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A NONSPECTRAL PROBLEM FOR PLANAR MORAN–SIERPINSKI MEASURES

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Abstract

Let $M = \begin{pmatrix} \rho^{-1} & 0 \\ 0 & \rho^{-1} \end{pmatrix}$ be an expanding real matrix with $0 < \rho < 1$, and let $\mathcal{D}_n = \{\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma_n \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \gamma_n \end{pmatrix}\}$ be digit sets with $\sigma_n, \gamma_n \in \{-1, 1\}$ for each $n \geq 1$. Then the infinite convolution

$$\mu_{M,\{\mathcal{D}_n\}} = \delta_{M^{-1}\mathcal{D}_1} * \delta_{M^{-2}\mathcal{D}_2} * \cdots$$

is called a Moran–Sierpinski measure. We give a necessary and sufficient condition for $L^2(\mu_{M,\{\mathcal{D}_n\}})$ to admit an infinite orthogonal set of exponential functions. Furthermore, we give the exact cardinality of orthogonal exponential functions in $L^2(\mu_{M,\{\mathcal{D}_n\}})$ when $L^2(\mu_{M,\{\mathcal{D}_n\}})$ does not admit any infinite orthogonal set of exponential functions based on whether ρ is a trinomial number or not.

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1. Introduction

A fundamental problem in harmonic analysis is whether $E_\Lambda := \{e^{-2\pi i\langle\lambda,x\rangle} : \lambda \in \Lambda\}$ forms an orthonormal basis for $L^2(\mu)$, the space of all square-integrable functions with respect to a probability measure μ. A Borel probability measure μ on $\mathbb{R}^d$ is called a *spectral measure* if we can find a countable set $\Lambda \subset \mathbb{R}^d$ such that the set of exponential functions $E_\Lambda := \{e^{-2\pi i\langle\lambda,x\rangle} : \lambda \in \Lambda\}$ forms an orthonormal basis for $L^2(\mu)$. If such Λ exists, then Λ is called a *spectrum* for μ.

Spectral theory has been studied extensively since it was initiated by Fuglede [11] in 1974. Jorgensen and Pedersen [13] related spectral measures to fractals and gave the first example of a singular, nonatomic, fractal spectral measure. They showed that the one-fourth Cantor measure is a spectral measure, but the one-third Cantor measure is not. Further research on the spectrality and nonspectrality of measures treats self-similar measures (see [2] for a recent example), self-affine measures (see [4, 5]) and Moran measures (see [1]).

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A Sierpinski-type measure $\mu_{M,\mathcal{D}}$ is defined by

$$\mu_{M,\mathcal{D}}(\cdot) = \frac{1}{\#\mathcal{D}} \sum_{d\in\mathcal{D}} \mu_{M,\mathcal{D}}(M(\cdot) - d),$$

where $M = \left(\begin{smallmatrix} b_1 & 0 \\ 0 & b_2 \end{smallmatrix}\right)$ with $b_1, b_2 > 1$, is an expanding matrix, and $\mathcal{D} = \{\left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)\}$. The Sierpinski-type measure plays an important role in fractal geometry and geometric measure theory (see [10, 12]). In [8], Deng and Lau considered the special case $b_1 = b_2 = b$ and proved that $L^2(\mu_{M,\mathcal{D}})$ admits an infinite orthogonal set of exponential functions if and only if $b = (p/q)^{1/r}$ for some $p, q, r \in \mathbb{N}$ with $3 \mid p$, and $\mu_{M,\mathcal{D}}$ is a spectral measure if and only if $3 \mid b$. Dai *et al.* [5] generalised the results under the assumption that $b_1 \neq b_2$.

The nonspectral problem on the singular measure μ may be the start of investigating the completeness of a family of exponential functions in $L^2(\mu)$. For the Sierpinski-type measure defined above, Dutkay and Jorgensen [9] discovered that if $b_1 = b_2 = b$ and $3 \nmid b$, then there exist at most 3 mutually orthogonal exponential functions in $L^2(\mu_{M,\mathcal{D}})$. Later, Li [14] proved that if $M = \left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right)$ with $ac \notin 3\mathbb{Z}$, then there exist at most 3 mutually orthogonal exponential functions in $L^2(\mu_{M,\mathcal{D}})$ and the number 3 is best possible. The more general setting with $a, b, c \in \mathbb{R}$ was considered by Chen *et al.* [3]. Recently, Liu *et al.* [15] considered the matrix $M = \left(\begin{smallmatrix} a & b \\ d & c \end{smallmatrix}\right)$ with $ac - bd \notin 3\mathbb{Z}$ and showed that there exist at most 9 mutually orthogonal exponential functions in $L^2(\mu_{M,\mathcal{D}})$ and the number 9 is best possible. All of the known results above are concentrated on the situation when a (or b, c, d) is the rth root of a rational for $r \geq 1$.

Motivated by the above results, we will study the nonspectrality of the planar Moran–Sierpinski measure $\mu_{M,\{\mathcal{D}_n\}}$. Let

$$M = \begin{pmatrix} \rho^{-1} & 0 \\ 0 & \rho^{-1} \end{pmatrix} \quad \text{with } 0 < \rho < 1, \tag{1.1}$$

and

$$\mathcal{D}_n = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma_n \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \gamma_n \end{pmatrix} \right\} \subset \mathbb{Z}^2, \tag{1.2}$$

where $\sigma_n, \gamma_n \in \{-1, 1\}$ for each $n \geq 1$. Then there exists a Borel probability measure with compact support defined by the infinite convolution

$$\mu_{M,\{\mathcal{D}_n\}} = \delta_{M^{-1}\mathcal{D}_1} * \delta_{M^{-2}\mathcal{D}_2} * \cdots, \tag{1.3}$$

where $\delta_E = (1/\#E) \sum_{e\in E} \delta_e$ for any finite set E, δ_e is the Dirac measure at the point e and the convergence is in the weak sense. The measure $\mu_{M,\{\mathcal{D}_n\}}$ is called a *Moran–Sierpinski measure* and its support is the *Moran set*

$$T(M, \{\mathcal{D}_n\}) := \left\{ \sum_{n=1}^{\infty} M^{-n} d_n : d_n \in \mathcal{D}_n \right\}.$$

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Throughout the paper, we make the convention that all fractions have the simplest form, that is, for a fraction q/p, we have $\gcd(p,q) = 1$. We denote by r the smallest integer such that $(p/q)^r \in \mathbb{Q}$ (for example, for $\rho = (4/9)^{1/4} = (2/3)^{1/2}$, we take $r = 2$).

Our first result is the following theorem.

THEOREM 1.1. *Let M and $\mathcal{D}_n$ be defined by (1.1) and (1.2) and define the Moran measure $\mu_{M,\{\mathcal{D}_n\}}$ by (1.3). Then $L^2(\mu_{M,\{\mathcal{D}_n\}})$ admits an infinite orthonormal set of exponential functions if and only if $\rho = (q/p)^{1/r}$ for some $p, q, r \in \mathbb{N}$ with $3 \mid p$.*

The theorem indicates some connections between number theory and spectral theory. We can conclude from Theorem 1.1 that if $\rho \neq (q/p)^{1/r}$ for any $p, q, r \in \mathbb{N}$ with $3 \mid p$, then any orthogonal set of exponential functions for $L^2(\mu_{M,\{\mathcal{D}_n\}})$ is finite. In this case, we want to estimate the number of orthogonal exponential functions in $L^2(\mu_{M,\{\mathcal{D}_n\}})$ exactly. Specifically, for $\rho \in (0,1)$, we will distinguish the following cases:

- $\rho = (q/p)^{1/r}$ with $p, q, r \in \mathbb{N}$ and $\gcd(p,3) = 1$;
- ρ does not have the form $(q/p)^{1/r}$ for any $p, q, r \in \mathbb{N}$.

For simplicity, we call Λ *an orthogonal set* (respectively *a maximal orthogonal set*) for $\mu_{M,\{\mathcal{D}_n\}}$ if $\{e^{-2\pi i\langle\lambda,x\rangle} : \lambda \in \Lambda\}$ is an orthonormal family (respectively a maximal orthonormal family) for $L^2(\mu_{M,\{\mathcal{D}_n\}})$.

For the first case above, we obtain the following conclusion.

THEOREM 1.2. *Let $\rho = (q/p)^{1/r}$ for some $p, q, r \in \mathbb{N}$ with $\gcd(p,3) = 1$ and let M, $\mathcal{D}_n$ and $\mu_{M,\{\mathcal{D}_n\}}$ be defined by (1.1), (1.2) and (1.3), respectively. If Λ is an orthogonal set of $\mu_{M,\{\mathcal{D}_n\}}$, then the following statements hold:*

(i) *if $\gcd(q,3) = 1$, then $\#\Lambda \leq 3$, and 3 is best possible;*
(ii) *if $3 \mid q$, then there may be any number of elements in an orthogonal exponential set in $L^2(\mu_{M,\{\mathcal{D}_n\}})$.*

If ρ does not have the form $(q/p)^{1/r}$ for any $p, q, r \in \mathbb{N}$, we introduce the concept of trinomial number (see Definition 4.1) and prove the following theorem.

THEOREM 1.3. *Let M, $\mathcal{D}_n$ and $\mu_{M,\{\mathcal{D}_n\}}$ be defined by (1.1), (1.2) and (1.3), respectively. Suppose Λ is an orthogonal set of $\mu_{M,\{\mathcal{D}_n\}}$. Suppose furthermore that ρ does not have the form $(q/p)^{1/r}$ for any $p, q, r \in \mathbb{N}$.*

(i) *If ρ is a trinomial number with degree m, then $\#\Lambda \leq 3^{m+1}$.*
(ii) *If ρ is not a trinomial number, then $\#\Lambda \leq 3$ and 3 is best possible.*

We organise this paper as follows. In Section 2, we give the proof of Theorem 1.1. In Section 3, we prove Theorem 1.2. Finally, we prove Theorem 1.3 in Section 4.

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2. The proof of Theorem 1.1

Let μ be a Borel probability measure with compact support on $\mathbb{R}^2$. The Fourier transform of μ is defined as usual by

$$\widehat{\mu}(\xi) = \int e^{-2\pi i\langle \xi, x\rangle}\, d\mu(x)$$

for any $\xi \in \mathbb{R}^2$. We denote the zero set of $\widehat{\mu}$ by $\mathcal{Z}(\widehat{\mu})$, that is,

$$\mathcal{Z}(\widehat{\mu}) = \{\xi : \widehat{\mu}(\xi) = 0\}.$$

It is easy to show that Λ is an orthogonal set for μ if and only if

$$(\Lambda - \Lambda) \setminus \{0\} \subset \mathcal{Z}(\widehat{\mu}). \tag{2.1}$$

As the orthogonality of the set Λ is invariant under translations, without loss of generality, we always assume that $0 \in \Lambda$.

By the definition of Fourier transform of $\mu_{M,\{\mathcal{D}_n\}}$ and (1.3), for any $\xi \in \mathbb{R}^2$,

$$\widehat{\mu}_{M,\{\mathcal{D}_n\}}(\xi) = \prod_{j=1}^{\infty} \widehat{\delta}_{M^{-j}\mathcal{D}_j}(\xi).$$

Hence,

$$\mathcal{Z}(\widehat{\mu}_{M,\{\mathcal{D}_n\}}) = \bigcup_{j=1}^{\infty} M^j \mathcal{Z}(\widehat{\delta}_{\mathcal{D}_j}). \tag{2.2}$$

By a simple calculation,

$$\mathcal{Z}(\widehat{\delta}_{\mathcal{D}_n}) = \begin{cases} \frac{1}{3}\left\{\begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}2\\1\end{pmatrix}\right\} + \mathbb{Z}^2, & \sigma_n, \gamma_n = -1 \text{ or } \sigma_n, \gamma_n = 1; \\ \frac{1}{3}\left\{\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}2\\2\end{pmatrix}\right\} + \mathbb{Z}^2, & \sigma_n = -1, \gamma_n = 1 \text{ or } \sigma_n = 1, \gamma_n = -1. \end{cases} \tag{2.3}$$

For convenience, we denote

$$\mathcal{A}_1 = \frac{1}{3}\left(\begin{pmatrix}1\\2\end{pmatrix} + 3\mathbb{Z}^2\right), \quad \mathcal{A}_2 = \frac{1}{3}\left(\begin{pmatrix}2\\1\end{pmatrix} + 3\mathbb{Z}^2\right),$$

and

$$\mathcal{A}_3 = \frac{1}{3}\left(\begin{pmatrix}1\\1\end{pmatrix} + 3\mathbb{Z}^2\right), \quad \mathcal{A}_4 = \frac{1}{3}\left(\begin{pmatrix}2\\2\end{pmatrix} + 3\mathbb{Z}^2\right).$$

Then (2.2) and (2.3) imply that

$$\mathcal{Z}(\widehat{\mu}_{M,\{\mathcal{D}_n\}}) \subset \bigcup_{j=1}^{\infty} M^j(\mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3 \cup \mathcal{A}_4) = \bigcup_{j=1}^{\infty} \rho^{-j}(\mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3 \cup \mathcal{A}_4). \tag{2.4}$$

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For the one-dimensional self-similar measure $\mu_{\rho,m}$,

$$\mu_{\rho,m}(\cdot) = \frac{1}{m}\sum_{j=0}^{m-1}\mu_{\rho,m}(\rho^{-1}(\cdot) - j), \tag{2.5}$$

where $0 < \rho < 1$ and $m > 1$ is an integer, Deng [7] studied when $L^2(\mu_{\rho,m})$ admits an infinite orthogonal set of exponential functions and obtained the following conclusion.

Lemma 2.1 [7]. *Let $\mu_{\rho,m}$ be defined by (2.5). If m is a prime, then $L^2(\mu_{\rho,m})$ admits an infinite orthogonal set of exponential functions if and only if $\rho = (q/p)^{1/r}$ for $p, q, r \in \mathbb{N}$ with $m \mid p$.*

For any $\xi \in \mathbb{R}^2$,

$$\widehat{\mu}_{M,\{\mathcal{D}_n\}}(\xi) = \prod_{n=1}^{\infty}\widehat{\delta}_{\mathcal{D}_n}(M^{-n}\xi) = \prod_{i=1}^{r}\prod_{j=0}^{\infty}\widehat{\delta}_{\mathcal{D}_{jr+i}}(M^{-(jr+i)}\xi) = \prod_{i=1}^{r}\prod_{j=0}^{\infty}\widehat{\delta}_{\mathcal{D}_{jr+i}}(\rho^{jr+i}\xi).$$

Take $\nu_i = *_{j=0}^{\infty}\delta_{M^{-(jr+i)}\mathcal{D}_{jr+i}}$ for $1 \le i \le r$. Then

$$\mu_{M,\{\mathcal{D}_n\}} = \nu_1 * \nu_2 \cdots * \nu_r,$$

and we have

$$\mathcal{Z}(\widehat{\nu}_i) = \bigcup_{j=0}^{\infty}\rho^{-(jr+i)}\mathcal{Z}(\widehat{\delta}_{\mathcal{D}_{jr+i}}) = \rho^{-i}\bigcup_{j=0}^{\infty}\rho^{-jr}\mathcal{Z}(\widehat{\delta}_{\mathcal{D}_{jr+i}}). \tag{2.6}$$

Moreover,

$$\mathcal{Z}(\widehat{\mu}_{M,\{\mathcal{D}_n\}}) = \bigcup_{i=1}^{r}\mathcal{Z}(\widehat{\nu}_i) = \bigcup_{i=1}^{r}\rho^{-i}\bigcup_{j=0}^{\infty}\rho^{-jr}\mathcal{Z}(\widehat{\delta}_{\mathcal{D}_{jr+i}}). \tag{2.7}$$

Proof of Theorem 1.1. Suppose Λ is an infinite orthogonal set of $\mu_{M,\{\mathcal{D}_n\}}$ with $0 \in \Lambda$. Set $\Lambda = \binom{\Lambda^{(1)}}{\Lambda^{(2)}}$, where $\Lambda^{(1)}$ is the first coordinate of Λ and $\Lambda^{(2)}$ is the second coordinate. By the orthogonality of Λ, we have $(\Lambda - \Lambda) \setminus \{0\} \subset \mathcal{Z}(\widehat{\mu}_{M,\{\mathcal{D}_n\}})$. This, together with (2.3) and (2.4), implies that

$$(\Lambda^{(i)} - \Lambda^{(i)}) \setminus \{0\} \subset \bigcup_{j=1}^{\infty}\rho^{-j}\frac{\mathbb{Z}\setminus 3\mathbb{Z}}{3} \quad \text{for } i = 1, 2.$$

It follows that $(\Lambda^{(i)} - \Lambda^{(i)}) \setminus \{0\} \subset \mathcal{Z}(\widehat{\mu}_{\rho,3})$ for $i = 1, 2$. Therefore, $\Lambda^{(i)}$ $(i = 1, 2)$ is an orthogonal set of $\mu_{\rho,3}$.

We now claim that $\Lambda^{(i)}$ is infinite for $i = 1, 2$. It is enough to prove that $\Lambda^{(1)}$ is infinite, since the proof for $\Lambda^{(2)}$ is similar. Suppose to the contrary that $\Lambda^{(1)}$ is finite. By the pigeonhole principle, there exist two distinct elements $\lambda, \lambda' \in \Lambda$ with $\lambda = \binom{\lambda_1}{\lambda_2}$, $\lambda' = \binom{\lambda'_1}{\lambda'_2}$, such that $\lambda_1 = \lambda'_1$. Then

$$\lambda - \lambda' = \begin{pmatrix} 0 \\ \lambda_2 - \lambda'_2 \end{pmatrix} \notin \mathcal{Z}(\widehat{\mu}_{M,\{\mathcal{D}_n\}}).$$

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This is a contradiction. Hence, the claim follows and we conclude that $\Lambda^{(i)}$ $(i = 1, 2)$ is an infinite orthogonal set of $\mu_{\rho,3}$. By Lemma 2.1, $\rho = (q/p)^{1/r}$ for some $p, q, r \in \mathbb{N}$ with $3 \mid p$.

For the converse, suppose that $\rho = (q/p)^{1/r}$ for some $p, q, r \in \mathbb{N}$ with $3 \mid p$. Fix $i \in \{1, 2, \ldots, r\}$. By the pigeonhole principle, there exists an infinite set $\mathcal{T}$ such that for any distinct $j, j' \in \mathcal{T}$, we have $\mathcal{D}_{jr+i} = \mathcal{D}_{j'r+i}$. Without loss of generality, assume that $\mathcal{D}_{jr+i} = \{\binom{0}{0}, \binom{1}{0}, \binom{0}{1}\}$ for $j \in \mathcal{T}$. Set

$$\Lambda = \rho^{-i}\Big\{p^j a : a = \frac{1}{3}\begin{pmatrix}1\\2\end{pmatrix}, j \in \mathcal{T}\Big\} \cup \{0\}.$$

It is clear that $\Lambda \setminus \{0\} \subset \mathcal{Z}(\widehat{\mu}_{M,\{\mathcal{D}_n\}})$ and Λ is an infinite set. Now, it is enough to prove that Λ is an orthogonal set of $\mu_{M,\{\mathcal{D}_n\}}$. For any distinct $\lambda_1, \lambda_2 \in \Lambda$, we can write

$$\lambda_1 = \rho^{-i} p^{j_1} a, \quad \lambda_2 = \rho^{-i} p^{j_2} a$$

with $j_1 > j_2$. Then

$$\lambda_1 - \lambda_2 = \rho^{-i} p^{j_1} a - \rho^{-i} p^{j_2} a = \rho^{-(j_2 r + i)}(p^{j_1 - j_2} q^{j_2} - q^{j_2})a.$$

Since $\gcd(3, q) = 1$, we have $(p^{j_1 - j_2} q^{j_2} - q^{j_2})a \in \mathcal{Z}(\widehat{\delta}_{\mathcal{D}_{j_2 r+i}})$, and thus $\lambda_1 - \lambda_2 \in \mathcal{Z}(\widehat{\mu}_{M,\{\mathcal{D}_n\}})$. Hence,

$$(\Lambda \setminus \{0\} \subset \mathcal{Z}(\widehat{\mu}_{M,\{\mathcal{D}_n\}}).$$

Therefore, Λ is an infinite orthogonal set of $\mu_{M,\{\mathcal{D}_n\}}$. □

3. The proof of Theorem 1.2

We now turn to Theorem 1.2. To prove it, we need the following lemmas.

LEMMA 3.1 [8]. *Suppose that $b \in \mathbb{R}$ admits a minimal integer polynomial $qx^r - p$ $(r > 1)$ and satisfies $a_1 b^l + a_2 b^m = a_3 b^n$, where $a_1, a_2, a_3 \in \mathbb{Z} \setminus \{0\}$ and l, m, n are nonnegative integers. Then $l \equiv m \equiv n \pmod r$.*

LEMMA 3.2. *If $b \in \mathbb{R}$ has a minimal integer polynomial $qx^r - p$ with $r > 1$ and satisfies*

$$b^{n_1}\begin{pmatrix}a_{11}\\a_{12}\end{pmatrix} - b^{n_2}\begin{pmatrix}a_{21}\\a_{22}\end{pmatrix} = b^{n_3}\begin{pmatrix}a_{31}\\a_{32}\end{pmatrix}, \tag{3.1}$$

where $\binom{a_{i1}}{a_{i2}} \in \mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3 \cup \mathcal{A}_4$ for $1 \le i \le 3$ and n_1, n_2, n_3 are nonnegative integers, then $n_1 \equiv n_2 \equiv n_3 \pmod r$.

PROOF. From (3.1),

$$\begin{cases} 3b^{n_1} a_{11} - 3b^{n_2} a_{21} = 3b^{n_3} a_{31}, \\ 3b^{n_1} a_{12} - 3b^{n_2} a_{22} = 3b^{n_3} a_{32}. \end{cases}$$

It follows from (2.3) that $3a_{ij} \neq 0$ for all $i = 1, 2, 3$ and $j = 1, 2$. Applying Lemma 3.1, we have $n_1 \equiv n_2 \equiv n_3 \pmod r$. □

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PROOF OF THEOREM 1.2. (i) Assume for contradiction's sake that $\#\Lambda > 3$. Let $\Lambda = \{0, \lambda_1, \lambda_2, \lambda_3\}$ be an orthogonal set for $\mu_{M,\{\mathcal{D}_n\}}$. We claim that

$$(\Lambda - \Lambda) \setminus \{0\} \subset \mathcal{Z}(\widehat{\nu_i}) \quad (\text{see}(2.6)) \tag{3.2}$$

for some i with $1 \le i \le r$. Indeed, for any two distinct $\lambda_i, \lambda_j \in \Lambda$, we can write

$$\lambda_i = \rho^{-n_i}\begin{pmatrix}\lambda_{i1}\\ \lambda_{i2}\end{pmatrix}, \quad \lambda_j = \rho^{-n_j}\begin{pmatrix}\lambda_{j1}\\ \lambda_{j2}\end{pmatrix},$$

where $n_i, n_j \ge 1$ and $\binom{\lambda_{i1}}{\lambda_{i2}}, \binom{\lambda_{j1}}{\lambda_{j2}} \in \mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3 \cup \mathcal{A}_4$. From the orthogonality of Λ, there exists $\rho^{-m}\binom{\lambda_1}{\lambda_2}$ with $m \ge 1$ and $\binom{\lambda_1}{\lambda_2} \in \mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3 \cup \mathcal{A}_4$ such that

$$\rho^{-n_i}\begin{pmatrix}\lambda_{i1}\\ \lambda_{i2}\end{pmatrix} - \rho^{-n_j}\begin{pmatrix}\lambda_{j1}\\ \lambda_{j2}\end{pmatrix} = \rho^{-m}\begin{pmatrix}\lambda_1\\ \lambda_2\end{pmatrix}.$$

By Lemma 3.2, $n_i \equiv n_j \equiv m \pmod r$. Therefore, $\lambda_i, \lambda_j, \lambda_i - \lambda_j \in \mathcal{Z}(\widehat{\nu_i})$ for some i with $1 \le i \le r$ and the claim follows.

It follows from the above claim that

$$\Lambda \setminus \{0\} \subset (\Lambda - \Lambda) \setminus \{0\} \subset \mathcal{Z}(\widehat{\nu_i}),$$

for some $i \in \{1, 2, \dots, r\}$. This together with (2.6) implies that we can rewrite λ_k as

$$\lambda_k = \frac{1}{3}\rho^{-i}\Big(\frac{p}{q}\Big)^{n_k}\begin{pmatrix}\lambda_{k1}\\ \lambda_{k2}\end{pmatrix},$$

where $n_k \ge 0$ and $\binom{\lambda_{k1}}{\lambda_{k2}} \in 3(\mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3 \cup \mathcal{A}_4)$ for $k = 1, 2, 3$. Introduce $N = \max\{n_k : k = 1, 2, 3\}$. Then

$$\Lambda \setminus \{0\} = \frac{1}{3q^N}\rho^{-i}\Big\{p^{n_k}q^{N-n_k}\begin{pmatrix}\lambda_{k1}\\ \lambda_{k2}\end{pmatrix} : k = 1, 2, 3\Big\}.$$

Set

$$\Lambda^{(1)} = \{p^{n_k}q^{N-n_k}\lambda_{k1} : k = 1, 2, 3\}.$$

Since $3 \nmid p^{n_k}q^{N-n_k}\lambda_{k1}$ for $k = 1, 2, 3$, by the pigeonhole principle, there exist $j \ne l \in \{1, 2, 3\}$ such that

$$p^{n_j}q^{N-n_j}\lambda_{j1} \equiv p^{n_l}q^{N-n_l}\lambda_{l1} \pmod 3.$$

That is, $3 \mid (p^{n_j}q^{N-n_j}\lambda_{j1} - p^{n_l}q^{N-n_l}\lambda_{l1})$. Then

$$\lambda_j - \lambda_l = \frac{1}{3q^N}\rho^{-i}\Big(p^{n_j}q^{N-n_j}\begin{pmatrix}\lambda_{j1}\\ \lambda_{j2}\end{pmatrix} - p^{n_l}q^{N-n_l}\begin{pmatrix}\lambda_{l1}\\ \lambda_{l2}\end{pmatrix}\Big) \notin \mathcal{Z}(\widehat{\nu_i}),$$

which contradicts (3.2). Hence, $\#\Lambda \le 3$.

Next, we construct an appropriate orthogonal set to show that 3 is best possible. Let

$$\Lambda_0 = \rho^{-(jr+i)}\{\lambda_k : \lambda_k \in \mathcal{Z}(\widehat{\delta_{\mathcal{D}_{jr+i}}})\} \cup \{0\}$$

for some $j \ge 0$ and $i \in \{1, 2, \dots, r\}$. It is obvious that $(\Lambda_0 - \Lambda_0) \setminus \{0\} \subset \mathcal{Z}(\widehat{\nu_i})$ and $\#\Lambda = 3$. Hence, the number 3 is best possible.

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(ii) For any $n \geq 1$, either $\mathcal{Z}(\widehat{\delta_{\mathcal{D}_n}}) = \mathcal{A}_1 \cup \mathcal{A}_2$ or $\mathcal{Z}(\widehat{\delta_{\mathcal{D}_n}}) = \mathcal{A}_3 \cup \mathcal{A}_4$. Applying the pigeonhole principle, there exists an infinite set $\mathcal{T}$ and $k \in \{1, 3\}$ such that $\mathcal{Z}(\widehat{\delta_{\mathcal{D}_n}}) = \mathcal{A}_k \cup \mathcal{A}_{k+1}$ for all $n \in \mathcal{T}$. Without loss of generality, we assume that $\mathcal{Z}(\widehat{\delta_{\mathcal{D}_n}}) = \mathcal{A}_1 \cup \mathcal{A}_2$ for all $n \in \mathcal{T}$. By the pigeonhole principle again, there exist $i \in \{1, 2, \dots, r\}$ and an infinite set $\mathcal{T}'$ such that $\mathcal{Z}(\widehat{\delta_{\mathcal{D}_{jr+i}}}) = \mathcal{A}_1 \cup \mathcal{A}_2$ for all $j \in \mathcal{T}'$. Set $\mathcal{T}' = \{j_n\}_{n=1}^{\infty}$ with $j_1 < j_2 < \cdots$. For any $N \geq 1$, define

$$\Lambda_N = \left\{ \lambda_n = \rho^{-i} \frac{p^{j_n + j_N}}{3q^{j_n}} \begin{pmatrix} \alpha_n \\ \beta_n \end{pmatrix} : \begin{pmatrix} \alpha_0 \\ \beta_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} \alpha_n \\ \beta_n \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \text{ for } 1 \leq n \leq N \right\}.$$

As $3 \nmid p$, we have $\Lambda_N \setminus \{0\} \subset \mathcal{Z}(\widehat{\nu_i}) \subset \mathcal{Z}(\widehat{\mu}_{M,\{\mathcal{D}_n\}})$ by (2.6) and (2.7). For any distinct elements $\lambda_n, \lambda_m \in \Lambda_N$, we can write

$$\lambda_n = \rho^{-i} \frac{p^{j_n + j_N}}{3q^{j_n}} \begin{pmatrix} \alpha_n \\ \beta_n \end{pmatrix}, \quad \lambda_m = \rho^{-i} \frac{p^{j_m + j_N}}{3q^{j_m}} \begin{pmatrix} \alpha_m \\ \beta_m \end{pmatrix}$$

with $n < m$. Then

$$\lambda_n - \lambda_m = \rho^{-i} \frac{p^{j_m}}{3q^{j_m}} \left(p^{j_n + j_N - j_m} q^{j_m - j_n} \begin{pmatrix} \alpha_n \\ \beta_n \end{pmatrix} - p^{j_N} \begin{pmatrix} \alpha_m \\ \beta_m \end{pmatrix} \right).$$

Since $3 \nmid p$ and $3 \mid q$,

$$\left(p^{j_n + j_N - j_m} q^{j_m - j_n} \begin{pmatrix} \alpha_n \\ \beta_n \end{pmatrix} - p^{j_N} \begin{pmatrix} \alpha_m \\ \beta_m \end{pmatrix} \right) \in 3(\mathcal{A}_1 \cup \mathcal{A}_2).$$

It follows that $\lambda_n - \lambda_m \in \mathcal{Z}(\widehat{\mu}_{M,\{\mathcal{D}_n\}})$. Hence, Λ_N is an orthogonal set of $\mu_{M,\{\mathcal{D}_n\}}$. By the arbitrariness of N, the proof is completed. □

4. The proof of Theorem 1.3

In this section, we prove Theorem 1.3. We begin with the important concept of a trinomial number.

DEFINITION 4.1. We say that $\rho \in (0, 1)$ is a *trinomial number* if there exist $\alpha, \beta, \gamma \in \mathbb{Z} \setminus 3\mathbb{Z}$ and $m, n \in \mathbb{N}$ with $m > n > 0$, such that

$$\alpha \rho^{-m} + \beta \rho^{-n} + \gamma = 0. \tag{4.1}$$

The smallest m satisfying (4.1) is called *the degree of the trinomial number* ρ.

The next lemma is inspired by [6, Lemma 4.1].

LEMMA 4.2. *Let $P(x)$ be an integer polynomial with all its coefficients divisible by 3 except for one, and let $Q(x)$ be an integer polynomial whose first and last coefficients are not divisible by 3. Then $P(x)$ and $Q(x)$ are coprime.*

PROOF. Note that $Q(x)$ is not monomial. We divide the proof into two cases.

Case I: $Q(x)$ is irreducible in $\mathbb{Q}[x]$. We argue by contradiction. Suppose that $P(x)$ and $Q(x)$ are not coprime. Then there exists $H(x) \in \mathbb{Z}[x]$, such that $P(x) = H(x)Q(x)$. Set

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$\mathbb{Z}_3 = \mathbb{Z} \setminus 3\mathbb{Z}$. Denote by $P'(x), Q'(x), H'(x) \in \mathbb{Z}_3[x]$ the polynomials whose respective coefficients are congruent to the coefficients of $P(x), Q(x), H(x)$ modulo 3. Then

$$P'(x) = H'(x)Q'(x).$$

The assumption of Lemma 4.2 implies that $P'(x)$ is a monomial but $Q'(x)$ is not, which gives a contradiction. Thus, $P(x)$ and $Q(x)$ are coprime.

Case II: $Q(x)$ is reducible. Then we can write $Q(x) = Q_1(x)Q_2(x)\cdots Q_k(x)$, where each $Q_i(x)$ is irreducible for $1 \le i \le k$. By Case I, $P(x)$ and $Q_i(x)$ are coprime for $i = 1, 2, \ldots, k$. Therefore, $P(x)$ and $Q(x)$ are coprime. □

Now we have all ingredients for the proof of Theorem 1.3.

PROOF OF THEOREM 1.3. (i) Assume that $\#\Lambda > 3^{m+1}$. Let $\Lambda = \{0, \lambda_1, \ldots, \lambda_{3^{m+1}}\}$ be an orthogonal set for $\mu_{M,\{\mathcal{D}_n\}}$. Since ρ is a trinomial number with degree m, there exist $\alpha, \beta, \gamma \in \mathbb{Z} \setminus 3\mathbb{Z}$ and $m, n \in \mathbb{N}$ with $m > n > 0$, such that

$$\alpha\rho^{-m} + \beta\rho^{-n} + \gamma = 0. \tag{4.2}$$

Now we claim that for any $k \in \mathbb{N}$, there exist $\{c_{k,0}, c_{k,1}, \ldots, c_{k,m-1}\} \subset \mathbb{Z}$ such that

$$\alpha^k\rho^{-k} = \sum_{s=0}^{m-1} c_{k,s}\rho^{-s}, \tag{4.3}$$

where $c_{k,s} \in \mathbb{Z} \setminus 3\mathbb{Z} \cup \{0\}$. Indeed, if $k < m$, then (4.3) obviously holds. If $k \ge m$, we denote $k = s_1m + t_1$ $(0 \le t_1 \le m-1)$, then (4.2) implies that

$$\alpha^k\rho^{-k} = (\alpha\rho^{-m})^{s_1}\rho^{-t_1}\alpha^{k-s_1} = (-\beta\rho^{-n} - \gamma)^{s_1}\rho^{-t_1}\alpha^{k-s_1}.$$

If $s_1n + t_1 < m$, then (4.3) follows. If $s_1n + t_1 \ge m$, we set $s_1n + t_1 = s_2m + t_2$ with $0 \le t_2 \le m-1$). Then

$$\alpha^{k-s_1}\rho^{-(s_1n+t_1)} = \alpha^{k-s_1}\rho^{-(s_2m+t_2)} = \alpha^{k-s_1-s_2}(\alpha\rho^{-m})^{s_2}\rho^{-t_2} = \alpha^{k-s_1-s_2}(-\beta\rho^{-n} - \gamma)^{s_2}\rho^{-t_2}.$$

After finitely many steps, we reach $r \in \mathbb{N}$, such that $s_rn + t_r < m$. Then the claim follows. By the pigeonhole principle, there exist mutually different $k_i, k_j, k_l \in \{1, \ldots, 3^{m+1}\}$, such that

$$c_{k_i,s} \equiv c_{k_j,s} \equiv c_{k_l,s} \pmod 3 \tag{4.4}$$

for $s = 0, 1, \ldots, m-1$. Denote the corresponding $\lambda_i, \lambda_j, \lambda_l$ by

$$\lambda_i = \tfrac{1}{3}\rho^{-k_i}a_i, \quad \lambda_j = \tfrac{1}{3}\rho^{-k_j}a_j, \quad \lambda_l = \tfrac{1}{3}\rho^{-k_l}a_l,$$

where $a_i, a_j, a_l \in \{\binom{1}{2}, \binom{2}{1}, \binom{1}{1}, \binom{2}{2}\} + 3\mathbb{Z}^2$. Let $N = \max\{k_i, k_j, k_l\}$. Denote by $a_i^{(1)}, a_j^{(1)}, a_l^{(1)}$ the first coordinates of a_i, a_j, a_l, respectively. Applying the pigeonhole principle again, there exist two elements of $\{\lambda_i, \lambda_j, \lambda_l\}$ (which we might as well denote as λ_i, λ_j), such that

$$\alpha^{N-k_i}a_i^{(1)} \equiv \alpha^{N-k_j}a_j^{(1)} \pmod 3. \tag{4.5}$$

By the orthogonality of Λ, there exists $\frac{1}{3}\rho^{-k_{ij}}a_{ij} \in \mathcal{Z}(\widehat{\mu}_{M,\{\mathcal{D}_n\}})$ with $k_{ij} \geq 1$ and $a_{ij} \in \{\binom{1}{2}, \binom{2}{1}, \binom{1}{1}, \binom{2}{2}\} + 3\mathbb{Z}^2$, such that

$$\tfrac{1}{3}\rho^{-k_i}a_i - \tfrac{1}{3}\rho^{-k_j}a_j = \tfrac{1}{3}\rho^{-k_{ij}}a_{ij}. \tag{4.6}$$

Applying (4.3), we have

$$\begin{aligned}\alpha^N\rho^{-k_i}a_i^{(1)} - \alpha^N\rho^{-k_j}a_j^{(1)} &= \alpha^{k_i}\rho^{-k_i}\alpha^{N-k_i}a_i^{(1)} - \alpha^{k_j}\rho^{-k_j}\alpha^{N-k_j}a_j^{(1)}\\ &= \sum_{s=0}^{m-1}(c_{k_i,s}\rho^{-s}\alpha^{N-k_i}a_i^{(1)} - c_{k_j,s}\rho^{-s}\alpha^{N-k_j}a_j^{(1)})\\ &= \sum_{s=0}^{m-1}(c_{k_i,s}\alpha^{N-k_i}a_i^{(1)} - c_{k_j,s}\alpha^{N-k_j}a_j^{(1)})\rho^{-s}.\end{aligned}$$

Combining this with (4.6) gives

$$\sum_{s=0}^{m-1}(c_{k_i,s}\alpha^{N-k_i}a_i^{(1)} - c_{k_j,s}\alpha^{N-k_j}a_j^{(1)})\rho^{-s} = \alpha^N\rho^{-k_{ij}}a_{ij}^{(1)}.$$

Define

$$P(x) = \sum_{s=0}^{m-1}(c_{k_i,s}\alpha^{N-k_i}a_i^{(1)} - c_{k_j,s}\alpha^{N-k_j}a_j^{(1)})x^{-s} - \alpha^N a_{ij}^{(1)}x^{-k_{ij}}, \quad Q(x) = \alpha x^m + \beta x^n + \gamma.$$

From (4.4) and (4.5), $c_{k_i,s}\alpha^{N-k_i}a_i^{(1)} - c_{k_j,s}\alpha^{N-k_j}a_j^{(1)} \in 3\mathbb{Z}$ for $0 \leq s \leq m-1$. Then by Lemma 4.2, $P(x)$ and $Q(x)$ are coprime. However, $P(\rho^{-1}) = 0$ and $Q(\rho^{-1}) = 0$. This gives a contradiction. Hence, $\#\Lambda \leq 3^{m+1}$.

(ii) We argue by contradiction. Suppose that $\#\Lambda \geq 4$, and let $\Lambda = \{0, \lambda_1, \lambda_2, \lambda_3\}$ be an orthogonal set for $\mu_{M,\{\mathcal{D}_n\}}$. By (2.1) and (2.4), we can write

$$\lambda_i = \frac{1}{3}\rho^{-k_i}a_i \quad \text{with } a_i \in \left\{\begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}2\\1\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}2\\2\end{pmatrix}\right\} + 3\mathbb{Z}^2,\ k_i \geq 1,$$

for $i = 1, 2, 3$. Applying the pigeonhole principle, there exist distinct λ_i, $\lambda_j \in \Lambda$, such that $a_i^{(1)} \equiv a_j^{(1)} \pmod 3$ and $k_i \geq k_j$. By the orthogonality of Λ, there exist $\frac{1}{3}\rho^{-k_{ij}}a_{ij} \in \mathcal{Z}(\widehat{\mu}_{M,\{\mathcal{D}_n\}})$ with $a_{ij} \in \{\binom{1}{2}, \binom{2}{1}, \binom{1}{1}, \binom{2}{2}\} + 3\mathbb{Z}^2$, such that

$$\tfrac{1}{3}\rho^{-k_i}a_i - \tfrac{1}{3}\rho^{-k_j}a_j = \tfrac{1}{3}\rho^{-k_{ij}}a_{ij}.$$

It follows that

$$\rho^{-k_i}a_i^{(1)} - \rho^{-k_j}a_j^{(1)} = \rho^{-k_{ij}}a_{ij}^{(1)}. \tag{4.7}$$

Now we distinguish three cases.

Case I: k_i, k_j, k_{ij} are mutually different. We might as well assume that $k_i > k_j > k_{ij}$ since the proof in the other cases is similar. Then (4.7) implies that

$$\rho^{-(k_i-k_{ij})}a_i^{(1)} - \rho^{-(k_j-k_{ij})}a_j^{(1)} = a_{ij}^{(1)}.$$

This means that ρ is a trinomial number, which is a contradiction.

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Case II: Only two elements of $\{k_i, k_j, k_{ij}\}$ are equal. We might as well assume that $k_i > k_j = k_{ij}$. From (4.7),

$$\rho^{-(k_i - k_{ij})} a_i^{(1)} - a_j^{(1)} = a_{ij}^{(1)}.$$

It follows that $\rho = (a_i^{(1)}/(a_j^{(1)} + a_{ij}^{(1)}))^{1/(k_i - k_{ij})}$, which contradicts the fact that ρ is not of the form $(q/p)^{1/r}$ for $p, q, r \in \mathbb{N}$.

Case III: $k_i = k_j = k_{ij}$. Then $a_i^{(1)} - a_j^{(1)} = a_{ij}^{(1)}$. As $a_i^{(1)} \equiv a_j^{(1)} \pmod 3$, we have $a_{ij}^{(1)} \equiv 0 \pmod 3$. This is impossible. Hence, $\#\Lambda \leq 3$.

Finally, let

$$\Lambda_0 = \begin{cases} \frac{1}{3}\rho^{-1}\left\{\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}2\\1\end{pmatrix}\right\} & \text{if } \mathcal{Z}(\widehat{\delta}_{\mathcal{D}_1}) = \frac{1}{3}\left\{\begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}2\\1\end{pmatrix}\right\} + \mathbb{Z}^2; \\ \frac{1}{3}\rho^{-1}\left\{\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}2\\2\end{pmatrix}\right\} & \text{if } \mathcal{Z}(\widehat{\delta}_{\mathcal{D}_1}) = \frac{1}{3}\left\{\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}2\\2\end{pmatrix}\right\} + \mathbb{Z}^2. \end{cases}$$

It is easy to verify that $(\Lambda_0 - \Lambda_0) \setminus \{0\} \subset \mathcal{Z}(\widehat{\mu}_{M,\{\mathcal{D}_n\}})$ and $\#\Lambda_0 = 3$. Therefore, the number 3 is best possible. □

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