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SOME INEQUALITIES INVOLVING THE SYMMETRIC FUNCTIONS

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1. Introduction

We firstly introduce some notation. Let $\boldsymbol{a}^{(t)} \in \boldsymbol{R}^n$ then we denote by $\boldsymbol{\alpha}^{(t)}$ a rearrangement of $\boldsymbol{a}^{(t)}$ in non-decreasing order. We write $\boldsymbol{a}^{(1)} \prec \boldsymbol{a}^{(2)}$ if

$$\sum_{i=1}^{k} \alpha_i^{(1)} \leq \sum_{i=1}^{k} \alpha_i^{(2)} \quad \text{for} \quad k = 1, 2, \ldots, n-1$$

and

$$\sum_{i=1}^{n} \alpha_i^{(1)} = \sum_{i=1}^{n} \alpha_i^{(2)}.$$

If the second condition is replaced by $\sum_{i=1}^{n} \alpha_i^{(1)} \leq \sum_{i=1}^{n} \alpha_i^{(2)}$ we write $\boldsymbol{a}^{(1)} \dot{\prec} \boldsymbol{a}^{(2)}$.

We shall also use E_r to denote the r^{th} elementary symmetric function and C_r the r^{th} completely symmetric function. In (3) Daykin proved the following result.

Theorem 1. *Let* $\boldsymbol{a}$ *and* $\boldsymbol{b}$ *be n-tuples of non-negative real numbers and* S *an integer such that* $2 \leq S \leq n$. *If* $\boldsymbol{a} \prec \boldsymbol{b}$ *then*

$$E_S(\boldsymbol{a}) \leq E_S(\boldsymbol{b}) \tag{1.1}$$

but

$$C_S(\boldsymbol{b}) \leq C_S(\boldsymbol{a}). \tag{1.2}$$

Equality holds in (1.1) *if and only if either both sides are zero or* $\boldsymbol{a}$ *is a rearrangement of* $\boldsymbol{b}$ *whilst equality holds in* (1.2) *if and only if* $\boldsymbol{a}$ *is a rearrangement of* $\boldsymbol{b}$.

Over the years there has been considerable interest in inequalities involving E_r and C_r (see (7), pp. 95–107) and the referee has pointed out that the above theorem can be obtained from known results as follows. If $\boldsymbol{a} \prec \boldsymbol{b}$ there is a doubly stochastic matrix M with $\boldsymbol{b} = M\boldsymbol{a}$ ((4), Theorem 46). Further the set of $(n \times n)$ doubly stochastic matrices is a convex polyhedron with permutation matrices as vertices (1). But $E_r^{1/r}$ is concave (5) and $C_r^{1/r}$ is convex (13) so, when they are defined on a convex polyhedron, min $E_r^{1/r}$ and max $C_r^{1/r}$ are realised at vertices of the polyhedron. Hence

$$E_r^{1/r}(\boldsymbol{b}) = E_r^{1/r}(M\boldsymbol{a}) \geq E_r^{1/r}(P_1\boldsymbol{a}) = E_r^{1/r}(\boldsymbol{a})$$

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and

$$C_r^{1/r}(\boldsymbol{b}) = C_r^{1/r}(M\boldsymbol{a}) \leqslant C_r^{1/r}(P_2\boldsymbol{a}) = C_r^{1/r}(\boldsymbol{a})$$

where P_1 and P_2 are suitable permutation matrices.

In this note we investigate how the theorem can be generalised and obtain simple proofs for generalisations of inequalities by Daykin, Minc and Oppenheim.

2. Generalisation of Theorem 1

When the condition $\boldsymbol{a} \prec \boldsymbol{b}$ of Theorem 1 is replaced by $\boldsymbol{a} \prec\!\!\prec \boldsymbol{b}$, (1.2) will clearly not hold in general; on the other hand (1.1) will continue to hold, for let $c_i = a_i$, $i = 1, \ldots, n-1$ and $c_n = a_n + \Sigma_{i=1}^n b_i - \Sigma_{i=1}^n a_i$ then clearly $E_S(\boldsymbol{a}) \leqslant E_S(\boldsymbol{c})$ and, since $\boldsymbol{c} \prec \boldsymbol{b}$, $E_S(\boldsymbol{c}) \leqslant E_S(\boldsymbol{b})$ by Theorem 1. When we have $\boldsymbol{a} \prec\!\!\prec \boldsymbol{b}$ in place of $\boldsymbol{a} \prec \boldsymbol{b}$ however, the following theorem shows that we can obtain an inequality corresponding to (1.2) and a sharper inequality than (1.1).

Theorem 2. *Let* $\boldsymbol{a}$ *and* $\boldsymbol{b}$ *be* n*-tuples of non-negative real numbers,* $a = \Sigma_{i=1}^n a_i$, $b = \Sigma_{i=1}^n b_i$ *and* S *an integer such that* $2 \leqslant S \leqslant n$. *If* $\boldsymbol{a} \prec\!\!\prec \boldsymbol{b}$ *then*

(i) $bE_S(\boldsymbol{a}) \leqslant aE_S(\boldsymbol{b})$,

(ii) $a^{nS}C_S(\boldsymbol{b}) \leqslant b^{nS}C_S(\boldsymbol{a})$.

In both cases there is equality if and only if both sides are zero or **a** is a rearrangement of **b**.

Proof. We may clearly suppose $b_1 \leqslant b_2 \leqslant \cdots \leqslant b_n$. Let $c_0 = 0$ and for $r = 0, 1, \ldots, n-1$ define

$$c_{r+1} = \min\left\{b_{r+1}, \left(a - \sum_{i=0}^{r} c_i\right) \Big/ (n-r)\right\}$$

then (c_i) is a non-decreasing sequence such that $\Sigma_{i=1}^n c_i = a$. Let $x_i = c_i$, $i = 1, \ldots, n-1$ and $x_n = c_n + b - a$. Since $\boldsymbol{a} \prec \boldsymbol{c}$ and $\boldsymbol{x} \prec \boldsymbol{b}$ we have by Theorem 1

$$E_S(\boldsymbol{a}) \leqslant E_S(\boldsymbol{c}) \qquad E_S(\boldsymbol{x}) \leqslant E_S(\boldsymbol{b}), \tag{2.1}$$

$$C_S(\boldsymbol{a}) \geqslant C_S(\boldsymbol{c}) \qquad C_S(\boldsymbol{x}) \geqslant C_S(\boldsymbol{b}). \tag{2.2}$$

(i) Let $\boldsymbol{x}' = (x_1, \ldots, x_{n-1})$ then, with the convention that $E_n(\boldsymbol{x}') = 0$, we have

$$\begin{aligned} aE_S(\boldsymbol{x}) - bE_S(\boldsymbol{c}) &= (a-b)E_S(\boldsymbol{x}') + a(b - a + c_n)E_{S-1}(\boldsymbol{x}') - bc_nE_{S-1}(\boldsymbol{x}') \\ &= (b-a)\{(x_1 + \cdots + x_{n-1})E_{S-1}(\boldsymbol{x}') - E_S(\boldsymbol{x}')\} \geqslant 0 \qquad (2.3) \end{aligned}$$

Thus $bE_S(\boldsymbol{a}) \leqslant bE_S(\boldsymbol{c}) \leqslant aE_S(\boldsymbol{x}) \leqslant aE_S(\boldsymbol{b})$ as required.

Suppose $bE_S(\boldsymbol{a}) = aE_S(\boldsymbol{b}) \neq 0$ then equalities hold in (2.1) and we must have that $\boldsymbol{a}$ is a rearrangement of $\boldsymbol{c}$ and $\boldsymbol{b}$ a rearrangement of $\boldsymbol{x}$. Now

$E_S(\boldsymbol{c}) \geq E_S(\boldsymbol{a}) > 0$ so $E_{S-1}(\boldsymbol{x}') > 0$ and, from (2.3), $bE_S(\boldsymbol{c}) = aE_S(\boldsymbol{x})$ can hold only if $b = a$, i.e. $\boldsymbol{x} = \boldsymbol{c}$ and the equality condition for (i) is established.
(ii) Now $b^n c_i \geq a^n x_i$ for $i = 1, \ldots, n-1$. Further

$$\begin{aligned} b^n c_n - a^n x_n &= b^n c_n - a^n b + a^{n+1} - a^n c_n \\ &= (b-a)\{(b^{n-1} + b^{n-2}a + \cdots + a^{n-1})c_n - a^n\} \\ &\geq (b-a)\{na^{n-1}c_n - a^n\} \geq 0 \end{aligned}$$

since $c_n \geq a/n$ because (c_n) is a non-decreasing sequence.

Thus $b^n c_i \geq a^n x_i$ $(i = 1, \ldots, n)$ so $C_S(b^n\boldsymbol{c}) \geq C_S(a^n\boldsymbol{x})$ i.e. $b^{nS}C_S(\boldsymbol{c}) \geq a^{nS}C_S(\boldsymbol{x})$ so from (2.2) $b^{nS}C_S(\boldsymbol{a}) \geq a^{nS}C_S(\boldsymbol{b})$.

The equality condition for (ii) can now be checked in a similar way to (i).

Remarks on Theorem 2. (1) (i) is a generalisation of a result of Oppenheim (**8**).

(2) The conclusion of (i) cannot be improved to $b^{1+\delta}E_S(\boldsymbol{a}) \leq a^{1+\delta}E_S(\boldsymbol{b})$ where $\delta > 0$, for let $a_i = 1 = b_i$ $i = 1, \ldots, n-1$, $a_n = 1$ and $b_n = 1 + X$, where X is large, then $a^{1+\delta}E_S(\boldsymbol{b})/b^{1+\delta}E_S(\boldsymbol{a}) = 0((1/X)\delta)$.

(3) The conclusion of (ii) can almost certainly be strengthened to $a^r C_S(\boldsymbol{b}) \leq b^r C_S(\boldsymbol{a})$ where $r < Sn$ and it would be of some interest to know the best possible value for r.

(4) The referee has pointed out that (i) follows from the fact that E_r/E_{r-1} is concave (**5**) and a result in (**2**).

3. An inequality of Oppenheim

Oppenheim (**9**) has shown that Theorem 2 (i) can be improved in the special case $S = n = 3$ and $\max_i b_i \leq \max_i a_i$. His result is a special case of the following theorem.

Theorem 3. *Let* $\boldsymbol{a}$ *and* $\boldsymbol{b}$ *be* n*-tuples of non-negative real numbers,* $a = \sum_{i=1}^n a_i$ *and* $b = \sum_{i=1}^n b_i$. *If* $\boldsymbol{a} \prec\!\!\prec \boldsymbol{b}$ *and* $\max_i b_i \leq \max_i a_i$, *then*

$$b^2 E_n(\boldsymbol{a}) \leq a^2 E_n(\boldsymbol{b}).$$

Equality holds if and only if either both sides are zero or $\boldsymbol{a}$ *is a rearrangement of* $\boldsymbol{b}$.

Proof. We may clearly suppose that $\boldsymbol{a}$ and $\boldsymbol{b}$ are arranged in non-decreasing order. Let $\boldsymbol{u} \in \mathbb{R}^{n+2}$ and $\boldsymbol{v} \in \mathbb{R}^{n+2}$ be non-decreasing rearrangements of $(\boldsymbol{a}, \frac{1}{2}b, \frac{1}{2}b)$ and $(\boldsymbol{b}, \frac{1}{2}a, \frac{1}{2}a)$ respectively. Clearly $\sum_{i=1}^{n+2} u_i = \sum_{i=1}^{n+2} v_i$. Since $a_{n-1} \leq a_n$, $2a_{n-1} \leq a \leq b$ so $a_{n-1} \leq \frac{1}{2}b$ and $u_i = a_i$ $(i = 1, \ldots, n-1)$.

For $\lambda = 0, 1, 2$ let $G_r(\lambda) = a_1 + \cdots + a_{n-r} - b_1 - b_2 - \cdots b_{n-r-\lambda} - \lambda a/2$, then, for $k = 1, \ldots, n-1$, $\sum_{i=1}^k (u_i - v_i) = G_{n-k}(\lambda)$ for some $\lambda = 0$, 1 or 2. For $r \geq 1$ we clearly have $G_r(\lambda) \leq 0$ for $\lambda = 0$, 1 or 2 so $\sum_{i=1}^k u_i \leq \sum_{i=1}^k v_i$ for $k = 1, \ldots, n-1$.

(i) Suppose $v_{n+2}=\frac{1}{2}a$ then $v_{n+2}\leqslant\frac{1}{2}b\leqslant u_{n+2}$ and $v_{n+1}+v_{n+2}=a\leqslant b\leqslant u_{n+1}+u_{n+2}$.

(ii) Suppose $v_{n+2}=b_n$ then $v_{n+2}\leqslant a_n\leqslant u_{n+2}$ and $v_{n+1}+v_{n+2}=b_n+\max(b_{n-1},\frac{1}{2}a)\leqslant a_n+\frac{1}{2}b\leqslant u_{n+1}+u_{n+2}$.

Thus in both cases $\boldsymbol{u}\prec\boldsymbol{v}$. Hence, by Theorem 1, $E_{n+2}(\boldsymbol{u})\leqslant E_{n+2}(\boldsymbol{v})$ i.e. $b^2E_n(\boldsymbol{a})\leqslant a^2E_n(\boldsymbol{b})$ as required.

Suppose $E_{n+2}(\boldsymbol{u})=E_{n+2}(\boldsymbol{v})\neq 0$ then, by Theorem 1, $u_i=v_i\neq 0$ $i=1,\ldots,n+2$. Thus $v_{n-1}=u_{n-1}=a_{n-1}\leqslant\frac{1}{2}a$ so $b_i=u_i=a_i$ for $i=1,\ldots,n-1$. But $u_i=v_i$ $(i=n,\ n+1,\ n+2)$ now gives $\frac{1}{2}a=\frac{1}{2}b$ and the theorem follows.

Remarks on Theorem 3. (1) The conclusion of Theorem 3 cannot be improved to $b^{2+\delta}E_n(\boldsymbol{a})\leqslant a^{2+\delta}E_n(\boldsymbol{b})$ where $\delta>0$, for let $a_i=b_i=1$ $i=1,\ldots,n-2$, $a_{n-1}=X-1$ and $a_n=X=b_{n-1}=b_n$, where X is large, then

$$\log\frac{b^{2+\delta}E_n(\boldsymbol{a})}{a^{2+\delta}E_n(\boldsymbol{b})}=\frac{\delta}{2X}+0\left(\frac{1}{X^2}\right).$$

(2) Under the conditions of Theorem 3 we do not in general have $b^2E_S(\boldsymbol{a})\leqslant a^2E_S(\boldsymbol{b})$ when $2\leqslant S\leqslant n$, for let $a_1=b_1=1$, $a_2=3$, $a_3=4=b_2=b_3$ then $b^2E_2(\boldsymbol{a})=1539$ but $a^2E_2(\boldsymbol{b})=1536$.

4. Inequalities of Ruderman, Minc and Daykin

We remind the reader that $\boldsymbol{\alpha}^{(t)}$ denotes a rearrangement of $\boldsymbol{a}^{(t)}$ in non-decreasing order.

Theorem 4. *If $\boldsymbol{a}^{(t)}$ $(t=1,\ldots,m)$ are n-tuples of non-negative real numbers, $a_i=\Sigma_{t=1}^m a_i^{(t)}$, $\alpha_i=\Sigma_{t=1}^m \alpha_i^{(t)}$ and S an integer satisfying $2\leqslant S\leqslant n$, then*

$$E_S(\boldsymbol{\alpha})\leqslant E_S(\boldsymbol{a})$$

where $\boldsymbol{\alpha}=(\alpha_i)$ and $\boldsymbol{a}=(a_i)$. Equality holds if and only if either both sides are zero or $\boldsymbol{a}^{(1)}+\cdots+\boldsymbol{a}^{(m)}$ is a rearrangement of $\boldsymbol{\alpha}^{(1)}+\cdots+\boldsymbol{\alpha}^{(m)}$.

Proof. Without loss of generality we may suppose that $a_1\leqslant a_2\leqslant\cdots\leqslant a_n$. Clearly $\alpha_1^{(t)}+\cdots+\alpha_r^{(t)}\leqslant a_1^{(t)}+\cdots+a_r^{(t)}$ for $1\leqslant t\leqslant m$ and $1\leqslant r\leqslant n$. Thus $\Sigma_{i=1}^r\alpha_i\leqslant\Sigma_{i=1}^r a_i$ and hence $\boldsymbol{\alpha}\prec\boldsymbol{a}$. The result now follows by Theorem 1.

Remarks on Theorem 4. (1) When $S=n$ we have an inequality of Ruderman **(10)**. Minc **(6)** later re-proved the inequality and obtained the conditions for equality.

(2) Daykin's Theorem 2 in **(3)** is a special case of this theorem with the $\boldsymbol{a}^{(t)}$ $(t=1,\ldots,m)$ just rearrangements of a single n-tuple $\boldsymbol{a}$, since Hall's Theorem on distinct representatives (see **(11)**) ensures that his result can be put in this form.

We now require:

Lemma. *If a and b are n-tuples such that* $a_1 \geq a_2 \geq \cdots \geq a_n > 0$ *and* $b_1 \geq a_1$, $b_1 b_2 \geq a_1 a_2, \ldots, b_1 b_2 \ldots b_n \geq a_1 a_2 \ldots a_n$ *then for* $p > 0$ *and* $r = 1, \ldots, n$, $b_1^p + b_2^p + \cdots + b_r^p \geq a_1^p + a_2^p + \cdots + a_r^p$.

Equality holds if and only if $a_i = b_i$ $(i = 1, \ldots, n)$.

Proof. The lemma clearly follows from the special case $p = 1$, $r = n$ which is well-known (see (**12**) pp. 145–146).

Theorem 5. *If* $\mathbf{a}^{(t)}$ $(t = 1, \ldots, m)$ *are n-tuples of non-negative real numbers,* $a_i = \Pi_{t=1}^m a_i^{(t)}$, $\alpha_i = \Pi_{t=1}^m \alpha_i^{(t)}$ *and* $p > 0$ *then*

$$\sum_{i=1}^n a_i^p \leq \sum_{i=1}^n \alpha_i^p.$$

Equality holds if and only if (a_i) *is a rearrangement of* (α_i).

Proof. Without loss of generality we may suppose that $a_1 \leq a_2 \leq \cdots \leq a_n$. For $1 \leq t \leq m$ and $0 \leq r < n$ $\alpha_n^{(t)} \alpha_{n-1}^{(t)} \ldots \alpha_{n-r}^{(t)} \geq a_n^{(t)} a_{n-1}^{(t)} \ldots a_{n-r}^{(t)}$ so $\alpha_n \alpha_{n-1} \ldots \alpha_{n-r} \geq a_n a_{n-1} \ldots a_{n-r}$. Let a_q be the first non-zero a_i $(i = 1, \ldots, n)$ then, by the lemma, $\alpha_n^p + \alpha_{n-1}^p + \cdots + \alpha_q^p \geq a_n^p + \cdots + a_q^p = \Sigma_{i=1}^n a_i^p$ and the required inequality follows.

Suppose $\Sigma_{i=1}^n a_i^p = \Sigma_{i=1}^n \alpha_i^p$ then, for q as defined above, we must have $\alpha_i = 0$ for $i < q$ and $\alpha_n^p + \cdots + \alpha_q^p = a_n^p + \cdots + a_q^p$ which implies $\alpha_i = a_i$ for $q \leq i \leq n$ by the lemma and the theorem follows.

Remark on Theorem 5. When $p = 1$ we have an inequality of Ruderman (**10**). Minc (**6**) later re-proved the inequality and obtained the conditions for equality.

From the following theorem we can deduce immediately Theorems 3, 4 and 5 of Minc (**6**).

Theorem 6. *If* $\mathbf{a}^{(t)} \in R^n$ $(t = 1, \ldots, m)$ *and* (m_i) *and* (M_i) *are rearrangements in non-decreasing order of* $(\min_t a_i^{(t)})$ *and* $(\max_t a_i^{(t)})$ *respectively then* $m_i \leq \min_t \alpha_i^{(t)}$ *and* $M_i \geq \max_t \alpha_i^{(t)}$ $(i = 1, \ldots, n)$.

Proof. Suppose there is a k with $1 \leq k \leq n$ such that $m_k > \min_t \alpha_k^{(t)}$. Let $\alpha_k^{(\tau)} = \min_t \alpha_k^{(t)}$ so that $m_k > \alpha_k^{(\tau)}$. Since $\alpha_i^{(\tau)} \leq \alpha_k^{(\tau)}$ for $1 \leq i \leq k$ at least k of the m_i must be less than or equal to $\alpha_k^{(\tau)}$ so there is a $j > k$ such that $m_j \leq \alpha_k^{(\tau)} < m_k$ and we have a contradiction.

The inequality $M_i \geq \max_t \alpha_i^{(t)}$ follows by a similar argument.

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