m\right)$$

we then obtain

$$(7) \qquad 2 \int_0^\infty N_\nu(x) \, dx = \frac{\frac{1}{2}\nu}{\pi^{1/2}} \sum_{m=0}^{\infty} \frac{\Gamma\left(\frac{1}{2}\nu+m\right) \Gamma\left(\frac{1}{2}\nu+\frac{1}{2}+m\right)}{\Gamma(\nu+1+m) \; m!} .$$

To justify the change of order assumed above we use theorem B § 176 [1]. This states that if (i) ${}_1F_1(\nu; \nu+1; x^2)$

converges absolutely for $x \geq 0$ and uniformly for $0 \leq x \leq b$, where b is arbitrary, and (ii) the series on the right of (7) converges absolutely then the change of order of integration and summation is justified even if the range of integration is infinite. Condition (i) is satisfied, since ${}_1F_1$ converges for all values of x, and the investigation on the convergence of Hypergeometric functions given in [8] § 2.38 shows that (ii) is also satisfied.

Finally, on using the hypergeometric summation formula given in [8] § 14.11 we see that the right hand side of (7) is equal to 1 and this establishes (4).

3. Proof of (5). We need the following result:

$$(8) \qquad \int_0^\infty y\, J_\nu(wy)\, N_0(xy)\, dy = x^{-2} N_\nu(w/x) .$$

This is proved in [9] p. 394 equation (3), but the result stated in [9] has been rewritten so as to make use of our N_ν terminology. The case when $x = 1$ is also needed and this is

$$(9) \qquad \int_0^\infty y\, J_\nu(wy)\, N_0(y)\, dy = N_\nu(w) .$$

In both (8) and (9) $J_\nu(y)$ denotes the Bessel function of order ν.

Equations (8) and (9) are examples of Hankel transforms, see [6] chapter 2, theorem 19. The conditions of theorem 19 are satisfied here because $N_0(x)$ is of bounded variation and is absolutely integrable from 0 to ∞.

Associated with the Hankel transform is the Parseval formula, given in [6] theorem 23, p. 60. This enables us to deduce from (8) and (9) that

$$(10) \qquad \int_0^\infty w\, x^{-2} N_\nu(w/x)\, N_\nu(w)\, dw = \int_0^\infty w\, N_0(wx)\, N_0(w)\, dw .$$

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On writing $N_0(w) = (2\pi)^{-1/2} \exp.(-w^2/2)$ in the right hand integral of (10) we find that this integral takes the value $1/2\pi(1+x^2)$. On writing $w = ux$ in the left hand integral of (10) we then obtain

$$(11) \quad 2\int_0^\infty u\, N_\nu(u)\, N_\nu(ux)\, du = \frac{1}{\pi(1+x^2)}$$

and so complete the proof of (5). The statement made just before equation (4), concerning the ratio U/V, is therefore proved.

It is also possible to deduce (11) from (8) by means of a result I have given in a previous paper, [2] theorem 3, p. 582. But the method used here is somewhat quicker.

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