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ON A THEOREM OF ARHANGEL'SKIĬ CONCERNING LINDELÖF p-SPACES

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1. Introduction. In [**4**] Arhangel'skiĭ proved the remarkable result that every regular space which is hereditarily a Lindelöf p-space has a countable base. As a consequence of the main theorem in this paper, we obtain an analogue of Arhangel'skiĭ's result, namely that every regular space which is hereditarily an $\aleph_1$-compact strong Σ-space has a countable net. Under the assumption of the generalized continuum hypothesis (GCH), the main theorem also yields an affirmative answer to Problem 2 in Arhangel'skiĭ's paper.

In § 3 we introduce and study a new cardinal function called the *discreteness character* of a space. The definition is based on a property first studied by Aquaro in [**1**], and for the class of T_1 spaces it extends the concept of $\aleph_1$-compactness to higher cardinals. (In general the discreteness character and the cellularity of a space are not related; however, the two functions agree hereditarily.) The main theorem is proved in § 4, and Arhangel'skiĭ's problem is discussed in § 5.

Throughout this paper $\mathfrak{m}$ and $\mathfrak{n}$ denote cardinal numbers; $\mathfrak{m}^+$ is the smallest cardinal greater than $\mathfrak{m}$; σ, τ, and ρ denote ordinal numbers; and $|A|$ denotes the cardinality of the set A. Unless otherwise stated, no separation axioms are assumed. However, paracompact spaces are always Hausdorff and regular spaces are always T_1.

2. Definitions and known results. We let w, L, h, d, z, c, s and ψ denote the following standard cardinal functions: weight, Lindelöf degree, height (= her. L), density, width (= her. d), cellularity, spread (= her. c), and pseudo-character. (For definitions, see Juhász [**12**].)

The *metrizability degree* of a space X, denoted $m(X)$, is $\aleph_0 \cdot \mathfrak{m}$, where $\mathfrak{m}$ is the smallest cardinal such that X has a base which is the union of $\mathfrak{m}$ discrete collections. See [**10**] for a study of this cardinal function. For any T_1 space X let $F(X) = \aleph_0 \cdot \mathfrak{m}$, where $\mathfrak{m}$ is the smallest cardinal such that every open subset of X is the union of $\leqq \mathfrak{m}$ closed sets. Note that $F(X) \leqq h(X)$ for any regular space X and $\psi(X) \leqq F(X)$ whenever X is T_1. A T_1 space X is *perfect* if $F(X) = \aleph_0$; i.e., every open set is a countable union of closed sets.

In [**9**] the concept of a p-space [**2**] was extended to higher cardinals as follows. A collection $\{\mathscr{G}_\alpha : \alpha \text{ in } A\}$ of open covers of a space X is a *pluming for* X if the following holds: if $p \in G_\alpha \in \mathscr{G}_\alpha$ for all α in A, then

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(a) $C^*(p) = \bigcap\{\bar{G}_\alpha : \alpha \text{ in } A\}$ is compact;

(b) $\{\bigcap_{\alpha \in F} \bar{G}_\alpha : F$ a finite subset of $A\}$ is a "base" for $C^*(p)$ in the sense that given any open set R containing $C^*(p)$, there is a finite subset F of A such that $\bigcap_{\alpha\in F} \bar{G}_\alpha \subseteq R$.

(See [**9**] for a proof that every regular space has a pluming.) For a regular space X, the *pluming degree* of X, denoted $p(X)$, is $\aleph_0 \cdot \mathfrak{m}$, where $\mathfrak{m}$ is the smallest cardinal such that X has a pluming $\{\mathscr{G}_\alpha : \alpha \text{ in } A\}$ with $|A| = \mathfrak{m}$. The definition of a pluming for X is based on an internal characterization of p-spaces given by Burke [**5**], and from Burke's theorem it follows that a Tychonoff space X is a p-space if and only if $p(X) = \aleph_0$. For any regular space X we write $pp(X) = \sup\{p(Y) : Y \subseteq X\}$.

According to Arhangel'skiĭ [**3**] a collection $\mathscr{N}$ of subsets of a space X is a *net* if given any point p in X and any neighborhood R of p, there is some N in $\mathscr{N}$ such that $p \in N \subseteq R$. The *net weight* of a space X, denoted $n(X)$, is $\aleph_0 \cdot \mathfrak{m}$, where $\mathfrak{m}$ is the smallest cardinal such that X has a net of cardinality $\mathfrak{m}$. It is easy to check that $d(X) \leqq n(X)$, $L(X) \leqq n(X)$, and $n(X) \leqq w(X)$. A space X has a countable net if and only if $n(X) = \aleph_0$.

Let X be a set, and let $\mathscr{S}$ be a cover of X. The cover $\mathscr{S}$ is said to be *separating* if given any two distinct points p and q in X, there is some S in $\mathscr{S}$ such that $p \in S$, $q \notin S$. For p in X, the *order of p with respect to $\mathscr{S}$*, denoted $\operatorname{ord}(p, \mathscr{S})$, is the cardinality of the set $\{S \text{ in } \mathscr{S} : p \in S\}$.

For a T_1 space X, the *point separating weight* of X, denoted $psw(X)$, is $\aleph_0 \cdot \mathfrak{m}$, where $\mathfrak{m}$ is the smallest cardinal such that X has a separating open cover $\mathscr{S}$ with $\operatorname{ord}(p, \mathscr{S}) \leqq \mathfrak{m}$ for all p in X. Note that $psw(X) \leqq n(X)$, and that $psw(X) = \aleph_0$ if and only if X has a point-countable separating open cover. (See [**15**; **18**].) In [**9**] it is proved that $w(X) = L(X) \cdot p(X) \cdot psw(X)$ for any regular space X.

For any space X the *point weight* of X, denoted $pw(X)$, is $\aleph_0 \cdot \mathfrak{m}$, where $\mathfrak{m}$ is the smallest cardinal such that X has a base $\mathscr{B}$ with $\operatorname{ord}(p, \mathscr{B}) \leqq \mathfrak{m}$ for all p in X. Clearly $pw(X) \leqq w(X)$ for any space X, $psw(X) \leqq pw(X)$ whenever X is T_1, and $pw(X) = \aleph_0$ if and only if X has a point-countable base.

3. The discreteness character. The *discreteness character* of a space X, denoted $\Delta(X)$, is $\aleph_0 \cdot \mathfrak{m}$, where

$$\mathfrak{m} = \sup\{|\mathscr{F}| : \mathscr{F} \text{ is a discrete collection of non-empty closed sets in } X\}.$$

We also write $\Delta\Delta(X) = \sup\{\Delta(Y) : Y \subseteq X\}$. Note that for any space X, $\Delta(X) \leqq L(X)$, $\Delta\Delta(X) = s(X)$, and $w(X) = m(X) \cdot \Delta(X)$. As for characterizations and other basic properties of $\Delta(X)$, we have the following propositions.

PROPOSITION 3.1. *Let X be a T_1 space, let $\mathfrak{n}$ be an infinite cardinal.*

(1) $\Delta(X) = \aleph_0 \cdot \sup\{|\mathscr{F}| : \mathscr{F}$ *is a locally finite collection of non-empty closed sets in* $X\}$;

(2) $\Delta(X) = \aleph_0 \cdot \sup\{|Y| : Y \subseteq X$ *and every subset of* Y *is closed in* $X\}$;

(3) $\Delta(X) \leqq \mathfrak{n}$ *if and only if every subset of X of cardinality* $> \mathfrak{n}$ *has a limit point.*

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Proof. We prove (1) only. Let $\Delta^*(X)$ denote the right hand side of (1). Clearly $\Delta(X) \leqq \Delta^*(X)$. Assume, then, that $\Delta(X) = \mathfrak{m}$, and let us show that $\Delta^*(X) \leqq \mathfrak{m}$. Let $\mathscr{F} = \{F_\alpha : \alpha \text{ in } A\}$ be a locally finite collection of closed sets in X such that $F_\alpha \neq \emptyset$ for all α in A and $F_\alpha \neq F_\beta$ whenever $\alpha \neq \beta$. Suppose $|A| > \mathfrak{m}$. For each α in A pick $x_\alpha \in F_\alpha$. By Zorn's lemma, there is a subset B of A which is maximal with respect to the property that if α and β are any two distinct elements of B, then $x_\alpha \neq x_\beta$. By the maximality of B, the point finiteness of $\mathscr{F}$, and the assumption that $|A| > \mathfrak{m}$, one can conclude that $|B| > \mathfrak{m}$. Then $\{\{x_\alpha\} : \alpha \text{ in } B\}$ is a discrete collection of non-empty closed sets in X such that $|B| > \mathfrak{m}$ and $\{x_\alpha\} \neq \{x_\beta\}$ for $\alpha \neq \beta$. This contradicts $\Delta(X) = \mathfrak{m}$. Hence $|A| \leqq \mathfrak{m}$, from which it follows that $\Delta^*(X) \leqq \mathfrak{m}$.

Remark 3.2. Recall that a space is $\aleph_1$-*compact* if every uncountable subset has a limit point. By the above proposition, a T_1-space X is $\aleph_1$-compact if and only if $\Delta(X) = \aleph_0$. In addition, a space X satisfies property (*) in [**1**] if and only if $\Delta(X) = \aleph_0$.

PROPOSITION 3.3. *Let X be a T_1-space. Then $s(X) \leqq \Delta(X) \cdot F(X)$. In particular, every perfect T_1 $\aleph_1$-compact space hereditarily satisfies the countable chain condition.*

Proof. Let $\Delta(X) \cdot F(X) = \mathfrak{m}$, let D be a discrete subspace of X, and let us show that $|D| \leqq \mathfrak{m}$. For each p in D let V_p be an open neighborhood of p such that $V_p \cap (D - \{p\}) = \emptyset$, and let $W = \bigcup\{V_p : p \text{ in } D\}$. Now W is open and $F(X) \leqq \mathfrak{m}$, so $W = \bigcup\{H_\sigma : 0 \leqq \sigma < \mathfrak{m}\}$, where each H_σ is a closed set. For each $\sigma < \mathfrak{m}$ let $K_\sigma = H_\sigma \cap D$, and note that $D = \bigcup\{K_\sigma : 0 \leqq \sigma < \mathfrak{m}\}$. The proof is complete if we can show that $|K_\sigma| \leqq \mathfrak{m}$ for each $\sigma < \mathfrak{m}$. So let $\sigma < \mathfrak{m}$ be fixed. Now $\mathscr{F} = \{\{x\} : x \text{ in } K_\sigma\}$ is a discrete collection of closed sets in X. (Let $p \in X$. If $p \notin H_\sigma$, then $(X - H_\sigma)$ is a neighborhood of p which misses all elements of $\mathscr{F}$. If $p \in H_\sigma$, then $p \in W$ and so there exists q in D such that $p \in V_q$. Thus V_q is a neighborhood of p which intersects at most one element of $\mathscr{F}$.) Since $\Delta(X) \leqq \mathfrak{m}$, it follows that $|K_\sigma| \leqq \mathfrak{m}$.

COROLLARY 3.4. *Let X be a T_1-space. Then* $|X| \leqq 2^{\Delta(X) \cdot F(X)}$.

Proof. Hajnal and Juhász [**8**] have proved that $|X| \leqq 2^{s(X) \cdot \psi(X)}$ for any T_1 space X. Since $s(X) \leqq \Delta(X) \cdot F(X)$ and $\psi(X) \leqq F(X)$, it follows that $|X| \leqq 2^{\Delta(X) \cdot F(X)}$.

Remark 3.5. Suppose X is a perfect T_1 $\aleph_1$-compact space. Then by Corollary 3.4, $|X| \leqq 2^{\aleph_0}$. This result generalizes a theorem of Stephenson [**20**].

Before proving our next result, we need a generalization of a lemma due to Aquaro [**1**]. No doubt this generalization is well known, and should be considered folklore by now. However, for the sake of completeness we sketch a proof.

LEMMA 3.6. *Let X be a topological space with* $\Delta(X) \leqq \mathfrak{m}$, *and let $\mathscr{V}$ be an open cover of X such that* $\operatorname{ord}(p, \mathscr{V}) \leqq \mathfrak{m}$ *for all p in X. Then there is a subcover of $\mathscr{V}$ of cardinality* $\leqq \mathfrak{m}$.

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Proof. Suppose no subcollection of $\mathscr{V}$ of cardinality $\leqq \mathfrak{m}$ covers X. By Zorn's lemma, there is a subset M of X which is maximal with respect to the property that if p and q are distinct elements of M, then $q \notin \operatorname{st}(p, \mathscr{V})$. By the maximality of M, the hypothesis that $\operatorname{ord}(p, \mathscr{V}) \leqq \mathfrak{m}$ for all p in X, and the assumption that no subcollection of $\mathscr{V}$ of cardinality $\leqq \mathfrak{m}$ covers X, one can show that $|M| > \mathfrak{m}$. Now $\mathscr{F} = \{\{p\}^- : p \text{ in } M\}$ is a discrete collection of closed sets in X. For, let $x \in X$, and let V be some element of $\mathscr{V}$ which contains x. Suppose $V \cap \{p\}^- \neq \emptyset$ and $V \cap \{q\}^- \neq \emptyset$, where p and q are distinct elements of M. Since V is open, $p \in V$ and $q \in V$, so $q \in \operatorname{st}(p, \mathscr{V})$, a contradiction. Thus $\mathscr{F}$ is a discrete collection of closed sets in X with $|\mathscr{F}| > \mathfrak{m}$, a contradiction of $\Delta(X) \leqq \mathfrak{m}$.

PROPOSITION 3.7. *Let X be a regular space. Then $w(X) = p(X) \cdot \Delta(X) \cdot pw(X)$. In particular, every regular $\aleph_1$-compact p-space with a point-countable base has a countable base.*

Proof. Clearly $p(X) \cdot \Delta(X) \cdot pw(X) \leqq w(X)$. Assume, then, that $p(X) \cdot \Delta(X) \cdot pw(X) = \mathfrak{m}$, and let us show that $w(X) \leqq \mathfrak{m}$. Since $pw(X) \leqq \mathfrak{m}$, X has the property that every open cover has an open refinement $\mathscr{V}$ such that $\operatorname{ord}(p, \mathscr{V}) \leqq \mathfrak{m}$ for all p in X. It follows from Lemma 3.6 that $L(X) \leqq \mathfrak{m}$. Since $w(X) \leqq p(X) \cdot L(X) \cdot psw(X)$ (see [**9**]), we conclude that $w(X) \leqq \mathfrak{m}$.

Problem 3.8. Does every regular $\aleph_1$-compact p-space with a point-countable separating open cover have a countable base?

4. The main theorem. We begin by introducing a cardinal function called the *Σ-degree*. This function extends the concept of a strong Σ-space (see [**17**]) to higher cardinality. Recall that for X a set, $p \in X$, and $\mathscr{F}$ a cover of X, $C(p, \mathscr{F}) = \bigcap\{F \in \mathscr{F} : p \in F\}$. A collection $\{\mathscr{F}_\alpha : \alpha \text{ in } A\}$ of locally finite closed covers of a space X is a *strong Σ-net* for X if the following hold for each p in X:

(a) $C(p) = \bigcap\{C(p, \mathscr{F}_\alpha) : \alpha \text{ in } A\}$ is compact;

(b) $\{C(p, \mathscr{F}_\alpha) : \alpha \text{ in } A\}$ is a "base" for $C(p)$ in the sense that given any open set R containing $C(p)$, there exists α in A such that $C(p, \mathscr{F}_\alpha) \subset R$.

Before defining the Σ-degree we need the following existence result.

PROPOSITION 4.1. *Let X be a regular space. Then X has a strong Σ-net $\{\mathscr{F}_\alpha : \alpha \text{ in } A\}$ with $|A| \leqq n(X)$.*

Proof. Let $\mathscr{N} = \{N_\alpha : \alpha \text{ in } A\}$ be a net for X with $|A| \leqq n(X)$. For each α in A let $\mathscr{F}_\alpha = \{\bar{N}_\alpha, X\}$. Then, as is easy to check, $\{\mathscr{F}_\alpha : \alpha \text{ in } A\}$ is a strong Σ-net for X.

The *Σ-degree* of a regular space X, denoted $\Sigma(X)$, is $\aleph_0 \cdot \mathfrak{m}$, where $\mathfrak{m}$ is the smallest cardinal such that X has a strong Σ-net $\{\mathscr{F}_\alpha : \alpha \text{ in } A\}$ with $|A| = \mathfrak{m}$. By the above proposition, $\Sigma(X) \leqq n(X)$. Note that a regular space X is a strong Σ-space if and only if $\Sigma(X) = \aleph_0$ (see [**15**; **17**]). We also define $\Sigma\Sigma(X) = \sup\{\Sigma(Y): Y \subseteq X\}$ for any regular space X.

Now we are ready to prove the main theorem in this paper, namely that $n(X) \leqq \Delta\Delta(X)\cdot\Sigma\Sigma(X)$ for any regular space X. The basic idea behind the proof is the same as that developed by Arhangel'skiĭ in [**4**], and can be briefly described as follows. Assume $\Delta\Delta(X)\cdot\Sigma\Sigma(X) = \mathfrak{m}$. First divide X into two subspaces X_1 and X_2 in such a way that every compact subset of X_i ($i = 1, 2$) has cardinality $\leqq\mathfrak{m}$. To complete the proof, it suffices to show that $n(X_i) \leqq \mathfrak{m}$, $i = 1, 2$. This is accomplished by showing that X_i is the union of $\leqq\mathfrak{m}$ subspaces, each of which has net weight $\leqq\mathfrak{m}$.

The proof of the main theorem (4.9) requires several propositions. The first of these makes use of the following set-theoretic result of Miščenko (see [**7**; **16**]).

LEMMA (Miščenko). *Let X be a set, let $\mathfrak{m}$ be an infinite cardinal, let $\mathscr{S}$ be a collection of subsets of X such that* $\operatorname{ord}(p, \mathscr{S}) \leqq \mathfrak{m}$ *for all p in X, and let H be a subset of X. Then the cardinality of the set of all finite minimal covers of H by elements of $\mathscr{S}$ does not exceed* $\mathfrak{m}$.

PROPOSITION 4.2. *Let X be a regular space. Then $n(X) = \Delta(X)\cdot\Sigma(X)\cdot psw(X)$.*

Proof. It is easy to check that $\Delta(X)\cdot\Sigma(X)\cdot psw(X) \leqq n(X)$. Suppose, then, that $\Delta(X)\cdot\Sigma(X)\cdot psw(X) = \mathfrak{m}$, and let us construct a net $\mathscr{N}$ for X with $|\mathscr{N}| \leqq \mathfrak{m}$. Let $\{\mathscr{F}_\alpha:\alpha \text{ in } A\}$ be a strong Σ-net for X with $|A| \leqq \mathfrak{m}$. Since $\Delta(X) \leqq \mathfrak{m}$, it follows from Proposition 3.1 that $|\mathscr{F}_\alpha| \leqq \mathfrak{m}$ for each α in A. Let $\mathscr{H}$ be all finite intersections of elements of $\bigcup\{\mathscr{F}_\alpha:\alpha \text{ in } A\}$, and note that $|\mathscr{H}| \leqq \mathfrak{m}$. Let $\mathscr{S}$ be a separating open cover of X such that $\operatorname{ord}(p, \mathscr{S}) \leqq \mathfrak{m}$ for all p in X. We may assume that $X \in \mathscr{S}$, and hence for any subset H of X there is at least one finite minimal cover of H by elements of $\mathscr{S}$, namely $\{X\}$.

First we prove that $|\mathscr{S}| \leqq \mathfrak{m}$. For each H in $\mathscr{H}$ let $\{\mathscr{S}(H, \sigma):0 \leqq \sigma < \mathfrak{n}_H \leqq \mathfrak{m}\}$ be all finite minimal covers of H by elements of $\mathscr{S}$ (use Miščenko's lemma), and let

$$\mathscr{S}' = \bigcup\{\mathscr{S}(H, \sigma):H \in \mathscr{H}, 0 \leqq \sigma < \mathfrak{n}_H\}.$$

We are going to show that $\mathscr{S} \subseteq \mathscr{S}'$, from which it follows that $|\mathscr{S}| \leqq \mathfrak{m}$. Let $S_0 \in \mathscr{S}$, and let $p \in S_0$. Recall that $C(p) = \bigcap\{C(p,\mathscr{F}_\alpha):\alpha \text{ in } A\}$ is compact. Obtain a finite subcollection $\mathscr{S}_0$ of $\mathscr{S}$ which covers $C(p)$ and has these properties: (1) $S_0 \in \mathscr{S}_0$; (2) if $S \in \mathscr{S}_0$ and $S \neq S_0$, then $p \notin S$. Choose α in A such that $H = C(p,\mathscr{F}_\alpha) \subseteq \bigcup\mathscr{S}_0$. Let $\mathscr{S}_1$ be a minimal subcollection of $\mathscr{S}_0$ which covers H, and note that $S_0 \in \mathscr{S}_1$. Now $\mathscr{S}_1 = \mathscr{S}(H, \sigma)$ for some $\sigma < \mathfrak{n}_H$, so $S_0 \in \mathscr{S}'$.

Now let

$$\mathscr{N} = \{H - W:H \in \mathscr{H}, W = \emptyset \text{ or } W \text{ a finite union of elements of } \mathscr{S}\}.$$

Then $|\mathscr{N}| \leqq \mathfrak{m}$, and so the proof is complete if we can show $\mathscr{N}$ a net for X. Let $p \in X$, let R be an open neighborhood of p. Let $Z = C(p) - R$, and note that Z is compact. (Recall that $C(p) = \bigcap \{C(p,\mathscr{F}_\alpha):\alpha \text{ in } A\}$ and is compact.) We may assume that $Z \neq \emptyset$, since the case $Z = \emptyset$ is trivial. Let $\mathscr{S}_0$ be a finite subcollection of $\mathscr{S}$ which covers Z such that $p \notin \bigcup\mathscr{S}_0 = W$. Now $C(p) \subseteq R \cup W$, so there exists α in A such that $C(p,\mathscr{F}_\alpha) \subseteq R \cup W$. Then $N = C(p,\mathscr{F}_\alpha) - W$ is an element of $\mathscr{N}$ such that $p \in N \subseteq R$.

COROLLARY 4.3. *Let X be a regular space. Then $w(X) = \Delta(X)\cdot\Sigma(X)\cdot pw(X)$.*

Proof. Let $\Delta(X)\cdot\Sigma(X)\cdot pw(X) = \mathfrak{m}$. Then by the above result, $n(X) = \mathfrak{m}$, from which it follows that $d(X) \leqq \mathfrak{m}$. Let D be a dense subset of X with $|D| \leqq \mathfrak{m}$, let $\mathscr{B}$ be a base for X such that $\mathrm{ord}(p, \mathscr{B}) \leqq \mathfrak{m}$ for all p in X and $\emptyset \notin \mathscr{B}$. Then $\mathscr{B} = \{B \text{ in } \mathscr{B}:B \cap D \neq \emptyset\}$, from which it easily follows that $|\mathscr{B}| \leqq \mathfrak{m}$.

PROPOSITION 4.4. *Let X be a regular space. Then $L(X) \leqq \Delta(X)\cdot\Sigma(X)$.*

Proof. Let $\Delta(X)\cdot\Sigma(X) = \mathfrak{m}$, and let $\{\mathscr{F}_\alpha:\alpha \text{ in } A\}$ be a strong Σ-net for X with $|A| \leqq \mathfrak{m}$. Since $\Delta(X) \leqq \mathfrak{m}$, it follows that $|\mathscr{F}_\alpha| \leqq \mathfrak{m}$ for all α in A. Let $\mathscr{H}$ be all finite intersections of elements of $\bigcup\{\mathscr{F}_\alpha:\alpha \text{ in } A\}$, and note that $|\mathscr{H}| \leqq \mathfrak{m}$.

Now let $\mathscr{V}$ be an open cover of X, and let us show that there is a subcover of $\mathscr{V}$ of cardinality $\leqq \mathfrak{m}$. Let $\mathscr{H}_0$ be all elements of $\mathscr{H}$ which are contained in a finite union of elements of $\mathscr{V}$. Clearly $|\mathscr{H}_0| \leqq \mathfrak{m}$, and so if we can show that $\mathscr{H}_0$ covers X, it easily follows that a subcollection of $\mathscr{V}$ of cardinality $\leqq \mathfrak{m}$ covers X. Let $p \in X$. Now $C(p)$ is compact, and $\mathscr{V}$ covers $C(p)$, so there is a finite subcollection $\mathscr{V}_0$ of $\mathscr{V}$ such that $C(p) \subseteq \bigcup\mathscr{V}_0 = W$. Choose α in A such that $C(p,\mathscr{F}_\alpha) \subseteq W$. Then $C(p,\mathscr{F}_\alpha)$ is an element of $\mathscr{H}$ which is contained in a finite union of elements of $\mathscr{V}$; i.e., $C(p,\mathscr{F}_\alpha) \in \mathscr{H}_0$. Since $p \in C(p, \mathscr{F}_\alpha)$, the proof is complete.

PROPOSITION 4.5. *Let X be a regular space. Then $z(X) \leqq \Delta\Delta(X)\cdot\Sigma\Sigma(X)$.*

Proof. Let $\Delta\Delta(X)\cdot\Sigma\Sigma(X) = \mathfrak{m}$. First note, by Proposition 4.4, that $h(X) \leqq \mathfrak{m}$ and hence $\psi(X) \leqq \mathfrak{m}$. For simplicity we show $d(X) \leqq \mathfrak{m}$. (A similar argument can be used to show $z(X) \leqq \mathfrak{m}$.) The technique we use is due to Ponomarev [**19**]. Suppose $d(X) > \mathfrak{m}$. Then there is a subset $Y = \{x_\sigma:0 \leqq \sigma < m^+\}$ of X such that for all $\sigma < m^+$, $x_\sigma \notin \{x_\tau:0 \leqq \tau < \sigma\}^-$. For each $\sigma < m^+$ let $\{V(\sigma, \rho):0 \leqq \rho < \mathfrak{m}\}$ be a collection of open neighborhoods of x_σ such that $\bigcap\{V(\sigma, \rho):0 \leqq \rho < \mathfrak{m}\} = \{x_\sigma\}$ and $V(\sigma, \rho) \cap \{x_\tau:0 \leqq \tau < \sigma\} = \emptyset$ for all $\rho < \mathfrak{m}$. Let

$$\mathscr{S} = \{V(\sigma, \rho) \cap Y:0 \leqq \sigma < \mathfrak{m}^+, 0 \leqq \rho < \mathfrak{m}\}.$$

Then $\mathscr{S}$ is a separating open cover of Y such that $\mathrm{ord}(x_\sigma, \mathscr{S}) \leqq \mathfrak{m}$ for all $\sigma < \mathfrak{m}^+$. Now $\Delta(Y)\cdot\Sigma(Y) \leqq \mathfrak{m}$, and so by Proposition 4.2 we have $n(Y) \leqq \mathfrak{m}$. This is a contradiction.

PROPOSITION 4.6. *Let X be a regular space, let $\Delta\Delta(X)\cdot\Sigma\Sigma(X) \leqq \mathfrak{m}$. Then the number of compact subsets of X is $\leqq 2^{\mathfrak{m}}$.*

Proof. First, $h(X) \leqq \mathfrak{m}$, and so every closed subset of X is the intersection of $\leqq \mathfrak{m}$ open sets. By Proposition 4.5, $d(X) \leqq \mathfrak{m}$, and so by a well known result $w(X) \leqq 2^{\mathfrak{m}}$. (See [**12**, p. 10].) Let $\mathscr{B}$ be a base for X with $|\mathscr{B}| \leqq 2^{\mathfrak{m}}$, let $\mathscr{V}$ be all finite unions of elements of $\mathscr{B}$, and let $\mathscr{W}$ be all intersections of $\leqq \mathfrak{m}$ elements of $\mathscr{V}$. Note that $|\mathscr{W}| \leqq 2^{\mathfrak{m}}$. Now let K be any compact subset of X, and let us show that $K \in \mathscr{W}$. First, $K = \bigcap\{U_\sigma : 0 \leqq \sigma < \mathfrak{m}\}$, where each U_σ is an open set. For each $\sigma < \mathfrak{m}$ there exists V_σ in $\mathscr{V}$ such that $K \subseteq V_\sigma \subseteq U_\sigma$. Hence K is the intersection of $\leqq \mathfrak{m}$ elements of $\mathscr{V}$, and so $K \in \mathscr{W}$.

The following set-theoretic lemma extends the result found in [**13**, Chapter 3, § 40, Lemma 2]. The proof is similar and so is omitted.

LEMMA 4.7. *Let X be a set, let $\mathfrak{m}$ be an infinite cardinal, and let $\{K_\sigma : 0 \leqq \sigma < \mathfrak{n} \leqq \mathfrak{m}\}$ be a collection of subsets of X such that $|K_\sigma| = \mathfrak{m}$ for all $\sigma < \mathfrak{n}$. Then there is a subset Z of X such that $Z \cap K_\sigma \neq \emptyset$ and $(X - Z) \cap K_\sigma \neq \emptyset$ for all $\sigma < \mathfrak{n}$.*

PROPOSITION 4.8. *Let $\mathfrak{m}$ be an infinite cardinal, let X be a regular space such that $\Delta\Delta(X) \cdot \Sigma\Sigma(X) \leqq \mathfrak{m}$, and assume that every compact subset of X has cardinality $\leqq \mathfrak{m}$. Then $n(X) \leqq \mathfrak{m}$.*

Proof. Let $\{\mathscr{F}_\alpha : \alpha \text{ in } A\}$ be a strong Σ-net for X with $|A| \leqq \mathfrak{m}$, and let $\mathscr{H}$ be all finite intersections of elements of $\bigcup\{\mathscr{F}_\alpha : \alpha \text{ in } A\}$. As noted before, $|\mathscr{H}| \leqq \mathfrak{m}$. By Proposition 4.4, $h(X) \leqq \mathfrak{m}$, and so every closed subset of X is the intersection of $\leqq \mathfrak{m}$ open sets. For each H in $\mathscr{H}$ let $\{W(H, \sigma) : 0 \leqq \sigma < \mathfrak{m}\}$ be a collection of open sets such that $H = \bigcap\{W(H, \sigma) : 0 \leqq \sigma < \mathfrak{m}\}$, and let

$$\mathscr{S} = \{X - H : H \text{ in } \mathscr{H}\} \cup \{W(H, \sigma) : H \text{ in } \mathscr{H}, 0 \leqq \sigma < \mathfrak{m}\}.$$

Note that $|\mathscr{S}| \leqq \mathfrak{m}$.

The idea of the proof is to express X as the union of $\leqq \mathfrak{m}$ subspaces, say $X = \bigcup\{Y_\tau : 0 \leqq \tau < \mathfrak{m}\}$, in such a way that $\mathscr{S}$, when relativized to each Y_τ, is a separating open cover. Suppose, for a moment, that this is accomplished. Then for each $\tau < \mathfrak{m}$, $\Delta(Y_\tau) \cdot \Sigma(Y_\tau) \cdot psw(Y_\tau) \leqq \mathfrak{m}$, and so by Proposition 4.2 $n(Y_\tau) \leqq \mathfrak{m}$. It then follows that $n(X) \leqq \mathfrak{m}$.

The proof is complete if we can construct the required subspaces of X. Define a relation $\sim$ on X as follows: $p \sim q$ if and only if $p \in C(q, \mathscr{H})$ and $q \in C(p, \mathscr{H})$. Now $\sim$ is an equivalence relation on X, so there is a cover $\{E_t : t \text{ in } T\}$ of X by non-empty sets such that (1) $E_s \cap E_t = \emptyset$ whenever s and t are distinct elements of T; (2) for any two points p and q in X, $\{p, q\} \subseteq E_t$ if and only if $p \sim q$. Now for each t in T, $|E_t| \leqq \mathfrak{m}$. (Let p be any point of E_t. Then E_t is a subset of the compact set $C(p) = \bigcap\{C(p, \mathscr{F}_\alpha) : \alpha \text{ in } A\}$, and $|C(p)| \leqq \mathfrak{m}$ by hypothesis.) Let $E_t = \{x(t, \tau) : 0 \leqq \tau < \mathfrak{m}\}$, and for each $\tau < \mathfrak{m}$ let $Y_\tau = \{x(t, \tau) : t \text{ in } T\}$. Note that $X = \bigcup\{Y_\tau : 0 \leqq \tau < \mathfrak{m}\}$, and so it remains to show that $\mathscr{S}$ is a separating open cover of each Y_τ. Let $\tau < \mathfrak{m}$ be fixed, and let p and q be distinct points of Y_τ. Then there exist s, t in T, $s \neq t$, such that $p = x(s, \tau)$ and $q = x(t, \tau)$. Since $s \neq t$, it follows that $p \sim q$ is

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false, and hence $p \notin C(q, \mathscr{H})$ or $q \notin C(p, \mathscr{H})$. First suppose $p \notin C(q, \mathscr{H})$. Then there exists H in $\mathscr{H}$ such that $q \in H, p \notin H$. Then $(X - H)$ is an element of $\mathscr{S}$ which contains p and not q. Next suppose $q \notin C(p, \mathscr{H})$. Then there exists H in $\mathscr{H}$ such that $p \in H, q \notin H$. Since $q \notin H$, there exists $\sigma < \mathfrak{m}$ such that $q \notin W(H, \sigma)$. Hence $W(H, \sigma)$ is an element of $\mathscr{S}$ which contains p and not q.

THEOREM 4.9. *Assume* GCH. *Then* $n(X) = \Delta\Delta(X) \cdot \Sigma\Sigma(X)$ *for any regular space* X.

Proof. Clearly $\Delta\Delta(X) \cdot \Sigma\Sigma(X) \leqq n(X)$. Assume, then, that $\Delta\Delta(X) \cdot \Sigma\Sigma(X) = \mathfrak{m}$, and let us show $n(X) \leqq \mathfrak{m}$. By Proposition 4.6, the number of compact subsets of X is $\leqq 2^{\mathfrak{m}}$. Note also that $|X| \leqq 2^{\mathfrak{m}}$. Let $\mathscr{K} = \{K_\sigma : 0 \leqq \sigma < \mathfrak{n} \leqq 2^{\mathfrak{m}}\}$ be all compact subsets of X of cardinality $2^{\mathfrak{m}}$. By Lemma 4.7, there is a subset Z of X such that $Z \cap K_\sigma \neq \emptyset$ and $(X - Z) \cap K_\sigma \neq \emptyset$ for all $\sigma < \mathfrak{n}$.

Let us show that $n(Z) \leqq \mathfrak{m}$. A similar argument establishes $n(X - Z) \leqq \mathfrak{m}$, from which it follows that $n(X) \leqq \mathfrak{m}$. To show $n(Z) \leqq \mathfrak{m}$, it suffices, by Proposition 4.8, to show that every compact subset of Z has cardinality $\leqq \mathfrak{m}$. So let K be a compact subset of Z, but suppose $|K| > \mathfrak{m}$. Now $K \subseteq X$ and $|X| \leqq 2^{\mathfrak{m}}$ so $|K| \leqq 2^{\mathfrak{m}}$. By GCH, we conclude that $|K| = 2^{\mathfrak{m}}$. Now K is a compact subset of X, so $K = K_\sigma$ for some $\sigma < \mathfrak{n}$. But then $K \cap (X - Z) \neq \emptyset$, a contradiction. Hence we have $|K| \leqq \mathfrak{m}$, and the proof is complete.

Remark 4.10. Consider the above proof for the special case $\mathfrak{m} = \aleph_0$. It is well known (see [**12**, p. 33]) that every compact Hausdorff space in which every point is a G_δ has cardinality $\leqq \aleph_0$ or $2^{\aleph_0}$. Consequently the continuum hypothesis is not needed to prove that $|K| = 2^{\aleph_0}$ under the assumption $|K| > \aleph_0$. This leads to the following corollary of 4.9.

COROLLARY 4.11. *Let X be a regular space which is hereditarily an $\aleph_1$-compact strong Σ-space. Then X has a countable net.*

COROLLARY 4.12. *Let X be a regular space. Suppose that X is hereditarily a strong Σ-space and hereditarily satisfies the countable chain condition. Then X has a countable net.*

COROLLARY 4.13. *Let X be a regular space which is hereditarily a Lindelöf Σ-space. Then X has a countable net.*

5. Arhangel'skiĭ's Problem. In [**4**] Arhangel'skiĭ proved that every regular space which is hereditarily a paracompact p-space and satisfies the countable chain condition has a countable base. He then asked if this result can be generalized in the following natural way.

Problem [**4**]. Let X be a regular space which is hereditarily a paracompact p-space. Is it true that $c(X) = w(X)$?

In this section we show that the answer is "yes" under the assumption of

GCH. We begin by extending to higher cardinality Arhangel'skiĭ's result that a regular space has a countable base if it is hereditarily a Lindelöf p-space.

LEMMA 5.1. *Let X be a regular space, let $\{\mathscr{F}_\alpha : \alpha \text{ in } A\}$ be a collection of locally finite closed covers of X with $|A| \leqq \mathfrak{m}$, and let Γ be all finite subsets of A. Assume the following hold for each p in X:*

(a) $C(p) = \bigcap\{C(p, \mathscr{F}_\alpha) : \alpha \text{ in } A\}$ *is compact;*

(b) $\{\bigcap_{\alpha \in \gamma} C(p, \mathscr{F}_\alpha) : \gamma \text{ in } \Gamma\}$ *is a "base" for* $C(p)$.

Then $\Sigma(X) \leqq \mathfrak{m}$.

Proof. For each γ in Γ let $\mathscr{H}_\gamma = \wedge\{\mathscr{F}_\alpha : \alpha \text{ in } \gamma\}$. Then each $\mathscr{H}_\gamma$ is a locally finite closed cover of X, and it is easy to check that $\{\mathscr{H}_\gamma : \gamma \text{ in } \Gamma\}$ is a strong Σ-net for X.

PROPOSITION 5.2. *Let X be a regular space. Then* $\Sigma(X) \leqq L(X) \cdot p(X)$.

Proof. Let $L(X) \cdot p(X) = \mathfrak{m}$. Then there is a pluming $\{\mathscr{G}_\alpha : \alpha \text{ in } A\}$ for X such that $|A| \leqq \mathfrak{m}$ and $|\mathscr{G}_\alpha| \leqq \mathfrak{m}$ for all α in A. Let $\mathscr{H}$ be all finite intersections of elements of $\bigcup\{\mathscr{G}_\alpha : \alpha \text{ in } A\}$, and note that $|\mathscr{H}| \leqq \mathfrak{m}$. For each H in $\mathscr{H}$ let $\mathscr{F}(H) = \{\bar{H}, X\}$. By the lemma above, the proof is complete if we can show that for each p in X:

(a) $C(p) = \bigcap\{C(p, \mathscr{F}(H)) : H \text{ in } \mathscr{H}\}$ is compact;

(b) if W is open and $C(p) \subseteq W$, then there exist $H_0, H_1, \ldots, H_k$ in $\mathscr{H}$ such that $\bigcap_{i=0}^k C(p, \mathscr{F}(H_i)) \subseteq W$.

First note that $C(p, \mathscr{F}(H)) = \bar{H}$ whenever $p \in \bar{H}$, and that $C(p) = \bigcap\{\bar{H} : H \text{ in } \mathscr{H},\ p \in \bar{H}\}$. For each α in A choose G_α in $\mathscr{G}_\alpha$ such that $p \in G_\alpha$. Now $\bigcap\{\bar{G}_\alpha : \alpha \text{ in } A\} = C^*(p)$ is compact and contains $C(p)$, so (a) is proved. To prove (b), let W be an open set with $C(p) \subseteq W$. Set $Z = C^*(p) - W$. Now Z is compact, and $\{X - \bar{H} : H \text{ in } \mathscr{H},\ p \in \bar{H}\}$ covers Z, so there exist $H_1, \ldots, H_k$ in $\mathscr{H}$ with $p \in \bar{H}_i$, $i = 1, \ldots, k$, such that $Z \subseteq \bigcup_{i=1}^k (X - \bar{H}_i) = V$. Let $R = V \cup W$. Then R is an open set containing $C^*(p)$, so there is a finite subset F of A such that $\bigcap_{\alpha \in F} \bar{G}_\alpha \subseteq R$. Let $H_0 = \bigcap_{\alpha \in F} G_\alpha$. Note that $H_0 \in \mathscr{H}$ and $p \in \bar{H}_0$. For $i = 0, 1, \ldots, k$, $C(p, \mathscr{F}(H_i)) = \bar{H}_i$, and since $\bigcap_{i=0}^k \bar{H}_i \subseteq W$, the proof of (b) is complete.

THEOREM 5.3. *Assume* GCH. *Then* $w(X) = h(X) \cdot pp(X)$ *for any regular space* X.

Proof. Let $h(X) \cdot pp(X) = \mathfrak{m}$, and let us show $w(X) \leqq \mathfrak{m}$. By the previous result $\Sigma\Sigma(X) \leqq \mathfrak{m}$, and since $\Delta\Delta(X) \leqq h(X) \leqq \mathfrak{m}$, it follows from Theorem 4.9 that $n(X) \leqq \mathfrak{m}$. Clearly $psw(X) \leqq n(X)$ for any regular space X, and so we have $w(X) = L(X) \cdot p(X) \cdot psw(X) \leqq \mathfrak{m}$. (See [**9**].)

Remark 5.4. Consider the above proof for $\mathfrak{m} = \aleph_0$. In this case we can use Corollary 4.11 instead of Theorem 4.9, thereby avoiding the continuum hypothesis. Thus we obtain Arhangel'skiĭ's result that every regular space which is hereditarily a Lindelöf p-space has a countable base.

COROLLARY 5.5. *Assume* GCH. *If X is a hereditarily paracompact space, then $w(X) = c(X) \cdot pp(X)$. In particular, if X is hereditarily a paracompact p-space, then $w(X) = c(X)$.*

Proof. Let $c(X) \cdot pp(X) = \mathfrak{m}$, and let us show $w(X) \leqq \mathfrak{m}$. By the above theorem, it suffices to show $h(X) \leqq \mathfrak{m}$. So let $Y \subseteq X$ and let us show $L(Y) \leqq \mathfrak{m}$. We may assume that Y is open. Let $\mathscr{V}$ be an open cover of Y. Now Y is paracompact, so $\mathscr{V}$ has a σ-disjoint open refinement $\bigcup_{k=1}^{\infty} \mathscr{W}_k$ (see [**14**]). Since $c(X) \leqq \mathfrak{m}$, it follows that $|\mathscr{W}_k| \leqq \mathfrak{m}$, $k = 1, 2, \ldots$, and so $|\bigcup_{k=1}^{\infty} \mathscr{W}_k| \leqq \mathfrak{m}$. It easily follows that $\mathscr{V}$ has a subcollection of cardinality $\leqq \mathfrak{m}$ which covers Y.

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