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A curve selection lemma in spaces of arcs and the image of the Nash map

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Abstract

We prove a finiteness result in the space of arcs X_∞ of a singular variety X, which is an extension of the stability result of Denef and Loeser (*Germs of arcs on singular algebraic varieties and motivic integration*, Invent. Math. **135** (1999), 201–232). From this follows a curve selection lemma for generically stable subsets of X_∞. As a consequence, we extend to all dimensions the problem of wedges, proposed by Lejeune-Jalabert (*Arcs analytiques et résolution minimale des singularités des surfaces quasi-homogénes*, Lecture Notes in Mathematics, vol. 777 (Springer, Berlin, 1980), 303–336), and we obtain that an affirmative answer to this problem is equivalent to the surjectivity of the Nash map. This implies, for instance, that the Nash map is bijective for sandwiched surface singularities.

1. Introduction

Let X be a singular variety over a field k. The space of arcs X_∞ on X is a k-scheme which is not of finite type. However, in this article we will show some finiteness properties for X_∞. More precisely, we deal with *irreducible generically stable* subsets N of the space of arcs X_∞ (Definition 3.1). We will prove that, if z is the generic point of such a set N, then the ring $\widehat{\mathcal{O}_{X_\infty,z}}$ is Noetherian and the Curve selection lemma can be applied in *Spec* $\widehat{\mathcal{O}_{X_\infty,z}}$ (Corollaries 4.6 and 4.8).

The irreducible generically stable subsets of X_∞ can be understood as those irreducible subsets N of X_∞ for which the *stability* property in [DL99] holds in a nonempty open subset [DL99, Lemma 4.1] (see also Lemma 3.2 in this article). For example, the Zariski closure of the set of arcs lifting to a fixed exceptional component of a given resolution of singularities is generically stable. The concept of stability has appeared in works on arc spaces previous to the introduction of motivic integration such as [Lej90, Hic93], and plays a central role in motivic integration in arc spaces, after [DL99].

Our finiteness result on the irreducible generically stable subsets of X_∞ follows from an improvement of the stability result in [DL99]. In [DL99], the morphisms from the space of $(n+1)$-truncations of arcs $j_{n+1}(X_\infty)$ to the space of n-truncations, for $n \gg 0$, were studied. Our result deals with the induced morphisms $\overline{j}_{n+1,n} : \overline{j_{n+1}(X_\infty)} \to \overline{j_n(X_\infty)}$ on the *Zariski closures* of the truncation spaces. Given a generically stable subset N, for $n \gg 0$, the fiber of $\overline{j}_{n+1,n}$ over the generic point of $j_n(N)$ has, in general, dimension larger than the expected value $\dim X$. There is a finiteness property in the way the fiber over $j_n(N)$ of $j_{n+1}(X_\infty) \to j_n(X_\infty)$ is embedded in $\overline{j_{n+1}(X_\infty)}$, that can be expressed as follows: for each irreducible generically stable subset N, there exists an open affine subscheme W of X_∞ such that $N \cap W$ is a nonempty closed subset of W whose defining ideal is finitely generated (Theorem 4.1).

As an application, we extend to varieties X of any dimension, the *problem of wedges* on surfaces proposed by Lejeune-Jalabert in 1980 [Lej80]. A wedge can be defined as a formal arc in

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the space of arcs X_∞. Our finiteness result provides necessary and sufficient conditions for an *essential component* to belong to the image of the *Nash map* in terms of wedges (Theorem 5.1; see also [Reg04] where the result was already announced).

The Nash map, established by Nash in a preprint in 1968 (published in [Nas95]), is an injective map between the set of irreducible components of X_∞^{Sing} not concentrated on $Sing\, X$ and the set of essential components of the resolutions of singularities of X. Here, by X_∞^{Sing} we mean the closed subset of X_∞ of the arcs centered in some point of $Sing\, X$, and by essential component we mean an irreducible component of the exceptional locus which appears on every resolution of singularities up to birational equivalence (see § 2). The Nash map is known to be surjective if X is a surface with minimal singularities [Reg95] and if X is a toric variety of any dimension [IK03], and there is a four-dimensional example for which the Nash map is not surjective [IK03]. The problem of wedges has been affirmatively solved for sandwiched surface singularities in [LR99]. From Theorem 5.1 it follows that the Nash map is bijective if X is a surface with sandwiched singularities.

2. Preliminaries on the space of arcs and the Nash map

Let X be an algebraic variety of dimension d over a field k. Given an extension of fields $k \subseteq K$, and $n \in \mathbb{N}$, a *K-arc of order n* is a k-morphism $Spec\, K[t]/(t)^{n+1} \to X$, and a *$K$-arc* is a k-morphism $Spec\, K[[t]] \to X$. There exists a k-scheme of finite type X_n whose K-rational points are the K-arcs of order n, for any field K containing k, and there exist natural *affine* morphisms $j_{n',n} : X_{n'} \to X_n$ for $n' \geqslant n$. The projective limit $X_\infty := \lim_{\leftarrow} X_n$ is a (not of finite type) k-scheme whose K-rational points are the K-arcs in X, for any extension of fields $k \subseteq K$, and, for any k-scheme Y, we have $Hom_k(Y, X_\infty) \cong Hom_k(Y \hat{\times} Spec\, k[[t]], X)$ (see [IK03]). Let us denote by $j_n : X_\infty \to X_n$ and $j : X_\infty \to X$ the natural morphisms, and by $j_\infty : X_\infty \widehat{\times} Spec\, k[[t]] \to X$ the universal family. Given $z \in X_\infty$, with residue field k_z, let $h_z : Spec\, k_z[[t]] \to X$ be the corresponding arc. Note that $j(z) = h_z(0)$ where 0 is the closed point of $Spec\, k_z[[t]]$, we say that h_z *is centered in* $h_z(0)$.

By a *resolution of singularities* of X we mean a proper, birational morphism $p : Y \to X$ with Y nonsingular, such that $Y \setminus p^{-1}(Sing\, X) \to X \setminus Sing\, X$ is an isomorphism. Let $X_\infty^{Sing} := j^{-1}(Sing\, X)$ be the set of arcs centered in some singular point of X. An irreducible component of X_∞^{Sing} is said to be *good* if it is not contained in $(Sing\, X)_\infty$. The *Nash map* is a canonical map $\mathcal{N}$ from the set of good irreducible components of X_∞^{Sing} to the set of essential divisors over X. An *essential divisor* over X is a divisorial valuation ν of the function field $k(X)$ of X centered in $Sing\, X$ such that the center of ν on any resolution of singularities $p : Y \to X$ is an irreducible component of the exceptional locus of p [Nas95, see p. 35] and [IK03].

Let us describe the map $\mathcal{N}$ in suitable terms for the rest of the paper. *In this section and in § 5, we assume the existence of resolution of singularities in the birational equivalence class of X.* For example, if $char\, k = 0$, or $\dim X \leqslant 2$, or $\dim X = 3$ and k an algebraically closed field with $char\, k \geqslant 7$, then such a resolution of singularities exists. Let us fix a resolution $p : Y \to X$. There is a one-to-one correspondence between the set $\{\nu_\alpha\}_{\alpha \in \mathcal{E}}$ of essential divisors over X and the set of their centers $\{E_\alpha\}_{\alpha \in \mathcal{E}}$ on Y, which are called *essential components* on Y. Given $z \in X_\infty \setminus (Sing\, X)_\infty$, i.e. such that the arc h_z maps the generic point η of $Spec\, k_x[[t]]$ into $X \setminus Sing\, X$, by the properness of p, there is a unique lifting $\widetilde{h}_z : Spec\, k_z[[t]] \to Y$ of h_z to Y.

Definition 2.1. Let E be an irreducible component of the exceptional locus of p. We define

$$N_E(Y) := \{z \in X_\infty \setminus (Sing\, X)_\infty / \widetilde{h}_z(0) \in E\}$$

and N_E to be the closure of $N_E(Y)$ in X_∞.

Proposition 2.2 ([Nas95] and [IK03]). (i) *For each essential component E_α on Y, the set N_{E_α} is irreducible and does not depend on Y. Given an essential divisor ν_α, we set $N_\alpha := N_{E_\alpha}$ where E_α is the center of ν_α on Y.*

(ii) *Any good irreducible component of* X_∞^{Sing} *is of the form* N_α *for a unique essential divisor* ν_α*. This determines the Nash map*

$$\{\text{good irreducible components of } X_\infty^{Sing}\} \xrightarrow{\mathcal{N}} \{\text{essential divisors over } X\} = \{\nu_\alpha\}_{\alpha\in\mathcal{E}}.$$

This map is injective.

(iii) *If* $char\, k = 0$*, then all the irreducible components of* X_∞^{Sing} *are good, i.e.*

$$X_\infty^{Sing} = \bigcup_{\alpha\in\mathcal{E}} N_\alpha.$$

Proof. For any irreducible component E of the exceptional locus of p, $N_E(Y)$ is the image, by the morphism $Y_\infty \to X_\infty$ induced by p, of the set Y_∞^E of arcs in Y centered in some point of E. Since E is irreducible and Y is nonsingular, Y_∞^E, and hence N_E, are irreducible subsets of Y_∞ and X_∞, respectively. If z_E is the generic point of N_E, then $z_E \in X_\infty \setminus (Sing\, X)_\infty$, hence h_{z_E} lifts to any resolution of singularities of X, and if $\widetilde{h}_{z_E}$ is the lifting to Y, then $\widetilde{h}_{z_E}(0)$ is the generic point of E. This implies assertion (i). It also implies that the essential divisor ν_α is univocally defined from the set $N_\alpha = N_{E_\alpha}$, since we may choose Y such that E_α is a divisor of Y. Since any good component of X_∞^{Sing} is some N_α for $\alpha \in \mathcal{E}$, assertion (ii) follows. Assertion (iii) follows from [IK03, Lemma 2.12]. □

Remark 2.3. An essential divisor ν_α belongs to the image of the Nash map if $N_\alpha \not\subset N_\beta$ for all $\beta \in \mathcal{E}$, $\beta \neq \alpha$. The map $\mathcal{N}$ is bijective if and only if $N_\alpha \not\subset N_\beta$ for any $\alpha, \beta \in \mathcal{E}$, $\alpha \neq \beta$. If $char\, k = 0$ this is equivalent to saying that the equality in assertion (iii) is the decomposition of X_∞^{Sing} into its irreducible components.

Let N be an irreducible subset of X_∞ and z its generic point. For any $f \in \mathcal{O}_{X,h_z(0)}$, we define $ord_N f := ord_t h_z^\sharp(f)$, where, for any morphism of affine schemes $\varphi : Z \to Y$, we denote the induced morphism of rings by $\varphi^\sharp : \mathcal{O}_Y \to \mathcal{O}_Z$. Note that $ord_N f = \min\{ord_t h_x^\sharp(f)/x \in N\}$. Therefore, if ν_α is an essential divisor, and z_α the generic point of N_α, then

$$ord_{N_\alpha} f := ord_t h_{z_\alpha}^\sharp(f) = \min\{ord_t h_x^\sharp(f)/x \in N_\alpha\} = \nu_\alpha(f) \tag{1}$$

for any $f \in \mathcal{O}_{X,h_{z_\alpha}(0)}$. The following criterion determines a sufficient condition for an essential divisor ν_α to belong to the image of the Nash map.

Proposition 2.4 (See [Reg95, Theorem 1.10]). *Let* ν_α *and* ν_β *be two essential divisors of* X*. Suppose that* $N_\alpha \subseteq N_\beta$*. Then*

$$\nu_\alpha(f) \geqslant \nu_\beta(f) \quad \text{for any } f \in \mathcal{O}_{X,h_{z_\alpha}(0)}.$$

Proof. Let z_α and z_β be the generic points of the sets N_α and N_β respectively. The inclusion $(N_\beta, z_\alpha) \subseteq (X_\infty, z_\alpha)$ determines a morphism $\varphi_{\beta,\alpha} : Spec\, \mathcal{O}_{N_\beta,z_\alpha}[[t]] \to X$. The residue field k_α of z_α in X_∞ is the residue field of the local ring $\mathcal{O}_{N_\beta,z_\alpha}$, and the residue field of z_β in X_∞ is the fraction field of $\mathcal{O}_{N_\beta,z_\alpha}$. This defines the following commutative diagram.

$$\begin{array}{ccc} Spec\, k_\beta[[t]] & \searrow^{h_{z_\beta}} & \\ \downarrow & & \\ Spec\, \mathcal{O}_{N_\beta,z_\alpha}[[t]] & \xrightarrow[\varphi_{\beta,\alpha}]{} & X \\ \uparrow & & \\ Spec\, k_\alpha[[t]] & \nearrow_{h_{z_\alpha}} & \end{array} \tag{2}$$

The result follows from the induced diagram of morphisms of rings and (1). □

3. Generically stable subsets of the space of arcs

Definition 3.1. An irreducible subset N of X_∞ will be called *generically stable* if there exists an open affine subscheme W_0 of X_∞, such that $N \cap W_0$ is a nonempty closed subset of W_0 whose defining ideal is the radical of a finitely generated ideal.

In [DL99], Lemma 4.1, a decomposition of $X_\infty \setminus (Sing\, X)_\infty$ is given by subsets A of X_∞ such that, for fixed A, for $n \gg 0$, $j_n(A)$ is a locally closed subset of X_n and the maps $j_{n+1}(X_\infty) \to j_n(X_\infty)$ are trivial fibrations over $j_n(A)$ with fiber $\mathbb{A}^d_k$, where we consider $j_n(A)$ with the reduced scheme structure and $d = \dim X$. Moreover, these subsets are of the type $A = C \cap W$ where $C = j_{e'}^{-1}(C_{e'})$ and $W = j_{e'}^{-1}(W_{e'})$, for some $e' \in \mathbb{N}$, some closed subset $C_{e'}$ of $X_{e'}$, and some open affine subset $W_{e'}$ of $X_{e'}$, and we have $\overline{j_n(C)} \cap j_{n,e'}^{-1}(W_{e'}) \subseteq j_n(X_\infty)$ for $n \geqslant e'$ (see also Remark 4.3). From this, Lemma 3.2 follows. An improvement of this result will be given in Lemma 4.2.

Lemma 3.2 [DL99, Lemma 4.1]. *Let N be an irreducible generically stable subset of X_∞. Then, there exist an integer n_1 and an open subset W_{n_1} of X_{n_1} such that:*

(i) *for $n \geqslant n_1$, $j_n(N) \cap j_{n,n_1}^{-1}(W_{n_1})$ is a locally closed subset of X_n, and the map $j_{n+1}(X_\infty) \longrightarrow j_n(X_\infty)$ induces a trivial fibration*
$$j_{n+1}(N) \cap j_{n+1,n_1}^{-1}(W_{n_1}) \to j_n(N) \cap j_{n,n_1}^{-1}(W_{n_1})$$
with fiber $\mathbb{A}^d_k$, where the reduced structure is considered in both schemes;

(ii) *for $n \geqslant n_1$, we have $\overline{j_n(N)} \cap j_{n,n_1}^{-1}(W_{n_1}) = j_n(N) \cap j_{n,n_1}^{-1}(W_{n_1})$.*

Definition 3.3. Let N be an irreducible generically stable subset of X_∞. We define the codimension of N to be $\operatorname{codim}_{X_\infty} N := \operatorname{codim}_{X_n} j_n(N)$ for $n \gg 0$.

3.4 A subset N of X_∞ is said to be *semi-algebraic* if there exists a covering of X by open affine sets U such that $N \cap U_\infty$ is the intersection with U_∞ of a semi-algebraic subset of the space of arcs in the affine space. A semi-algebraic subset of $(\mathbb{A}^m_k)_\infty$ is a finite boolean combination of sets of the form $\rho^{-1}(Y \times L)$ where, given $r, s \in \mathbb{N}$, and $g_1, \ldots, g_{r+s} \in k[x_1, \ldots, x_m]$, ρ is the map
$$\rho : (\mathbb{A}^m_k)_\infty \to \mathbb{A}^r_k \times \mathbb{Z}^s, \quad x \mapsto ((\overline{ac}\; h^\sharp_x(g_i))_{i=1}^r, (ord_t h^\sharp_x(g_{r+j}))_{j=1}^s,) \tag{3}$$
where $\overline{ac}\; h^\sharp_x(g_i)$ is the coefficient of lowest degree of $h^\sharp_x(g_i) \in k_x[[t]]$, Y is a closed subset of $\mathbb{A}^s_k$ and $L \subseteq \mathbb{Z}^r$ is the intersection of a semi-space and a sublattice of $\mathbb{Z}^r$ (see [DL99, (2.1)]).

3.5 Let us introduce some notation. Let $X \subseteq \mathbb{A}^m_k$ be an affine variety. We have $\mathcal{O}_{\mathbb{A}^m_\infty} = k[\underline{\mathbf{X}}_0, \underline{\mathbf{X}}_1, \ldots, \underline{\mathbf{X}}_n, \ldots]$, where for $i \geqslant 0$, $\underline{\mathbf{X}}_i = (\mathbf{X}_{1,i}, \ldots, \mathbf{X}_{m,i})$ is an m-uple of variables. Let $\mathbf{j}_\infty : \mathbb{A}^m_\infty \widehat{\times} Spec\, k[[t]] \to \mathbb{A}^m$ be the universal family and, for any $f \in k[x_1, \ldots, x_m]$ in the ideal I_X defining X, let $\mathbf{j}^\sharp_\infty(f) = \sum_{n=0}^\infty \mathbf{F}_n t^n$, where $\mathbf{F}_n \in k[\underline{\mathbf{X}}_0, \ldots, \underline{\mathbf{X}}_n]$. Then
$$X_\infty = Spec\, k[\underline{\mathbf{X}}_0, \underline{\mathbf{X}}_1, \ldots, \underline{\mathbf{X}}_n, \ldots]/(\{\mathbf{F}_n\}_{n \geqslant 0, f \in I_X}).$$
We will denote with bold capital letters the elements in $\mathcal{O}_{\mathbb{A}^m_\infty}$, with capital letters the elements in $\mathcal{O}_{X_\infty}$ and, for any $\mathbf{G} \in \mathcal{O}_{\mathbb{A}^m_\infty}$, we will denote by G its class in $\mathcal{O}_{X_\infty}$. Given $G \in \mathcal{O}_{X_\infty}$, let $D(G)$ be the open set $(G \neq 0)$ in X_∞.

Lemma 3.6 (See [DL99, Lemma 3.1]). *Let N be an irreducible semi-algebraic subset of X_∞. Let z be the generic point of N and suppose that the image of h_z is dense in X. Then N is a generically stable subset of X_∞.*

Proof. We may assume that X is affine. Let $\{g_i\}_{i \in I}$ be the finite number of functions in $\mathcal{O}_X$ appearing in a definition of N (see (3)). Let z be the generic point of N, and $b_i := ord_t h^\sharp_z(g_i)$ for $i \in I$.

Then $b_i < \infty$ for all i, since the image of h_z is dense in X. Let $j_\infty^\sharp(g_i) = \sum_n G_{i,n} t^n$, $G = \prod_i G_{i,b_i}$ and C the closed subset of X_∞ defined by $G_{i,n} = 0$ for $i \in I$, $n \leqslant b_i$. Then $N \cap D(G)$ is the nonempty intersection with $D(G) \cap C$ of a subset of X_∞ defined by a finite Boolean combination of sets given by $p(G_{1,b_1}, \ldots, G_{s,b_s}) = 0$, where p is a polynomial over k, $g_1, \ldots, g_s \in \mathcal{O}_X$ and $b_j = ord_t h_z^\sharp(g_j)$ for $j = 1, \ldots, s$. From this, the result follows. ☐

Remark 3.7. We have shown (Lemma 3.2) that Definition 3.1 extends the notion of stability in [DL99, 2.7]. In [DL99] only semi-algebraic subsets of X_∞ are considered. For instance, from the proof of Lemma 3.6 it follows that, if $\mathcal{O}_{\mathbb{A}^1_\infty} = k[\mathbf{X}_0, \mathbf{X}_1, \ldots, \mathbf{X}_n, \ldots]$, then the subset $\{\mathbf{X}_1 = 0\}$ of $\mathbb{A}^1_\infty$ is a generically stable subset for which no nonempty open subset is semi-algebraic.

Let ν_α be an essential divisor over X, and N_α the irreducible subset of X_∞ in Proposition 2.2.

Proposition 3.8. *The set N_α is an irreducible generically stable subset of X_∞. In addition, there exists an open affine subscheme W_0 of X_∞, such that $N \cap W_0$ is a nonempty semi-algebraic subset of X_∞.*

Proof. We may assume that X is affine. Let $p : Y \to X$ be a resolution of singularities such that the center E_α of ν_α is a divisor of Y. Let V be an affine chart in Y such that $V \cap E_\alpha \neq \emptyset$ and E_α is defined in V by a single equation $\ell \in \mathcal{O}(V)$. Let $y_i = g_i/g_0$, $1 \leqslant i \leqslant d$, be local coordinates in V, where $g_i \in \mathcal{O}_X$. There exist $a \in \mathbb{N}$ and a polynomial p in $d+1$ variables with coefficients in k such that

$$\ell = \frac{p(g_0, \ldots, g_d)}{g_0^a}. \tag{4}$$

By (1), any $x \in X_\infty$ such that $ord_t h_x^\sharp(g_i) = \nu_\alpha(g_i)$ for $0 \leqslant i \leqslant d$, lifts to an arc $\widetilde{h}_x$ on Y. For such an $x \in X_\infty$, the condition $ord_t h_x^\sharp(p(g_0, \ldots, g_d)) = 1 + a\nu_\alpha(g_0)$ is equivalent to $ord_t \widetilde{h}_x^\sharp(\ell) = 1$. Let us consider the set

$$\Omega_V = \{x \in X_\infty / ord_t h_x^\sharp(g_i) = \nu_\alpha(g_i), \text{ for } 0 \leqslant i \leqslant d, ord_t h_x^\sharp(p(g_0, \ldots, g_d)) = 1 + a\nu_\alpha(g_0)\}.$$

We have that Ω_V is a nonempty set contained in $N_{E_\alpha}(Y)$, hence in N_α. In addition, $\Omega_V = D(G_0) \cap C$ where $G_0 \in \mathcal{O}_{X_\infty} \setminus \mathcal{P}$, and $C \supseteq N_\alpha$ is a closed set of X_∞ defined by a finite number of conditions of the type $ord_t h_x^\sharp(g) \geqslant c$, where $g \in \mathcal{O}_X$ and $c \in \mathbb{N}$, since $\nu_\alpha(g_i) < \infty$ for $0 \leqslant i \leqslant d$. Therefore, $N_\alpha \cap D(G_0) = C \cap D(G_0)$ and the proposition follows. ☐

Given a resolution of singularities $p : Y \to X$ and an irreducible component E of the exceptional locus of p, let

$$N_E^0(Y) := \{x \in X_\infty \backslash (Sing\, X)_\infty / \widetilde{h}_x \text{ intersects } E \text{ transversally at a nonsingular point of } p^{-1}(Sing\, X)_{\mathrm{red}}\}$$

where $\widetilde{h}_x$ is the arc on Y lifting h_x.

Corollary 3.9. *Let $p : Y \to X$ be a resolution of singularities, and E_α an essential component of Y. Then, $N_{E_\alpha}^0(Y)$ is a nonempty open subset of N_α.*

Proof. We may suppose that X is affine and E_α is a divisor of Y, and we have to prove that, if S is a closed subset of E_α, $S \neq E_\alpha$, then the set $N_{E_\alpha,S}^0(Y)$ of arcs $x \in X_\infty \setminus (Sing\, X)_\infty$ such that $\widetilde{h}_x$ intersects E_α transversally at a point of $E_\alpha \setminus S$ is a nonempty open subset of N_α. Let $z \in N_{E_\alpha,S}^0(Y)$. Let V be an affine chart as in the proof of Proposition 3.8 such that $\widetilde{h}_z(0) \in V$, then we may choose $g_0, \ldots, g_d$ as in the previous proof and such that $ord_t \widetilde{h}_z^\sharp(g_i) = \nu_\alpha(g_i)$. Let $l_1, \ldots, l_r$ be generators of the ideal defining S in V, and $l_i = q_i(g_0, \ldots, g_d)/g_0^n$ as in (4). Then $\Omega_V \cap (\cup_i \{x / ord_t q_i(g_0, \ldots, g_d) = n\nu_\alpha(g_0)\})$ is an open subset of N_α containing z and contained in $N_{E_\alpha,S}^0$. Therefore, the result follows. ☐

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4. A finiteness theorem on the ideals defining irreducible generically stable sets

Let us introduce the notation needed to prove our finiteness result on the generically stable subsets of the space of arcs. Given an affine variety X over k, let d be its dimension and m its embedding dimension, and let us keep the notation in § 3.5. For $n \in \mathbb{N}$, let $\mathcal{O}_n := \mathcal{O}_{\overline{j_n(X_\infty)}}$, where $\overline{j_n(X_\infty)}$ is the closure of $j_n(X_\infty)$ in X_n, considered with its reduced structure. Given $G \in \mathcal{O}_n$, we will denote by $D_n(G)$ the open subset $(G \neq 0)$ of X_n, so that $D(G) = j_n^{-1}(D_n(G))$. We have $\mathcal{O}_n \subseteq \mathcal{O}_{n+1}$ and $\mathcal{O}_{X_\infty} = \bigcup_n \mathcal{O}_n$.

Let J be the Jacobian ideal of X. Given an ideal J' of $\mathcal{O}_X$ and an irreducible subset N of X_∞, let $\mathcal{J}'_N$ be the ideal of $\mathcal{O}_{X_\infty}$ generated by $\bigcup_{q\in J'}\{Q_0, \dots, Q_{e'-1}\}$, where $e' = ord_N J'$ and, for any $q \in \mathcal{O}_X$, we set $j_\infty^\sharp(q) = \sum_n Q_n t^n$, hence $Q_n \in \mathcal{O}_n$. If $\mathcal{P}$ is the prime ideal of $\mathcal{O}_{X_\infty}$ defining N then, by the definition of e', the ideal $\mathcal{J}'_N$ of $\mathcal{O}_{X_\infty}$ is contained in $\mathcal{P}$. In addition, $\mathcal{J}'_N$ is a finitely generated ideal, since it is generated by elements in $\mathcal{O}_{e'-1}$. We will consider generically stable subsets N of X_∞ and ideals J' which are the Jacobian ideal of a complete intersection subscheme X' of $\mathbb{A}^m_k$ containing X, hence $J' \subseteq J$.

Theorem 4.1. *Let X be a variety over k and let N be an irreducible generically stable subset of X_∞. There exists an open affine subscheme W of X_∞ such that $N \cap W$ is a nonempty closed subset of W whose defining ideal is finitely generated.*

Proof. We may assume that X is affine. Let us keep the notation at the beginning of the section. Let $\mathcal{P}$ be the prime ideal of $\mathcal{O}_{X_\infty}$ defining N, with the reduced structure, and let $\mathcal{P}_n$ be the prime ideal of $\mathcal{O}_n$ defining $\overline{j_n(N)}$, also considered with the reduced structure, then $\mathcal{P}_n = \mathcal{P} \cap \mathcal{O}_n$. Theorem 4.1 will be a consequence of Lemma 4.2 below. □

Lemma 4.2. *Let N be an irreducible generically stable subset of X_∞. Then, there exist $n_1 \in \mathbb{N}$, $G \in \mathcal{O}_{n_1}$, $G \in \mathcal{O}_{X_\infty} \setminus \mathcal{P}$ and an ideal J' of $\mathcal{O}_X$, $J' \subseteq J$, such that, for all $n \geqslant n_1$, there exist $H_{1,n+1}, \dots, H_{m-d,n+1}$ in $\mathcal{O}_{n+1}$ satisfying:*

(i) $\mathcal{P}_{n+1}(\mathcal{O}_{n+1})_G = (\mathcal{P}_n + (H_{1,n+1}, \dots, H_{m-d,n+1}))(\mathcal{O}_{n+1})_G$;

(ii) $H_{r,n+1} \in \mathcal{J}'_N$ *for* $r = 1, \dots, m-d$.

Assume that the lemma is proved. Then, since $\mathcal{J}'_N \subseteq \mathcal{P}$, we have that $\mathcal{P}(\mathcal{O}_{X_\infty})_G$ is finitely generated by the generators of $\mathcal{P}_{n_1}$ and the generators of $\mathcal{J}'_N$. Since $\mathcal{J}'_N$ is generated by a finite number of elements of $\mathcal{P}_{e'-1}$, we have $\mathcal{P}(\mathcal{O}_{X_\infty})_G = \mathcal{P}_n(\mathcal{O}_{X_\infty})_G$ for $n \geqslant \sup\{n_1, e'-1\}$.

Proof of Lemma 4.2. We will show that we may take the integer n_1 and $G \in \mathcal{O}_{n_1}$ so that n_1 and the open subset $W_{n_1} = D_{n_1}(G)$ of X_{n_1} are as in Lemma 3.2. In fact, we will describe how to obtain both n_1 and G in Lemma 3.2.

Let us first reduce the proof to work on a complete intersection scheme, in the same way as in the proof of [DL99, Lemma 4.1]: the singular locus $Sing\, X$ of X is the intersection of a finite number of hypersurfaces $l = 0$, where each $l \in \mathcal{O}_X$ determines $f_1, \dots, f_{m-d} \in k[x_1, \dots, x_m]$ and $c \in \mathbb{N}$ such that

$$(\mathcal{O}_X)_l \cong (k[x_1, \dots, x_m]/(f_1, \dots, f_{m-d}))_l \quad \text{and} \quad l^{c-1}J \subseteq J' \subseteq J \tag{5}$$

where J' is the Jacobian ideal of the complete intersection subscheme X' of $\mathbb{A}^m_k$ defined by $f_1, \dots, f_{m-d}$. Let $e = ord_N J$, then there exists one of such elements l such that $ord_N l = e$. We have $(\mathcal{O}_{X_\infty})_{L_e} \cong (\mathcal{O}_{X'_\infty})_{L_e}$, where by L_e we mean the coefficient in degree e of the images of l by the morphisms induced by the respective universal families. Let $e' := ord_N J'$, then $e \leqslant e' \leqslant ce$. Let $q_1, \dots, q_s$ be all $(m-d) \times (m-d)$-minors of the matrix $(\partial f_i/\partial x_j)_{1\leqslant i\leqslant m-d, 1\leqslant j\leqslant m}$. There exists $i \in \{1, \dots, s\}$ such that $ord_N q_i = e'$. Let us consider the set

$$A = D(L_e) \cap \{x \in X'_\infty / ord_x q_i = e', ord_x q_j \geqslant e' \text{ for } j = 1, \dots, s\}. \tag{6}$$

We have that $A \subseteq X_\infty$ and $N \cap A \neq \emptyset$.

We may suppose that $q := q_i$ is the determinant of the matrix $(\partial f_i/\partial x_j)_{1\leqslant i,j\leqslant m-d}$. Let M be its adjoint matrix, then

$$M\cdot\left(\frac{\partial f_i}{\partial x_j}\right)_{1\leqslant i\leqslant m-d,1\leqslant j\leqslant m} = \big(qI_{m-d} \mid (q_{r,j})_{1\leqslant r\leqslant m-d<j\leqslant m}\big)$$

where I_{m-d} is the identity $(m-d)\times(m-d)$-matrix, and $q_{r,j}\in J'$. The set A in (6) is equal to $D(L_e.Q_{e'})\cap V(\mathcal{J}'_N)$ where $V(\mathcal{J}'_N)$ is the closed set given by zeroes of the ideal $\mathcal{J}'_N$ defined from J' (see the beginning of § 4), hence $V(\mathcal{J}'_N)\supseteq N$. For $n\geqslant e'$, let us define $H_{1,n+1},\dots,H_{m-d,n+1}\in\mathcal{O}_{n+1}$ such that $H_{r,n+1}\in\mathcal{J}'_N$. The definition of $H_{1,n+1},\dots,H_{m-d,n+1}$ will only depend on the choice made to define the set A. Let $\mathbf{j}_\infty$ be the universal family in $\mathbb{A}^m_k$. If we multiply M by the column vector $(\mathbf{j}^\sharp_\infty(f_i))_{i=1}^{m-d}$, then, by Taylor's development, for $n\geqslant e'$ the coefficient in degree $n+e'+1$ is

$$\left(\mathbf{G}_{r,n}+\sum_{i=0}^{e'}\mathbf{Q}_i\mathbf{X}_{r,n+e'+1-i}+\sum_{j=m-d+1}^{m}\sum_{i=0}^{e'}\mathbf{Q}_{(r,j),i}\mathbf{X}_{j,n+e'+1-i}\right)_{r=1}^{m-d} \tag{7}$$

where $\mathbf{G}_{r,n}\in k[\underline{\mathbf{X}}_0,\dots,\underline{\mathbf{X}}_n]$. We recall that we use bold capital letters for the elements in $\mathcal{O}_{\mathbb{A}^m_\infty}$ and capital letters for the corresponding classes in $\mathcal{O}_{X_\infty}$, and that $\mathbf{j}^\sharp_\infty(q)=\sum_i\mathbf{Q}_it^i$, $\mathbf{j}^\sharp_\infty(q_{r,j})=\sum_i\mathbf{Q}_{(r,j),i}t^i$. Let

$$H_{r,n+1} := G_{r,n}+Q_{e'}X_{r,n+1}+\sum_{j=m-d+1}^{m}Q_{(r,j),e'}X_{j,n+1}\in\mathcal{O}_{X_\infty} \tag{8}$$

for $1\leqslant r\leqslant m-d$. Then $H_{r,n+1}\in\mathcal{O}_{n+1}$ and, by (7),

$$H_{r,n+1}+\sum_{i=0}^{e'-1}Q_iX_{r,n+e'+1-i}+\sum_{j=m-d+1}^{m}\sum_{i=0}^{e'-1}Q_{(r,j),i}X_{j,n+e'+1-i}=0 \tag{9}$$

in $\mathcal{O}_{X_\infty}$. Since $q,q_{r,j}\in J'$ and $e'=ord_NJ'$, we have $H_{r,n+1}\in\mathcal{J}'_N$. Therefore, (ii) holds and, since $\mathcal{J}'_N\subseteq\mathcal{P}$, we have $(\mathcal{P}_n+(H_{1,n+1},\dots,H_{m-d,n+1}))\mathcal{O}_{n+1}\subseteq\mathcal{P}_{n+1}$ for $n\geqslant e'$.

Let W_0 be an open subset of X_∞ as in Definition 3.1. We may suppose that $W_0=D(G_0)$ for some $G_0\in\mathcal{O}_{X_\infty}\setminus\mathcal{P}$. There exists a finitely generated ideal $\mathcal{I}$ of $\mathcal{O}_{X_\infty}$ such that $\mathcal{P}(\mathcal{O}_{X_\infty})_{G_0}$ is the radical of $\mathcal{I}(\mathcal{O}_{X_\infty})_{G_0}$. Let $n_0\in\mathbb{N}$ be such that G_0 and a system of generators of $\mathcal{I}$ are in $\mathcal{O}_{n_0}$. Let $n_1=sup\{n_0,e'\}$ and $G=G_0.L_e.Q_{e'}$. We will show that they satisfy Lemmas 4.2 and 3.2. The functions in (7) restricted to $V(\mathcal{J}'_N)$ are $(H_{r,n+1})_{r=1}^{m-d}$. Since $N\subseteq V(\mathcal{J}'_N)$, Hensel's lemma implies that, for $n\geqslant n_1$, a $\overline{k}$-rational point in $(\overline{j_n(N)}\cap D_n(G))\times\mathbb{A}^{d+1}_k$ is in $\overline{j_{n+1}(X_\infty)}$ if and only if it is in the zero locus of $\{H_{r,n+1}\}_{r=1}^{m-d}$ (see [DL99, p. 219]). It also implies that $\overline{j_n(N)}\cap D_n(G)\subseteq j_n(X_\infty)$, hence $\overline{j_n(N)}\cap D_n(G)=j_n(N)\cap D_n(G)$ for $n\geqslant n_1$. In addition, since we can eliminate $X_{r,n+1}$ in (8) because $Q_{e'}$ is a unit, we have

$$((\mathcal{O}_n)_G/\mathcal{P}_n)[\underline{\mathbf{X}}_{n+1}]/(\{H_{r,n+1}\}_{r=1}^{m-d})\cong((\mathcal{O}_n)_G/\mathcal{P}_n)[\mathbf{X}_{m-d+1,n+1},\dots,\mathbf{X}_{m,n+1}].$$

Therefore, $Spec(\mathcal{O}_{n+1})_G/\mathcal{P}_{n+1}\subseteq Spec((\mathcal{O}_n)_G/\mathcal{P}_n)[\underline{\mathbf{X}}_{n+1}]/(\{H_{r,n+1}\}_{r=1}^{m-d})$ are reduced schemes with the same $\overline{k}$-rational points, thus they are isomorphic. Hence,

$$\begin{aligned}(\mathcal{O}_{n+1})_G/\mathcal{P}_{n+1}&\cong(\mathcal{O}_{n+1})_G/(\mathcal{P}_n+(\{H_{r,n+1}\}_{r=1}^{m-d}))\\&\cong((\mathcal{O}_n)_G/\mathcal{P}_n)[\mathbf{X}_{m-d+1,n+1},\dots,\mathbf{X}_{m,n+1}]\end{aligned} \tag{10}$$

and the result follows. ∎

Remark 4.3. We have followed the ideas in the proof of [DL99, Lemma 4.1]. Note that, for fixed $e_0\in\mathbb{N}$, there is a finite number of sets A as in (6), where all possible choices of $e\leqslant e_0$, of l as in (5), of e' with $e\leqslant e'\leqslant ce$, and of q_i, a $(m-d)\times(m-d)$-minor of the Jacobian matrix of $f_1,\dots,f_{m-d}$

in (5), are considered. These sets A cover $j_n(X_\infty \setminus j_e^{-1}((Sing\, X)_\infty))$, and $j_{n+1}(X_\infty) \to j_n(X_\infty)$ induces a trivial fibration over $j_n(A)$ with fiber $\mathbb{A}_k^d$ for $n \geqslant ce$, where $j_n(A)$ is considered with the reduced scheme structure. This is [DL99, Lemma 4.1].

Our improvement of the result in [DL] is based in the following two ideas: the equalities (9), which are the reason why all of the elements $\{H_{r,n+1}\}_{1\leqslant r\leqslant m-d, n\geqslant n_1}$ belong to the finitely generated ideal $\mathcal{J}'_N$, and our expression (10) of the stability in terms of the rings of the reduced schemes $\overline{j_n(N)}$, which is Lemma 4.2(i).

From Lemma 4.2 we deduce the following result, which compares the fiber over $j_n(N)$ of the morphism $\overline{j_{n+1}(X_\infty)} \to \overline{j_n(X_\infty)}$ with the fiber over $j_n(N)$ of $j_{n+1}(X_\infty) \longrightarrow j_n(X_\infty)$, which is isomorphic to $\mathbb{A}^d_{k(j_n(N))}$.

Corollary 4.4. *Let N be an irreducible generically stable subset of X_∞. For $n \in \mathbb{N}$, let Γ_{n+1} be the fiber over $\overline{j_n(N)}$ of the morphism $\overline{j_{n+1}(X_\infty)} \longrightarrow \overline{j_n(X_\infty)}$. Then there exist $n_1 \in \mathbb{N}$ and a finitely generated ideal $\mathcal{J}'_N$ of $\mathcal{O}_{X_\infty}$ such that, for each $n \geqslant n_1$, there exist $H_{1,n+1}, \ldots, H_{m-d,n+1} \in \mathcal{O}_{n+1}$ in $\mathcal{J}'_N$ satisfying*

$$\mathcal{O}_{\Gamma_{n+1}}/(H_{1,n+1}, \ldots, H_{m-d,n+1})\mathcal{O}_{\Gamma_{n+1}} \cong k(j_n(N))[Z_1, \ldots, Z_d]$$

where $k(j_n(N))[Z_1, \ldots, Z_d]$ is the polynomial ring in d variables over the residue field of $j_n(N)$.

Remark 4.5. In equality (i) in Lemma 4.2 one might expect the stronger property that $\{H_{r,n+1}\}_{r=1}^{m-d} \in \mathcal{P}_{n+1}(\mathcal{O}_{n+1})_G$, i.e. $\mathcal{P}_n(\mathcal{O}_{n+1})_G = \mathcal{P}_{n+1}(\mathcal{O}_{n+1})_G$. Let us show that this is not true in general. Let X be the four-dimensional hypersurface singularity in [IK03, Example 4.5], defined by $x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^6 = 0$ over a field of characteristic $\neq 2, 3$. The blowing-up Y of X at the origin has a unique singular point, and its exceptional locus is irreducible and defines an essential valuation ν_β. The blowing-up of Y at its singular point is nonsingular, and its exceptional locus is irreducible and defines an essential valuation ν_α, $\nu_\alpha \neq \nu_\beta$. We have $N_\alpha \subset N_\beta = X_\infty$ (see [IK03, Theorem 4.3]). Let $N = N_\alpha$ and keep the notation in Lemma 4.2 and Corollary 4.4. Then $\mathcal{P}_n(\mathcal{O}_{n+1})_{\mathcal{P}_{n+1}} \neq \mathcal{P}_{n+1}(\mathcal{O}_{n+1})_{\mathcal{P}_{n+1}}$ for $n \gg 0$. This is equivalent to $\Gamma_{n+1} \cong \mathbb{A}^{d+1}_{k(j_n(N))}$, i.e. $H_{1,n+1}$ is not zero in $\mathcal{O}_{\Gamma_{n+1}}$.

Corollary 4.6. *Let N be an irreducible generically stable subset of X_∞, and let z be its generic point. Then:*

(i) *the ring $\widehat{\mathcal{O}_{X_\infty,z}}$ is Noetherian;*

(ii) *if N' is an irreducible subset of X_∞ such that $N' \supset N$, $N \neq N'$, then $\widehat{\mathcal{O}_{N',z}}$ is a Noetherian local ring of dimension $\geqslant 1$.*

All of the completions are with respect to the topology defined by the maximal ideal.

Proof. From Theorem 4.1 it follows that the maximal ideal $\mathcal{P}\mathcal{O}_{X_\infty,z}$ of $\mathcal{O}_{X_\infty,z}$ is finitely generated. Hence, so is the maximal ideal $\mathcal{P}\widehat{\mathcal{O}_{X_\infty,z}}$ of $\widehat{\mathcal{O}_{X_\infty,z}}$, and (i) follows from [Mat86, Theorem 29.4(i)]. Analogously, $\widehat{\mathcal{O}_{N',z}}$ is a Noetherian local ring. If (ii) does not hold, i.e. $\dim \widehat{\mathcal{O}_{N',z}} = 0$, then $\widehat{\mathcal{O}_{N',z}}$ is an artinian ring, hence $\mathcal{P}^n\widehat{\mathcal{O}_{N',z}} = (0)$ for some n. Equivalently, $\mathcal{P}^n = \mathcal{P}^{n+1}$, which implies $\mathcal{P}^n = (0)$ by Nakayama. This contradicts the fact that $\mathcal{O}_{N',z}$ is a domain, since N' is irreducible, and not a field, because $N \neq N'$. □

Remark 4.7. A related result is [Dri02] (see also [GK00]), where it is proved that, if $z \in X_\infty \setminus (Sing\, X)_\infty$ is a k-arc, then there exists a Noetherian complete local ring A such that $\widehat{\mathcal{O}_{X_\infty,z}}$ is isomorphic to the $(\{Z_i\}_{i\in I})$-adic completion of the ring $A[\{Z_i\}_{i\in I}]$, where I is a countably set and the Z_i are variables.

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Corollary 4.8 (Curve selection lemma). *Let N and N' be two irreducible subsets of X_∞ such that $N \subset N'$, $N \neq N'$. Suppose that N is generically stable, and let z be the generic point of N and k_z its residue field. Then, there exists a morphism*

$$\phi : \mathit{Spec}\, K[[\xi]] \to N'$$

where K is a finite algebraic extension of k_z, such that the image of the closed point of $\mathit{Spec}\, K[[\xi]]$ is z, and the image of the generic point belongs to $N' \setminus N$.

Proof. The inclusion $(N', z) \subseteq (X_\infty, z)$ determines a morphism $\varphi : \mathit{Spec}\, \mathcal{O}_{N',z} \to N'$ (it also determines a commutative diagram analogous to (2)). Since N is an irreducible generically stable subset of X_∞, the ring $\widehat{\mathcal{O}_{N',z}}$ is a Noetherian local ring of dimension $\geqslant 1$ (Corollary 4.6(ii)). It is equicharacteristic, since it contains the base field k, thus it has a coefficient field [Mat86, Theorem 28.3]. From Cohen's structure theorem, it follows that we can apply the curve selection lemma in $\mathit{Spec}\, \widehat{\mathcal{O}_{N',z}}$ and obtain a morphism $\psi : \mathit{Spec}\, K[[\xi]] \to \mathit{Spec}\, \widehat{\mathcal{O}_{N',z}}$, such that the image of the closed point is the closed point in $\mathit{Spec}\, \widehat{\mathcal{O}_{N',z}}$, the image of the generic point is not the closed point in $\mathit{Spec}\, \widehat{\mathcal{O}_{N',z}}$, and K is a finite algebraic extension of k_z. Then, we may take $\phi : \mathit{Spec}\, K[[\xi]] \to N'$ obtained by composition of ψ, the natural morphism $\mathit{Spec}\, \widehat{\mathcal{O}_{N',z}} \to \mathit{Spec}\, \mathcal{O}_{N',z}$ and φ. ☐

Corollary 4.8 is a weak version for arc spaces of the classical curve selection lemma [Mil68, § 3]. We ask whether the stronger version, i.e. the analogue to the classical curve selection lemma, holds. More precisely, we have the following.

Question. Let N be an irreducible generically stable subset of X_∞. Let N' be an irreducible subset of X_∞ such that $N \subset N'$, $N \neq N'$. Let z, z' be the generic points of N and N', respectively. Is it true that there is a morphism

$$\phi : \mathit{Spec}\, K[[\xi]] \to N'$$

where K is a finite algebraic extension of k_z, such that the image of the closed point of $\mathit{Spec}\, K[[\xi]]$ is z, and the image of the generic point is z'?

In general, if we do not assume that N is generically stable, we have the following weaker version of a curve selection lemma.

Proposition 4.9. *Assume $\mathit{char}\, k = 0$. Let N be an irreducible subset of X_∞^{Sing} strictly contained in an irreducible component of X_∞^{Sing}. Let z be the generic point of N and k_z its residue field. Then, there exists a morphism*

$$\phi : \mathit{Spec}\, K[[\xi]] \to X_\infty^{\mathit{Sing}}$$

where K is a finite algebraic extension of k_z, such that the image of the closed point of $\mathit{Spec}\, K[[\xi]]$ is z, and the image of the generic point belongs to $X_\infty^{\mathit{Sing}} \setminus N$.

Proof. If N is generically stable, the proposition follows from Corollary 4.8. Thus, let us suppose that N is not generically stable. We assume first that $z \notin (\mathit{Sing}\, X)_\infty$. Let $p : Y \to X$ be a resolution of singularities. The arc h_z has a unique lifting $\widetilde{h}_z : \mathit{Spec}\, k_z[[t]] \to Y$, let $\widetilde{z}$ be the corresponding point in Y_∞. Let E be an irreducible component of the exceptional locus of p which contains the center of $\widetilde{h}_z$. Then $N \subset N_E$, $N \neq N_E$ since N is not generically stable. Thus, there exists a morphism $\widetilde{\phi} : \mathit{Spec}\, k_z[[\xi]] \to Y_\infty$ such that the image of the closed point of $\mathit{Spec}\, K[[\xi]]$ is $\widetilde{z}$ and the image of the generic point is an arc on Y_∞ centered in E and different from $\widetilde{z}$, since Y is nonsingular. The composition of $\widetilde{\phi}$ with the morphism $p_\infty : Y_\infty \to X_\infty$ defines ϕ.

Finally, if $z \in (\mathit{Sing}\, X)_\infty$ then the result follows from the proof of [IK03, Lemma 2.12]. ☐

If the subset N of X_∞ is not generically stable, in general we do not have the stronger version of the curve selection lemma, as we show with the following example, which was communicated to me

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by M. Lejeune-Jalabert. Let X be the Whitney umbrella given by $x_3^2 = x_1 x_2^2$ in $\mathbb{A}^3_{\mathbb{C}}$, and let z be the point of X_∞ determined by the arc $x_1(t) = t, x_2(t) = x_3(t) = 0$. Then, z belongs to the closure of $X_\infty^{Sing} \setminus (Sing\,X)_\infty$ (see [IK03, Lemma 2.12]). However, there is no morphism $\phi : Spec\,K[[\xi]] \to X_\infty$ such that $\phi(0) = z$ and $\phi(\eta) \in X_\infty^{Sing} \setminus (Sing\,X)_\infty$. More precisely, if $x_1(\xi,t), x_2(\xi,t), x_3(\xi,t) \in K[[\xi,t]]$ satisfy $x_3^2 = x_1 x_2^2$ and $x_1(0,t) = t$, $x_2(0,t) = x_3(0,t) = 0$, then $x_2(\xi,t) = x_3(\xi,t) = 0$ since $ord_{(\xi,t)} x_1(\xi,t) = 1$.

5. The problem of wedges and the image of the Nash map

Given an extension of fields $k \subseteq K$, a *K-wedge* on X is a k-morphism $\phi : Spec\,K[[\xi,t]] \to X$. A K-wedge ϕ can be identified to a $K[[\xi]]$-point on X_∞ (see the first paragraph in § 2). We will call the *special arc of* ϕ, and will denote by ϕ_0, the image in X_∞ of the closed point 0 of $Spec\,K[[\xi]]$. We will call the *generic arc of* ϕ, and will denote by ϕ_η, the image in X_∞ of the generic point η of $Spec\,K[[\xi]]$.

The following problem is an extension of the problem of wedges proposed for surfaces in [Lej80].

Problem of wedges (see [Lej80]). Let ν_α be an essential divisor over X, let N_α be the irreducible subset of X_∞ defined by ν_α (Proposition 2.2(i)), z_α the generic point of N_α and k_α the residue field of z_α in X_∞. Let $p : Y \to X$ be a resolution of singularities of X, and K a field extension of k_α. Given a K-wedge ϕ on X whose special arc is z_α and whose generic arc belongs to X_∞^{Sing}, does ϕ lift to Y?

The following result characterizes the image of the Nash map in terms of wedges.

Theorem 5.1. *Let us assume the existence of resolution of singularities in the birational equivalence class of X. Let ν_α be an essential divisor over X. Let z_α be the generic point of N_α, and k_α its residue field. The following conditions are equivalent.*

(i) *ν_α belongs to the image of the Nash map $\mathcal{N}$.*

(ii) *For any resolution of singularities $p : Y \to X$ and for any field extension K of k_α, any K-wedge ϕ on X whose special arc is z_α and whose generic arc belongs to X_∞^{Sing}, lifts to Y.*

(ii′) *There exists a resolution of singularities $p : Y \to X$ satisfying condition (ii).*

Proof. To prove that condition (i) implies condition (ii), let $p : Y \to X$ be a resolution of singularities, and let ϕ be a K-wedge as in condition (ii). Since the special arc of ϕ is $z_\alpha \notin (Sing\,X)_\infty$, the generic arc ϕ_η does not belong to $(Sing\,X)_\infty$. Hence, ϕ_η belongs to a good irreducible component of X_∞^{Sing}. Since z_α is a specialization of ϕ_η, the assertion (i) implies that this component is N_α. Hence $\phi_\eta = z_\alpha$ and ϕ lifts trivially to Y.

It is clear that condition (ii) implies condition (ii′). For condition (ii′) implies condition (i), let us suppose that condition (i) does not hold. Then, there exists an essential divisor $\nu_\beta \neq \nu_\alpha$ such that $N_\alpha \subset N_\beta$, $N_\alpha \neq N_\beta$. Hence, we have the commutative diagram (2). By Proposition 3.8 and the curve selection lemma in $Spec\,\widehat{\mathcal{O}_{N_\beta,z_\alpha}}$ (Corollary 4.8), there exists a morphism $Spec\,K[[\xi]] \to N_\beta$ such that the image of the closed point is z_α, the image of the generic point is a point $z \in N_\beta \setminus N_\alpha$ and K is a finite algebraic extension of k_α. Let $\phi : Spec\,K[[\xi,t]] \to X$ be the induced wedge. The special arc of ϕ is z_α and the generic arc is $z \in N_\beta \setminus N_\alpha$. By condition (ii′) there exists a resolution of singularities $Y \to X$ such that ϕ lifts to Y. Then, h_z lifts to an arc $\widetilde{h}_z$ on Y such that $\widetilde{h}_z(0)$ does not belong to the center E_α of ν_α in Y. This implies that the generic point of E_α, which is $\widetilde{h}_{z_\alpha}(0)$, is a specialization of $\widetilde{h}_z(0) \notin E_\alpha$. Since E_α is an essential component on Y, this is a contradiction. □

Corollary 5.2. *Assuming the existence of resolution of singularities in the birational equivalence class of X, the Nash map is bijective if and only if the problem of wedges for X has an affirmative answer.*

A *sandwiched surface singularity* is the formal neighborhood of a singular point on the surface obtained by blowing-up a complete ideal of a local regular two-dimensional ring. A surface X has sandwiched singularities if, for each singular point P of X, the formal neighborhood of P on X is a sandwiched surface singularity.

Corollary 5.3. *Let X be a surface with sandwiched singularities. Then, the Nash map is bijective.*

Proof. In [LR99] we proved that the problem of wedges as it was stated in [Lej80] has an affirmative answer for X: if Y is the minimal resolution of singularities of X, E_α an essential component on Y, and ϕ a k-wedge whose special arc belongs to $N^0_{E_\alpha}(Y)$, then ϕ lifts to Y. The proof in [LR99] can be extended to a proof of Theorem 5.1(ii′), therefore Corollary 5.3 follows from Corollary 5.2. In fact, all statements and proofs in [LR] concerning k-arcs or k-wedges remain true if we consider K-arcs or K-wedges, where K is any field extension of k (see [LR99, § 2]). Analogously, [LR99, Theorem 3.3] remains true for any K-wedge ϕ whose special arc is a K-arc in $N^0_{E_\alpha}(Y)$. Since $z_\alpha \in N^0_{E_\alpha}(Y)$ (Corollary 3.7 or equality (1)), we conclude the result. □

In an analogous way, we may extend the proofs in [Lej80] and [GL97] to obtain the following consequences of Theorem 5.1.

Corollary 5.4. *The following holds.*

- (i) *Let X be a complex quasi-homogeneous surface singularity whose canonical equivariant resolution coincides with the minimal resolution. Let C be the projective curve quotient of X by the natural $\mathbb{C}^*$-action. One of the following holds:*
  - (1) *the genus of C is $\geqslant 1$, then the Nash map is bijective;*
  - (2) *the genus of C is 0, then C determines an essential divisor which belongs to the image of the Nash map.*
- (ii) *Let X be an algebraic surface over an uncountable algebraically closed field, and ν_α an essential divisor over X such that the arc h_{z_α} is smooth (i.e. $ord_t h^\sharp_{z_\alpha}(M) = 1$, where M is the maximal ideal of $\mathcal{O}_{X,h_{z_\alpha}(0)}$). Then ν_α belongs to the image of the Nash map.*

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