THE USE OF S-FUNCTIONS IN COMBINATORIAL ANALYSIS

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1. Introduction. The aim of this paper is to present a unified treatment of certain theorems in Combinatorial Analysis (particularly in enumerative graph theory), and their relations to various results concerning symmetric functions and the characters of the symmetric groups. In particular, it treats of the simplification that is achieved by working with S-functions in preference to other symmetric functions when dealing with combinatorial problems. In this way it helps to draw closer together the two subjects of Combinatorial Analysis and the theory of Finite Groups. The paper is mainly expository; it contains little that is really new, though it displays several old results in a new setting.

Much of the work on which this paper is based goes back to the remarkable paper "The theory of group-reduced distributions" by J. H. Redfield,[1] published in 1927, a paper which remained unrecognized for the pioneering paper that it was until Harary (**12**) drew attention to the fact that it had "apparently anticipated most of the major developments in enumerative techniques and results for the next thirty years". Redfield's paper has prompted several developments since its "rediscovery", and still contains many ideas suggesting further research.

As so often happens when two previously unrelated branches of mathematics are found to overlap, there are clashes between the notations that have been independently adopted. I have tried to develop a notation that will be a reasonable compromise between the two systems, but inevitably it has sometimes been necessary to modify or change the usual symbolism, and, on occasion, even to dispense with well-established notations. I have considered it worthwhile to give indications of the pros and cons of the possible notations.

Because the paper is expository, and straddles two branches of mathematics, it seemed desirable to include, even at the risk of being verbose, brief descriptions of some theorems which, while quite well established, may not be known to all readers.

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[1]Since this appears to have been Redfield's only publication, all references to Redfield in what follows will be to this paper (**31**).

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2. Definitions and notation.

2.1. Partitions. A partition of an integer n is a collection of positive integers (not necessarily distinct) whose sum is n. In specifying a partition we shall usually make use of one of two notations. Either the individual summands (parts) will be displayed, e.g.,

$$(\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_p),$$

where $\lambda_1 + \lambda_2 + \lambda_3 + \ldots + \lambda_p = n$, in which case the parts will usually be written in non-increasing order $\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 \geqslant \ldots \geqslant \lambda_p$; or we shall indicate the number of times each part occurs by means of a superscript. Thus the partition (6, 6, 3, 1, 1) of the integer 17 (written according to the first notation) will be written in the second notation as $(6^2\ 3\ 1^2)$. In the latter notation we may write the parts in non-descending order, so that this same partition could also be written as $(1^2\ 3\ 6^2)$.

In talking about partitions in general we shall denote them by Greek letters, writing, for example, (λ), (ρ), etc., for typical partitions, or even just λ, ρ, etc. Latin letters will stand for integers. Thus (n) will denote a specific partition, namely, the partition of the integer n into exactly one part.

With a partition $(\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_p)$ we can associate a diagram (the "Young" diagram) consisting of p horizontal rows of dots, containing $\lambda_1, \lambda_2, \ldots, \lambda_p$ dots, respectively, and left justified. Thus for the partition (6, 6, 3, 1, 1) we have the diagram of Figure 1, containing 6, 6, 3, 1, and 1 dots,

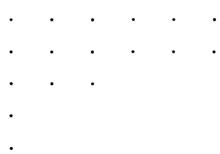

FIGURE 1

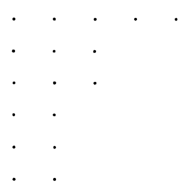

FIGURE 2

respectively, in the successive rows. If we interchange the rows and columns in a Young diagram, we obtain a diagram corresponding, in general, to a new partition, called the *conjugate* partition. Figure 2 is obtained from Figure 1 in this way; hence the partition (5, 3, 3, 2, 2, 2), or $(5\ 3^2\ 2^3)$, which it represents, is the conjugate of the partition (6, 6, 3, 1, 1), or $(6^2\ 3\ 1^2)$. The conjugate of a partition (λ) is denoted by $(\tilde{\lambda})$. If $(\tilde{\lambda}) = (\lambda)$, the partition is said to be self-conjugate. Figure 3 gives an example of a self-conjugate partition.

A third method of describing partitions, known as the Frobenius notation, will occasionally be used. In any Young diagram there is a diagonal of dots leading from the top left-hand corner. For each dot in this diagonal we write down the number of dots to its right, and, beneath that, we write the number of dots below it. If d is the number of dots in the diagonal we obtain a $2 \times d$

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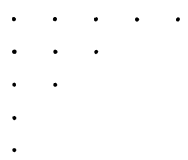

FIGURE 3

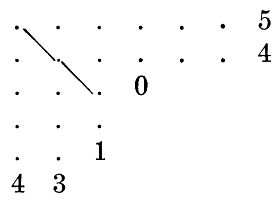

FIGURE 4

array of integers. This is illustrated by the partition in Figure 4, for which the array is

$$\begin{pmatrix} 5 & 4 & 0 \\ 4 & 3 & 1 \end{pmatrix}.$$

Note that the numbers in each row strictly decrease, and that zeros, even two zeros one above the other, are significant. For example, the arrays

$$\begin{pmatrix} 5 & 4 & 0 \\ 5 & 3 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 5 & 4 \\ 5 & 3 \end{pmatrix}$$

represent different partitions.

The conjugate of a partition expressed in the Frobenius notation is obtained by interchanging the two rows of the array.

2.2. Cycle-indices. If G is a permutation group of degree n, the cycle-index of G, denoted by $Z(G)$, is a polynomial in n indeterminates whose coefficients summarize the cycle-structure of G. Various letters have been used for these indeterminates, but for reasons that will appear later we shall use $s_1, s_2, \ldots, s_n$. Any element of G can be expressed as the product of disjoint cycles (unique except for order), and if this expression contains j_i cycles of length i ($i = 1, 2, \ldots, n$), we say that the element is of type $(1^{j_1} 2^{j_2} \ldots n^{j_n})$. Clearly, $j_1 + 2j_2 + \ldots + nj_n = n$. To form the cycle-index of G we write down the monomial $s_1^{j_1} s_2^{j_2} \ldots s_n^{j_n}$ for an element of type $(1^{j_1} 2^{j_2} \ldots n^{j_n})$, sum over all elements, and divide by the order of the group. Thus

$$Z(G) = \frac{1}{|G|} \sum_{(\rho)} A_\rho s_1^{j_1} s_2^{j_2} \ldots s_n^{j_n}, \tag{2.2.1}$$

where A_ρ is the number of elements of G whose type is given by the partition $(\rho) = (1^{j_1} 2^{j_2} \ldots n^{j_n})$ of n, and the summation is over all partitions of n.[2]

[2]The cycle-index of a permutation group was defined by Redfield under the name "Group Reduction Function", and denoted by the symbolism Grf(G). The notation $Z(G)$, due to Pólya, is now more or less standard, with $Z(G; s_1, s_2, \ldots, s_n)$ and $Z_G(s_1, s_2, \ldots, s_n)$ as useful variants when it is required to indicate the indeterminates.

It is well known that the total number of permutations of type $(1^{j_1} 2^{j_2} \ldots n^{j_n})$ is

$$\frac{n!}{1^{j_1} j_1!\, 2^{j_2} j_2! \ldots n^{j_n} j_n!}.$$

Hence the cycle-index of the symmetric group $\mathcal{S}_n$ of degree n is

$$(2.2.2) \qquad Z(\mathcal{S}_n) = \frac{1}{n!}\sum \frac{n!}{1^{j_1} j_1!\, 2^{j_2} j_2! \ldots n^{j_n} j_n!} s_1^{j_1} s_2^{j_2} \ldots s_n^{j_n},$$

summed over all partitions of n.

It will be convenient to abbreviate some of these expressions. Accordingly, if $(\rho) = (1^{j_1} 2^{j_2} \ldots n^{j_n})$, we shall write

$$s_\rho \text{ for } s_1^{j_1} s_2^{j_2} \ldots s_n^{j_n},$$

$$g_\rho \text{ for } \frac{n!}{1^{j_1} j_1!\, 2^{j_2} j_2! \ldots n^{j_n} j_n!},$$

and

$$\Pi(\rho) \text{ for } 1^{j_1} j_1!\, 2^{j_2} j_2! \ldots n^{j_n} j_n!.$$

We then have

$$(2.2.3) \qquad Z(G) = \frac{1}{|G|}\sum_\rho A_\rho s_\rho$$

and

$$(2.2.4) \qquad Z(\mathcal{S}_n) = \frac{1}{n!}\sum_\rho g_\rho s_\rho = \sum_\rho \frac{s_\rho}{\Pi(\rho)}.$$

If G and H are permutation groups of degrees m and n, respectively, acting on disjoint sets A and B, respectively, then there is a group $G \cdot H$, "the direct product" of G and H, which acts on $A \cup B$. A typical permutation of $G \cdot H$ is obtained by permuting the elements of A by a permutation belonging to G, and permuting the elements of B by a permutation belonging to H. Clearly, $|G \cdot H| = |G| \cdot |H|$.

This group is often written $G \times H$, and also, under the name of "direct sum", as $G + H$. It was shown by Pólya (**26**) that

$$(2.2.5) \qquad Z(G \cdot H) = Z(G) \cdot Z(H).$$

For this reason the notation "$G + H$" seems inappropriate. On the other hand "$G \times H$" is best reserved for the "Cartesian product" of two permutation groups (not used in this paper). This leaves "$G.H$" as about the only remaining possibility, if we are to stick to familiar symbols.

The "wreath product" (*Gruppenkranz*) of G and H, denoted by $G[H]$, is a permutation group of degree mn which acts on the elements of an $m \times n$ matrix. A typical permutation of $G[H]$ is obtained by permuting the elements in each row of the matrix by a permutation belonging to H (not necessarily the same permutation for each row) and then permuting the rows bodily by a permutation belonging to G.

It was shown by Pólya (**26**) that the cycle-index of $G[H]$ is obtained by "substituting" the cycle-index of H into that of G. This "substitution" requires definition, which we now give, at the same time generalizing it to apply to polynomials in $s_1, s_2, \ldots$ which are not necessarily cycle-indices.[3]

If A and B are polynomials in the indeterminates $s_1, s_2, \ldots,$ then by the "substitution of B into A" we shall mean the replacing of every s_r in A $(r = 1, 2, \ldots)$ by the polynomial obtained from B by multiplying by r the subscript of each of its indeterminates. Thus if

$$A = \tfrac{1}{2}(s_1^2 + s_2), \qquad B = \tfrac{1}{3}(s_1^3 + 2s_3),$$

the substitution is effected by replacing

$$s_1 \quad \text{by} \quad \tfrac{1}{3}(s_1^3 + 2s_3)$$

and

$$s_2 \quad \text{by} \quad \tfrac{1}{3}(s_2^3 + 2s_6)$$

in A. The result is

$$\tfrac{1}{2}\{\tfrac{1}{9}(s_1^3 + 2s_3)^2 + \tfrac{1}{3}(s_2^3 + 2s_6)\} = \tfrac{1}{8}(s_1^6 + 4s_1^3s_3 + 4s_3^2 + 3s_2^3 + 6s_6).$$

We shall denote the result of substituting B into A in this manner by $A[B]$. Pólya's expression for the cycle-index of the wreath product $G[H]$ can then be written as

$$Z(G[H]) = Z(G)[Z(H)].$$

2.3. Symmetric functions, alternants, bialternants. Let $\alpha_1, \alpha_2, \ldots, \alpha_k$ be any set of k indeterminates. A symmetric function of $\alpha_1, \alpha_2, \ldots, \alpha_k$ is one which is unchanged when these indeterminates are permuted in any way. An alternating function, or alternant, is one which is unchanged by an even permutation of the indeterminates, and changed in sign by an odd permutation of them. In what follows we shall confine ourselves to polynomial functions.

The following symmetric functions are of particular importance:

(i) the *power sums*, defined by

$$s_n = \sum \alpha_1^n = \alpha_1^n + \alpha_2^n + \ldots + \alpha_k^n, \tag{2.3.1}$$

where, by convention, the summation is assumed to be over all permutations of the α's;

(ii) the *elementary symmetric functions*, defined by

$$a_n = \sum \alpha_1 \alpha_2 \ldots \alpha_n \tag{2.3.2}$$

that is, the sum of all products of the α's n at a time (if $n > k$, then $a_n = 0$); and

[3]Other notations have been used for the wreath product, but have little to recommend them. The link with the idea of "substitution" is so strong that any notation that does not reflect this is liable to be misleading. To be consistent one should also change the name (wreath "product"), but this is less important. It is convenient to read $G[H]$ as "G of H".

(iii) the *homogeneous product sums*, defined by

$$h_n = \sum \alpha_1^{n_1} \alpha_2^{n_2} \alpha_3^{n_3} \ldots \alpha_h^{n_k} \tag{2.3.3}$$

summed for all permutations of the α's, and over all partitions $(n_1, n_2, \ldots, n_k)$ of n. Thus h_n is the sum of all products of the α's of total degree n.

The following results, being well known, are quoted without proof:

$$\begin{aligned} f(x) &\equiv \prod_{r=1}^{k} (1 - \alpha_r x) \\ &= 1 - a_1 x + a_2 x^2 - \ldots + (-1)^k a_k x^k, \end{aligned} \tag{2.3.4}$$

$$1/f(x) = 1 + h_1 x + h_2 x^2 + \ldots + h_r x^r + \ldots, \tag{2.3.5}$$

$$n a_n = s_1 a_{n-1} - s_2 a_{n-2} + s_3 a_{n-3} - \ldots \pm s_n, \tag{2.3.6}$$

$$n h_n = s_1 h_{n-1} + s_2 h_{n-2} + s_3 h_{n-3} + \ldots + s_n. \tag{2.3.7}$$

Proofs can be found in (**18**; **19**). Another result that we shall use is that any symmetric function can be expressed as a polynomial in the power sums s_n.

The expression

$$\sum \epsilon \alpha_1^{p_1} \alpha_2^{p_2} \ldots \alpha_k^{p_k}, \tag{2.3.8}$$

where the summation is over all permutations of the α's, and where $\epsilon = +1$ or -1 according as the permutation is even or odd, is easily seen to be an alternating function. It is readily verified that if two exponents are equal, (2.3.8) vanishes identically. Hence any such alternating function which does not vanish is of the form

$$\sum \epsilon \alpha_1^{\lambda_1+k-1} \alpha_2^{\lambda_2+k-2} \ldots \alpha_k^{\lambda_k} \tag{2.3.9}$$

where $\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 \geqslant \ldots \geqslant \lambda_k \geqslant 0$. Thus the alternant (2.3.9) can be associated with the partition $(\lambda) = (\lambda_1, \lambda_2, \ldots, \lambda_k)$. From the definition of a determinant it is seen that (2.3.9) can also be written as

$$|\alpha_s^{\lambda_t + n - t}|,$$

the displayed element being that in row s and column t.

The alternant (2.3.9) is zero if any two α's are equal. Hence it is divisible by $(\alpha_r - \alpha_s)$ for all r and s, and therefore by the product

$$\Delta(\alpha_1, \alpha_2, \ldots, \alpha_k) = \prod_{1 \leqslant r < s \leqslant k} (\alpha_r - \alpha_s). \tag{2.3.10}$$

This product is itself an alternant; in fact

$$\Delta(\alpha_1, \alpha_2, \ldots, \alpha_k) = \sum \epsilon \alpha_1^{k-1} \alpha_2^{k-2} \ldots \alpha_{k-1}. \tag{2.3.11}$$

Consequently, corresponding to any partition (λ) of n we may write the quotient of (2.3.9) and (2.3.10), and thus define a function which is written as $\{\lambda\}$. We have

$$(2.3.12) \qquad \{\lambda\} = \frac{\sum \pm \alpha_1^{\lambda_1+k-1}\ \alpha_2^{\lambda_2+k-2} \ldots \alpha_k^{\lambda_k}}{\Delta(\alpha_1, \alpha_2, \ldots, \alpha_k)}.$$

Now $\{\lambda\}$ is clearly a symmetric function, since any permutation of the α's either leaves both numerator and denominator of (2.3.12) unchanged, or changes the sign of both. These functions have been called *bialternants*.

It is possible to express $\{\lambda\}$ as a single determinant whose elements are either the homogeneous product sums h_r, or the elementary symmetric functions a_r. The results are

$$(2.3.13) \qquad \{\lambda\} = |h_{\lambda_s - s+t}| = |a_{\mu_s - s+t}|,$$

where $(\mu) = (\mu_1, \mu_2, \ldots)$ is $(\tilde{\lambda})$, the partition conjugate to (λ), and h_0 is interpreted as 1, and $h_r = 0$ if $r < 0$. For proofs of (2.3.13), see (**18**, p. 89).

Thus

$$\{3^2\,2\,1\} = \begin{vmatrix} h_3 & h_4 & h_5 & h_6 \\ h_2 & h_3 & h_4 & h_5 \\ 1 & h_1 & h_2 & h_3 \\ 0 & 0 & 1 & h_1 \end{vmatrix} = \begin{vmatrix} a_4 & a_5 & a_6 \\ a_2 & a_3 & a_4 \\ 1 & a_1 & a_2 \end{vmatrix}.$$

2.4. Group characters and S-functions. The bialternants just defined in terms of alternating functions also appeared in connection with the theory of group characters, bearing the alternative name of *Schur characteristic functions* or S-functions. For a full treatment of S-functions a knowledge of the definition and properties of the characteristics of the symmetric groups is required; but for the purposes of this paper we need to know very little about them. A characteristic of the symmetric group $\mathscr{S}_n$ is an integer (positive, negative, or zero), and there is a characteristic corresponding to every pair of partitions of the integer n. The standard notation is $\chi_\rho^{(\lambda)}$ for the characteristic corresponding to the partitions ρ and λ. The characteristics for $\mathscr{S}_n$ can therefore be displayed in a two-way table, of which Figure 5 is an example (for $n = 4$). The set of characteristics for a given λ and all partitions ρ is known as a group character.

$\lambda \backslash \rho$	1^4	$1^2 2$	13	4	2^2
4	1	1	1	1	1
31	3	1	0	−1	−1
2^2	2	0	−1	0	2
21^2	3	−1	0	1	−1
1^4	1	−1	1	−1	1

FIGURE 5

The group characteristics of $\mathscr{S}_n$ have an important orthogonality property (see **18**, p. 46) expressed by the following equations:

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$$\text{(2.4.1)} \qquad \sum_{\rho} \frac{g_\rho}{n!} \chi_\rho^{(\lambda)} \chi_\rho^{(\mu)} = \begin{cases} 1 & \text{if } \lambda = \mu, \\ 0 & \text{if } \lambda \neq \mu, \end{cases}$$

$$\text{(2.4.2)} \qquad \frac{g_\rho}{n!} \sum_{\lambda} \chi_\rho^{(\lambda)} \chi_\sigma^{(\lambda)} = \begin{cases} 1 & \text{if } \rho = \sigma, \\ 0 & \text{if } \rho \neq \sigma, \end{cases}$$

where g_ρ is defined as in §2.2.

The S-function $\{\lambda\}$ associated with a partition (λ) of n is defined in terms of characteristics by

$$\text{(2.4.3)} \qquad \{\lambda\} = \sum_{\rho} \frac{g_\rho}{n!} \chi_\rho^{(\lambda)} s_\rho = \sum_{\rho} \chi_\rho^{(\lambda)} \frac{s_\rho}{\Pi(\rho)}.$$

It can be shown that (as the notation has anticipated) the S-function defined by (2.4.3) and the bialternant defined by (2.3.12) are identical. For the proof, see (**18**, p. 87).

By multiplying both sides of (2.4.3) by $\chi_\sigma^{(\lambda)}$, summing over all λ, and using (2.4.2), we find that

$$\text{(2.4.4)} \qquad s_\sigma = \sum_{\lambda} \chi_\sigma^{(\lambda)} \{\lambda\}.$$

Since any symmetric function is expressible as a polynomial in the power sums, and hence as a linear function of the s_ρ's, it follows from (2.4.4) that any symmetric function is a linear combination of S-functions. We shall return to this result later.

3. Enumeration theorems. In this section we give a brief presentation of three theorems of importance in combinatorial analysis: Pólya's theorem, De Bruijn's theorem, and the superposition theorem. These three theorems have much in common; indeed, as we shall see later, it is almost true to say that they are equivalent, and hence can be regarded as different formulations of the same basic theorem. Nevertheless, as formulations, they are sufficiently different, that for specific applications, one of them usually turns out to be more convenient to use than the others.

3.1. Pólya's theorem. Let D and R be two given sets, and consider the set of all mappings f of D into R. With every element of R we shall associate a "content" which is in general a vector of non-negative integers, say $(n_1, n_2, \ldots, n_k)$. The content of a mapping is defined to be

$$\sum_{r \in R} n(r) \mathbf{C}_r,$$

where $n(r)$ is the number of elements of D that map onto r, $\mathbf{C}_r$ is the (vector) content of r, and the summation is a vector summation.

With the set D we associate a permutation group G of degree $|D|$, acting on the elements of D, and we regard two mappings as equivalent if we can convert one mapping into the other by permuting the elements of D by a permutation in G. In other words, f_1 and f_2 are equivalent if there is a $g \in G$ such that

$$f_1(d) = f_2(gd) \quad \text{for all } d \in D.$$

The problem then is: "What is the number of inequivalent mappings with a given content?"

Less formally, one can think of the elements of D as boxes, into each of which is to be placed one of a certain set of "figures". Repetitions are allowed in the sense that the same figure may appear in two or more boxes, i.e., several elements of D may map onto the same element of R. The result is a "configuration" whose content is the sum of the contents of the figures in the boxes. Certain permutations of the boxes, forming a group, are regarded as inessential, in that their application gives a configuration which will be regarded as essentially the same as the original.

The information concerning the figures, configurations, and their contents can be succinctly expressed by two formal series. The "figure-counting series" is defined as the polynomial

$$\phi(x_1, x_2, \ldots, x_k) = \sum f(n_1, n_2, \ldots, n_k) x_1^{n_1} x_2^{n_2} \ldots x_k^{n_k},$$

where $f(n_1, n_2, \ldots, n_k)$ is the number of figures having content $(n_1, n_2, \ldots, n_k)$. The "configuration-counting series" is defined as

$$\Phi(x_1, x_2, \ldots, x_k) = \sum F(n_1, n_2, \ldots, n_k) x_1^{n_1} x_2^{n_2} \ldots x_k^{n_k},$$

where $F(n_1, n_2, \ldots, n_k)$ is the number of configurations which have content $(n_1, n_2, \ldots, n_k)$.

We can now state Pólya's theorem.

PÓLYA'S THEOREM. *The configuration-counting series is obtained by substituting the figure-counting series in the cycle-index $Z(G)$; by which is meant replacing every s_r in $Z(G)$ by $\phi(x_1^r, x_2^r, \ldots, x_k^r)$.*

The result of this substitution is usually denoted by $Z(G; \phi(x_1, x_2, \ldots, x_k))$, or by $Z_G(\phi(x_1, x_2, \ldots, x_k))$. Hence Pólya's theorem can be written as

$$\Phi(x_1, x_2, \ldots, x_k) = Z(G; \phi(x_1, x_2, \ldots, x_k)).$$

Examples of Pólya's theorem are plentiful in the literature of combinatorial analysis (see, for example, **11**; **26**; **32**). As a fairly typical example we may take Harary's enumeration of unlabelled graphs on a given number n of nodes. Here the boxes are the $\frac{1}{2}n(n-1)$ pairs of nodes in (or rather, between) which may be placed either the figure "no edge" or the figure "one edge", which we take to have contents 0 and 1, respectively. A configuration is then a graph, and its content is the number of edges. The figure-counting series is just $1 + x$. Since permutations of the nodes are inessential, the appropriate group is the group $\mathscr{S}_n^{(2)}$ of permutations of the edges which is induced by the symmetric group $\mathscr{S}_n$ of all permutations of the nodes. Pólya's theorem then gives

$$Z(\mathscr{S}_n^{(2)}; 1 + x) \tag{3.1.1}$$

as the configuration-counting series. The number of graphs on n nodes and k edges is the coefficient of x^k in (3.1.1). For the evaluation of $Z(\mathscr{S}_n^{(2)})$ and the consideration of related problems, see Harary's paper (**11**).

3.2. De Bruijn's theorem. The theorem of De Bruijn (**5**) is a generalization of Pólya's theorem in two respects. First, the concept of the "content" of a mapping $f\colon D \to R$ is broadened to that of the "weight" of a mapping. This weight can be an element of a commutative ring, and is subject only to the restriction that equivalent mappings (in a sense to be defined) have the same weight. When, as is often the case, this ring is the ring of polynomials in k variables, we have the same situation as with Pólya's theorem, with the monomial weight $x_1^{n_1}x_2^{n_2} \ldots x_k^{n_k}$ corresponding to the vector content $(n_1, n_2, \ldots, n_k)$. Secondly, an extra permutation group H is introduced which acts on the elements of R. Two mappings f_1 and f_2 are now considered to be equivalent if one can be converted into the other by applying permutations in G and H to D and R, respectively; in other words, if there exist $g \in G$ and $h \in H$ such that

$$f_1(d) = hf_2(gd) \quad \text{for all } d \in D.$$

The problem is to find the sum of the weights associated with each equivalence class of mappings. This is analogous to finding the configuration-counting series when using Pólya's theorem.

The statement of De Bruijn's theorem is rather more complicated than that of Pólya's theorem. We first take R to be the union of p pairwise disjoint sets,

$$R = R_1 \cup R_2 \cup \ldots \cup R_p,$$

and H to be the direct product of p permutation groups,

$$H = H_1 \cdot H_2 \cdot \ldots \cdot H_p,$$

where H_i acts on R_i (there is no loss of generality in this). We further assume that the weight of a mapping $f\colon D \to R$ is defined by

$$W(f) = \textstyle\prod_{r \in R} \psi_i(n_r), \tag{3.2.1}$$

where $\psi_1(n), \psi_2(n), \ldots, \psi_p(n)$ are arbitrary functions associated with $R_1, R_2, \ldots, R_p$ and n_r is the number of elements of D that map onto r. (Since each element r of R belongs to exactly one R_i, the particular ψ_i to be used in (3.2.1) is always uniquely determined.) We now define

$$\eta_{it} = \sum_{n=0}^{\infty} Z(\mathscr{S}_n; ts_t, 2ts_{2t}, 3ts_{3t}, \ldots)[\psi_i(n)]^t,$$

where $i = 1, 2, \ldots, p$; $t = 1, 2, \ldots$. We can now state

De Bruijn's theorem. *The sum of the weights of the equivalence classes of mappings is given by*

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$$(3.2.2) \qquad Z(G;\partial/\partial s_1, \partial/\partial s_2, \ldots)_0 \prod_{i=1}^{k} Z(H_i;\eta_{i1}, \eta_{i2}, \ldots),$$

where the subscript zero indicates that after the differentiations are performed, we put $s_1 = s_2 = s_3 = \ldots = 0$.

It is easily verified that by taking each H_i to be the identity group, and each $\psi_i(n)$ to be x_i^n, we arrive at Pólya's theorem as a special case of De Bruijn's theorem. The sum of the weights is just the configuration-counting series.

As an example of the use of De Bruijn's theorem we may cite the enumeration of self-complementary graphs. We proceed as for the enumeration of ordinary graphs described above, but now introduce the extra group $H = \mathscr{S}_2$, which acts on the elements of R. This means that the complement of a graph, obtained by interchanging "edge" and "no edge", is now regarded as equivalent to the original graph. By comparing the result of the above with the configuration-counting series obtained from Pólya's theorem, the number of self-complementary graphs on a given number of nodes can be found. For details see (**29**; **30**).

Fairly recently it has been shown by Harary and Palmer (**14**) that by using Pólya's theorem in conjunction with a rather complicated permutation group (the power group, H^G) derived from two given groups G and H, the same results can be obtained as with De Bruijn's theorem using the groups G and H as above. Thus it appears that De Bruijn's theorem can be deduced from Pólya's theorem, at least when the weight function is defined by taking $\psi_i(n) = x_i^n$. Whether De Bruijn's theorem in its full generality (with $\psi_i(n)$ defined arbitrarily in a commutative field) can be derived from Pólya's theorem seems to be still an open question. It should be mentioned, too, that the cycle index of H^G is very difficult to compute by direct methods; a relatively easy method does exist for finding it, but amounts virtually to the use of De Bruijn's theorem.

3.3. The superposition theorem. Let us suppose that we have k sets $M_1, M_2, \ldots, M_k$, each of n elements. With each set M_i we associate a group G_i of permutations of its elements, and call this group the "automorphism group" of M_i. It will be convenient to think of the elements in each set as being strung out in a row, in some order or other. We shall regard two orderings of the elements in row i as equivalent if one is obtained from the other by application of a permutation in G_i. The number of inequivalent ways of arranging the elements of M_i in a row is then the number of cosets of G_i relative to $\mathscr{S}_n$.

We now "superpose" these k rows, writing them one under the other to form a matrix of k rows and n columns. (More formally, we establish a one-to-one correspondence between each M_i and the integers $1, 2, \ldots, n$). There are $n!$ ways of arranging the elements in each row, and hence $(n!)^k$ possible superpositions. Two superpositions (i.e., matrices) Σ_1 and Σ_2 will be said to be

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L-similar if every row of Σ_1 is equivalent to the corresponding row of Σ_2. The two superpositions will be said to be T-similar if there is a permutation of the columns of Σ_2 which converts it into a superposition which is L-similar to Σ_1. Both L-similarity and T-similarity are equivalence relations.

The problem is : Given $G_1, G_2, \ldots, G_k$, what is the number of equivalence classes determined by the relation of T-similarity? The answer to this problem was discovered by Redfield in 1927, and independently by the author in 1958 (**27**). Before stating the superposition theorem, which gives the answer, we must introduce another operation on polynomials.

Let A and B be two polynomials in $s_1, s_2, \ldots, s_n$, say

$$A = \sum_{(j)} A_{(j)} s_1^{j1} s_2^{j2} \ldots s_n^{jn},$$

$$B = \sum_{(j)} B_{(j)} s_1^{j1} s_2^{j2} \ldots s_n^{jn},$$

where the summations are over all partitions $(j) = (1^{j_1} 2^{j_2} \ldots n^{j_n})$ of n. This assumes that A and B are "isobaric", i.e., that $j_1 + 2j_2 + \ldots + nj_n$ is the same for all terms of the polynomials. There is no real loss of generality in this assumption; polynomials for which it is not true are sums of isobaric polynomials of differing weight (the value of n), and can be handled accordingly. We define the "inner product" $A * B$ by

$$A * B = \sum_{(j)} A_{(j)} B_{(j)} 1^{j_1} 2^{j_2} \ldots n^{jn} s_1^{j1} s_2^{j2} \ldots s_n^{jn} \tag{3.3.1}$$

or

$$\sum_{\rho} A_\rho B_\rho \Pi(\rho) s_\rho,$$

using the notation of §2. Thus the inner product of two isobaric polynomials is an isobaric polynomial of the same weight.[4] It is readily verified that the operation $*$ is commutative, associative, and distributive over addition, and that the cycle-index $Z(\mathscr{S}_n)$ acts as an identity element with respect to this operation. Since $*$ is associative, we can form the inner product of any number of polynomials, viz.,

$$A^{(1)} * A^{(2)} * \ldots * A^{(k)} = \sum_\rho A_\rho^{(1)} A_\rho^{(2)} \ldots A_\rho^{(k)} [\Pi(\rho)]^{k-1} s_\rho. \tag{3.3.2}$$

If A and B do not have the same weight, we define $A * B = 0$.

We denote by $N(A)$ the sum of the coefficients in the polynomial A, that is

$$N(A) = \sum_\rho A_\rho. \tag{3.3.3}$$

Thus $N(A)$ is the result of putting $s_1 = s_2 = \ldots s_n = 1$ in A. We can now state the superposition theorem.

SUPERPOSITION THEOREM. *The number of dissimilar superpositions, that is, the number of equivalence classes under the relation of T-similarity, is*

[4]The term "inner product" is due to Littlewood (**21**), who arrived at it in a completely different connection, which we shall consider later.

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$$N(Z(G_1) * Z(G_2) * \ldots * Z(G_k)). \tag{3.3.4}$$

A comment is in order concerning the notation just used, which is essentially that which was used in (**27**; **28**). Instead of our

$$A * B * \ldots * K$$

Redfield used

$$A \mathbin{\text{☋}} B \mathbin{\text{☋}} \ldots \mathbin{\text{☋}} K,$$

and apart from the outlandish choice of symbol (which both Foulkes (**10**), and Harary and Palmer (**15**) have replaced by $\cup$) there is nothing to dispute. The retention of $*$ is a matter of personal preference. But for

$$N(A * B * \ldots * K)$$

Redfield wrote

$$A \mathbin{\text{☊}} B \mathbin{\text{☊}} \ldots \mathbin{\text{☊}} K, \tag{3.3.5}$$

which can reasonably be judged to be a bad notation. For unlike ☋, the operation ☊ cannot be regarded as being a generalization, by associativity, of a binary operator. To be logical one would have to regard the whole assemblage of ☊'s in (3.3.5) as forming a sort of composite operator, acting on the set of polynomials $A, B, \ldots, K$ to yield a number. By using "$N(\ )$" together with $*$, no logical or notational difficulty arises; the operation denoted by "$N(\ )$" is a mapping from the set of polynomials into (usually) the rational numbers, and the standard functional notation for mappings has been adhered to. The choice of N, rather than some other letter, is arbitrary, but it ties in nicely with the notation $N(G_1, G_2)$ used by Littlewood (see **18**, p. 165) in a similar connection.[5]

If we consider a particular superposition, and regard it as a set of elements, namely the columns of the matrix, we see that it has an automorphism group, namely the group of permutations of the columns which convert it into a T-similar superposition. Thus with each superposition we associate a group, and hence a corresponding cycle-index. It was shown by Redfield that the superposition theorem is a consequence of a more general result, which deserves to be called the master theorem in this branch of combinatorics.

Redfield's master theorem. *The polynomial*

$$Z(G_1) * Z(G_2) * \ldots * Z(G_k)$$

is the sum of the cycle-indices of the automorphism groups of the distinct superpositions enumerated by the superposition theorem.

Since, by definition, $N(Z(G)) = 1$ for any permutation group G, it is clear how the superposition theorem follows from the master theorem; each cycle-index contributes 1 to the total.

[5]This is not the reason it was originally chosen; the agreement is coincidental.

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It follows from the master theorem that any inner product can be decomposed into the sum of cycle-indices. Redfield observed that this was often possible in many ways, yet only one decomposition is correct in a given application. He was unable to determine how to recognize the correct decomposition, and it was only recently that this problem was resolved by Foulkes (**10**), who showed how it can be done. In this paper we shall not be concerned with this problem.

The superposition theorem has many applications in enumerative graph theory. The sets $M_1, M_2, \ldots$ are the node sets of various graphs, each with its automorphism group, defined in the standard way (see, for example, **13**, p. 195). Superposition is effected by placing the graphs on top of each other so that their nodes coincide. L-similarity means similarity as labelled graphs, and T-similarity means similarity as unlabelled graphs (i.e., topological similarity). The superposition theorem then gives the number of superposed graphs. It should be noted that in the superposed graph the edges from different graphs are distinguished, since they correspond to different sets M_i. Various uses of the superposition theroem will be given in the following sections; others can be found in (**27**).

4. Derivation of Pólya's theorem from the superposition theorem.

4.1. In this section we shall see how Pólya's theorem can be deduced from the superposition theorem. Rather than taking space to give a full formal proof, we shall consider two kinds of figure-counting series, and derive Pólya's theorem for them. The second of these will, however, be sufficiently general that it will be clear how a fully general proof would go.

4.2. We consider first a common application of Pólya's theorem, in which there is only one figure, and each "box" either contains, or does not contain, the figure. The content of a box is 1 or 0 respectively, and the figure-counting series is $1 + x$. This is the situation considered in the example of §3. Our aim is to find the number of configurations with a specified content, say p.

We shall use the superposition theorem with $k = 2$, our two sets being

(i) a set of n boxes, with associated automorphism group G, and

(ii) a set of p 1's and $n - p$ 0's.

The automorphism group of the latter set is $\mathscr{S}_p \cdot \mathscr{S}_{n-p}$. Every superposition of these two sets clearly corresponds to an allocation of either a 1 (a figure) or a 0 (no figure) to every box. Since T-similar superpositions will correspond to indistinguishable configurations, it follows from the superposition theorem that

$$F_p = N(Z(G) * Z(\mathscr{S}_p \cdot \mathscr{S}_{n-p})) \tag{4.2.1}$$

is the number of essentially different configurations with content p. Note that Pólya's theorem gives a counting series, from which any individual

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coefficient has to be dug out, whereas here we have an expression for a single coefficient in that series (frequently a more useful result). We can, however, recover the configuration-counting series; it is

$$\Phi(x) = \textstyle\sum_p F_p x^p = \sum_p N(Z(G) * Z(\mathcal{S}_p \cdot \mathcal{S}_{n-p}))x^p = N(Z(G) * [\sum_p Z(\mathcal{S}_p \cdot \mathcal{S}_{n-p})x^p]) \tag{4.2.2}$$

by virtue of the distributivity of $*$ over addition.

Now the generating function for $Z(\mathcal{S}_m)$ is

$$\textstyle\sum_m Z(\mathcal{S}_m)t^m = \exp(s_1 t + \frac{1}{2}s_2 t^2 + \frac{1}{3}s_3 t^3 + \ldots). \tag{4.2.3}$$

Hence

$$\begin{aligned} \textstyle\sum Z(\mathcal{S}_m)t^m \cdot \sum_m Z(\mathcal{S}_m)x^m t^m &= \\ \exp(s_1 t + \tfrac{1}{2}s_2 t^2 + \ldots) \exp(s_1 xt + \tfrac{1}{2}s_2 x^2 t^2 + \ldots) &= \\ \exp\{s_1 t(1+x) + \tfrac{1}{2}s_2 t^2(1+x^2) + \ldots\}&, \end{aligned} \tag{4.2.4}$$

and in this expression

$$\textstyle\sum_p Z(\mathcal{S}_p \cdot \mathcal{S}_{n-p})x^p = \sum_p Z(\mathcal{S}_p) \cdot Z(\mathcal{S}_{n-p})x^p$$

is the coefficient of t^n. By (4.2.3) the coefficient of t^n in (4.2.4) is just $Z(\mathcal{S}_n)$ with each s_r replaced by $s_r(1 + x^r)$, i.e.,

$$Z(\mathcal{S}_n; s_1(1+x), s_2(1+x^2), \ldots).$$

Let us now consider

$$Z(G) * Z(\mathcal{S}_n; s_1(1+x), s_2(1+x^2), \ldots). \tag{4.2.5}$$

But for the factors $(1 + x^r)$, (4.2.5) would reduce to $Z(G)$, since $Z(\mathcal{S}_n)$ is an identity with respect to the operation $*$. As it is, the effect of these factors is to make (4.2.5) into the cycle-index $Z(G)$ with each s_r replaced by $s_r(1 + x^r)$, i.e.,

$$Z(G; s_1(1+x), s_2(1+x^2), \ldots).$$

Hence when we apply $N(\ )$ to (4.2.5), putting each $s_r = 1$, we obtain

$$Z(G; 1+x, 1+x^2, \ldots)$$

for the configuration-counting series, which is exactly the result of Pólya's theorem.

4.3. We now consider a figure-counting series of the form

$$\alpha_1 + \alpha_2 + \ldots + \alpha_p, \tag{4.3.1}$$

i.e., we can place any one of p distinct objects in each box. To find the number of configurations in which n_i boxes contain the object α_i $(i = 1, 2, \ldots, p)$, which will be the coefficient of $\alpha_1^{n_1}\alpha_2^{n_2} \ldots \alpha_p^{n_p}$ in the configuration-counting series, we arrange the boxes in a row, and superpose a set consisting of n_1 α_1's, n_2 α_2's, and so on. Proceeding as in §4.2, we obtain

$$N(Z(G) * Z(\mathscr{S}_{n_1}) \cdot Z(\mathscr{S}_{n_2}) \ldots Z(\mathscr{S}_{n_p})) \tag{4.3.2}$$

as the number of configurations. The configuration-counting series is therefore

$$\begin{aligned}(4.3.3)\quad \sum N(Z(G) * Z(\mathscr{S}_{n_1}) \cdot Z(\mathscr{S}_{n_2}) \ldots Z(\mathscr{S}_{n_p}))\alpha_1^{n_1}\alpha_2^{n_2} \ldots \alpha_p^{n_p} &= \\ N(Z(G) * (\sum Z(\mathscr{S}_{n_1}) \cdot Z(\mathscr{S}_{n_2}) \ldots Z(\mathscr{S}_{n_p})\alpha_1^{n_1}\alpha_2^{n_2} \ldots \alpha_p^{n_p})),&\end{aligned}$$

where the summations are for all n_i such that $n_1 + n_2 + \ldots + n_p = n$. The second operand in (4.3.3) is the coefficient of t^n in

$$\begin{aligned}\prod_{i=1}^{p} \sum Z(\mathscr{S}_{n_i})(t\alpha_i)^{n_i} &= \prod_{i=1}^{p} \exp\{s_1 t\alpha_i + \tfrac{1}{2}s_2 t^2\alpha_i^2 + \tfrac{1}{3}s_3 t^3\alpha_i^3 + \ldots\} \\ &= \exp\{s_1 t\sum\alpha_i + \tfrac{1}{2}t^2 s_2\sum\alpha_i^2 + \ldots\}\end{aligned}$$

and this coefficient is $Z(\mathscr{S}_n)$ with s_r replaced by $s_r(\sum\alpha_i^r)$. By the same reasoning as before the argument of $N(\ )$ in (4.3.3) reduces to $Z(G)$ with s_r replaced by $s_r(\sum\alpha_i^r)$. Hence (4.3.3) itself reduces to

$$Z(G; \sum\alpha_i, \sum\alpha_i^2, \ldots), \tag{4.3.4}$$

all summations being for $i = 1, 2, \ldots, p$. This again is the result given by Pólya's theorem.

Without going into details we can see how the proof for a general figure-counting series would go. If a_i of the objects are to have content i, we replace i of the α's in (4.3.1) by x^i. Thus we replace (4.3.1) by

$$\phi(x) = a_0 + a_1 x + a_2 x^2 + \ldots,$$

a general figure-counting series in one variable. An expression like $\sum\alpha_i^r$ is replaced by

$$a_0 + a_1 x^r + a_2 x^{2r} + \ldots = \phi(x^r)$$

and (4.3.4) becomes the Pólya result:

$$Z(G; \phi(x), \phi(x^2), \ldots).$$

The extension to polynomials in any number of variables is immediate. Infinite figure-counting series can be catered for by recalling that in any given application only a finite number of terms of the figure-counting series can play a part.

Thus it appears that, in a very real sense, the figure-counting series (4.3.1) is the most general; by suitable substitutions, any other figure-counting series can be derived from it. The importance of this remark will appear in the next section.

Hence the superposition theorem implies Pólya's theorem, which in turn implies De Bruijn's theorem, at least in its most common form. I have shown

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elsewhere (**30**) that De Bruijn's theorem implies the superposition theorem for two operands inside $N(\ )$, and by using some further extension made by De Bruijn to his theorem (see **6**) the general superposition theorem can be deduced.[6] Hence the remark at the beginning of §3 that the three theorems are (almost) equivalent.

5. A unified treatment. We shall now attempt to link together some of the preceding results. The main step is to give an interpretation to the variables s_r that occur in cycle-indices, and to this end we take s_r to stand for

$$\sum_{i=1}^{p} \alpha_i^{\,r},$$

where $\alpha_1, \alpha_2, \ldots, \alpha_p$ are indeterminates. This has, of course, been anticipated by the notation of §2, and means that every cycle-index is now a symmetric function. This idea is not new, since Redfield defined his group reduction functions in precisely this manner, but it does not seem that anyone else has found it advantageous to regard cycle-indices in this way.

At worst we cannot lose by giving this meaning to s_r, even if we never make use of it, and in fact there are many advantages in doing so. For the symmetric group we have

$$Z(\mathscr{S}_n) = \{n\} = h_n, \tag{5.1.1}$$

and from now on we shall use either $\{n\}$ or h_n to denote this symmetric function. Another function which has occurred frequently in combinatorial literature is

$$Z(\mathscr{A}_n) - Z(\mathscr{S}_n)$$

(where $\mathscr{A}_n$ is the alternating group). It has been variously written as $Z(U_n)$ (in **28**) and as $Z(A_n - S_n)$ (in **13**). It is easily shown that

$$Z(U_n) = Z(A_n) - Z(S_n) = \{1^n\} = a_n \tag{5.1.2}$$

and we shall use $\{1^n\}$ or a_n in place of the other, more cumbersome, notations.

The device of taking $s_r = \sum \alpha_i^{\,r}$ certainly leads to more elegance if nothing else. For example, if we take the figure-counting series in Pólya's theorem to be

$$\alpha_1 + \alpha_2 + \ldots + \alpha_p,$$

which, as we have seen, is effectively its most general form, then the configuration-counting series (4.3.4) becomes $Z(G; s_1, s_2, \ldots)$, and the statement of Pólya's theorem assumes the following remarkably simple form.

Revised statement of Pólya's theorem. *The configuration-counting series is the cycle-index.*

---

[6]A further link is provided by the very recent work of Sheehan (**34**) who has produced a generalization of Pólya's theorem from which the superposition theorem can be deduced.

More is to be gained than mere elegance, however. Since cycle-indices are now symmetric functions, they can be expressed as linear combinations of S-functions. Hence to evaluate an expression of the form

$$N(A * B)$$

it is sufficient to be able to evaluate it when the operands are S-functions. The rest follows from the distributivity of $*$ over addition. We therefore consider two S-functions

$$\{\lambda\} = \textstyle\sum_\rho \chi_\rho^{(\lambda)} s_\rho/\Pi(\rho)$$

and

$$\{\mu\} = \textstyle\sum_\rho \chi_\rho^{(\mu)} s_\rho/\Pi(\rho)$$

corresponding to partitions (λ) and (μ) of n. Then by (3.3.1)

$$\begin{aligned}\{\lambda\} * \{\mu\} &= \sum_\rho \chi_\rho^{(\lambda)} \frac{1}{\Pi(\rho)} \cdot \chi_\rho^{(\mu)} \frac{1}{\Pi(\rho)} \Pi(\rho) \\ &= \frac{1}{n!} \sum g_\rho \chi_\rho^{(\lambda)} \chi_\rho^{(\mu)} s_\rho\end{aligned} \tag{5.1.3}$$

so that

$$N(\{\lambda\} * \{\mu\}) = \frac{1}{n!} \sum_\rho g_\rho \chi_\rho^{(\lambda)} \chi_\rho^{(\mu)}.$$

Hence, by (2.4.1),

$$N(\{\lambda\} * \{\mu\}) = \begin{cases} 1 & \text{if } (\lambda) = (\mu), \\ 0 & \text{if } (\lambda) \neq (\mu). \end{cases} \tag{5.1.4}$$

This "orthogonal" property of S-functions means that non-zero contributions to $N(A * B)$ come only from S-functions occurring in both operands. If

$$A = \textstyle\sum_\rho A_\rho \{\rho\} \quad \text{and} \quad B = \sum_\rho B_\rho\{\rho\},$$

then

$$N(A * B) = \textstyle\sum_\rho A_\rho B_\rho. \tag{5.1.5}$$

This parallels the original definition of $N(A * B)$ but has the advantage that since the coefficients A_ρ and B_ρ are integers, the computation is much easier.

This raises the question of how easy it is to express a symmetric function in terms of S-functions. It is certainly always possible, as was remarked in §2, by use of equation (2.4.4); but this method requires tables of group characters,[7] and these are conveniently available only up to $n = 14$. We therefore look for methods which will at any rate apply to the sort of symmetric functions which often occur in combinatorial analysis. Two common kinds are the cycle-

[7]For tables of group characters up to $n = 10$, see (**18**); for $n = 11$, see (**37**); for $n = 12, 13$, see (**38**); for $n = 14$, see (**16**). For information on characters for $n = 15, 16$, see (**1**). Tables for $n \leqslant 20$ on paper tape and printed sheets are kept in the library of Mathematikmaskinnamnden, Stockholm 6, Sweden; see (**3**).

indices of direct products and wreath products of symmetric groups. We shall consider them separately.

6. Direct products of symmetric groups. We have

$$
\begin{aligned}
Z(\mathscr{S}_{n1}\cdot\mathscr{S}_{n2}\cdot \ldots \cdot\mathscr{S}_{nk}) &= Z(\mathscr{S}_{n1})\cdot Z(\mathscr{S}_{n2})\cdot \ldots \cdot Z(\mathscr{S}_{nk}) \\
&= h_{n_1}\, h_{n2} \ldots h_{n_k}
\end{aligned}
\tag{6.1.1}
$$

by (2.2.5). We therefore require a method of expressing $h_{n_1}h_{n_2} \ldots h_{n_k}$ in terms of S-functions.

If we have the expansion of $h_{n_1}h_{n_2} \ldots h_{n_{k-1}}$ in terms of S-functions, we can complete the expansion by multiplying each S-function already obtained by h_{n_k}. What is needed therefore is a method for evaluating a product of the form $\{\lambda\}h_r$, where $\{\lambda\}$ is any S-function. Such a rule is given in (**18**) and elsewhere, and will now be described.

We write down the Young diagram for the partition (λ), and call the dots that compose it "old dots". Now, in every possible way, we add r "new dots" to this diagram, subject to the two conditions:

(a) the resulting diagram is proper, i.e., the number of dots in successive rows is non-increasing.

(b) no two new dots lie in the same vertical line.

The required expansion is then the sum of the S-functions corresponding to the diagrams thus obtained.

To illustrate this we evaluate $\{31\}h_2$. We must add two new dots (denoted by .) to the diagram

$$
\begin{matrix} x & x & x \\ x & & \end{matrix}
$$

where each x denotes an old dot. The possible ways are

$$
\begin{matrix} x & x & x & . & . \\ x & & & & \\ & & & & \end{matrix}
\qquad
\begin{matrix} x & x & x & . \\ x & . & & \\ & & & \end{matrix}
\qquad
\begin{matrix} x & x & x & . \\ x & & & \\ . & & & \end{matrix}
\qquad
\begin{matrix} x & x & x \\ x & . & . \\ & & \end{matrix}
\qquad
\begin{matrix} x & x & x \\ x & . & \\ . & & \end{matrix}
$$

Hence we have

$$
\{31\}h_2 = \{51\} + \{42\} + \{41^2\} + \{3^2\} + \{321\}.
$$

It should be remarked that a somewhat more complicated rule will give the product of any two S-functions (see **18** for details), but we shall not have occasion to perform such multiplication.

An application to graphical enumeration will show the power of this method. A formula was given in (**28**) for finding the number of bigraphs with labelled nodes of given valenceis. If we consider bigraphs on $3 + 3$ nodes, with valencies 3, 2, and 1 for one set of nodes and 2, 2, 2 for the other, then the number of different bigraphs is given by

$$
N(h_3\, h_2\, h_1 * h_2{}^3)
$$

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in the notation we are now employing. Using the multiplication rule given above, it is readily verified that

$$h_3 h_2 = \{5\} + \{41\} + \{32\},$$

and hence that

$$(6.1.2) \quad h_3 h_2 h_1 = \{6\} + 2\{51\} + 2\{42\} + \{41^2\} + \{3^2\} + \{321\}.$$

Furthermore, $$h_2^2 = \{4\} + \{31\} + \{2^2\}$$

and $$h_2^3 = \{6\} + 2\{51\} + 3\{42\} + \{41^2\} + \{3^2\} + 2\{321\} + \{2^3\}.$$

Hence, by (5.1.4), we have

$$\begin{aligned} N(h_3 h_2 h_1 * h_2^3) &= 1\cdot 1 + 2\cdot 2 + 2\cdot 3 + 1\cdot 1 + 1\cdot 1 + 1\cdot 2 \\ &= 15. \end{aligned}$$

This is a special case of the problem of constructing matrices with integer elements having given row and column totals, a problem solved in a different way by McMahon (**24**). The methods described in this paper are applicable to many of the problems discussed by McMahon in his book.

This method of multiplying h_n's may seem bizarre and unfamiliar at first sight, but it is easier than multiplying the corresponding symmetric functions (as polynomials in s_r), and when it is performed, the resulting expansions are more easily operated on by $N(*)$. This advantage is even more marked if the operations are performed on a computer (see §10).

7. Wreath products of symmetric groups.

7.1. Wreath products of symmetric groups and their cycle-indices are of common occurrence in combinatorial problems. According to the notation introduced in §2 we may write

$$Z(S_m[S_n]) \quad \text{as} \quad Z(S_m)[Z(S_n)],$$

a special case of $A[B]$. We can now write this as $h_m[h_n]$, a much more concise notation. This form of combination of symmetric functions has appeared quite independently in the literature under the name of "plethysm", a term introduced by Littlewood (**17**). For the symmetric function which we have just written as $h_m[h_n]$, Littlewood writes $\{n\} \otimes \{m\}$, read "$\{n\}$ plethys $\{m\}$". Both notations can be used with any S-functions, so that

$$\{\lambda\}[\{\mu\}] \quad \text{is the same as} \quad \{\mu\} \otimes \{\lambda\}.$$

To give a proof that they are the same would require the presentation of too much background material, but a statement implying their equivalence is to be found on p. 276 of (**19**). As far as I know, no one has previously remarked that the operation "plethys" and that of forming the wreath product are one and the same thing. Zia-ud-Din (**36**) used the method of substitution described in §2 to compute several of the simpler plethysms, but the wreath product had not then been invented.

We have here a clash of notation. Do we write $A[B]$ or $B \otimes A$? The balance of evidence indicates that, well established though it is, the notation "$\otimes$" should yield. It has already been observed that the formation of the wreath "product" is intimately bound up with the idea of substitution. The notation $A[B]$ emphasizes this fact; the notation $B \otimes A$ obscures it. Indeed, the attempt to regard $B \otimes A$ as some other sort of operation automatically leads to the observation that some of its properties are atypical, as Littlewood himself observes. He says (**19**): "Plethysm has no analogue at all in arithmetic, though it resembles in some manner the taking of powers as e.g., in the expression a^b, yet the laws are entirely different." The laws quoted are

$$A \otimes (B + C) = A \otimes B + A \otimes C, \tag{7.1.1}$$

$$A \otimes (BC) \quad = A \otimes BA \otimes C, \tag{7.1.2}$$

together with the remark that $\otimes$ is not distributive on the left with either addition or multiplication.

In the wreath-product notation (7.1.1) and (7.1.2) become

$$(B + C)[A] = B[A] + C[A], \tag{7.1.3}$$

$$(BC)[A] \quad = B[A]C[A], \tag{7.1.4}$$

which are exactly what one expects of a process analogous to substitution. Further, one would not expect that $A[B + C]$ would be the same as $A[B] + A[C]$, or that $A[BC]$ would equal $A[B]A[C]$. Although it is possible to expand these two expressions, the expansions are not simple. Notice also that the operation $\otimes$ is associative:

$$(A \otimes B) \otimes C = A \otimes (B \otimes C), \tag{7.1.5}$$

and for this the analogy with taking powers fails, since $(a^b)^c \neq a^{(bc)}$. In our notation (7.1.5) becomes

$$C[B[A]] = C[B][A], \tag{7.1.6}$$

which is quite natural.

7.2. There are many results concerning plethysms, established for other purposes, which can be used in graph-theoretical applications. One of the simplest is the expansion of $h_m[h_2] = \{2\} \otimes m$ in terms of S-functions. It is implied by the results of (**18**, p. 238) that this symmetric function is the sum of all the S-functions corresponding to partitions of $2n$ into even parts only, each such S-function being counted exactly once. Thus, for example,

$$h_4[h_2] = \{8\} + \{62\} + \{4^2\} + \{42^2\} + \{2^4\}. \tag{7.2.1}$$

Also of importance are the functions $h_m[a_2]$, $a_m[h_2]$, and $a_m[a_2]$. The first of these can be found from a general result (see **17**, p. 359), namely, that if

$$\{\lambda\}[\{\mu\}] = \sum_\nu c_\nu\{\nu\},$$

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then

$$(7.2.2) \qquad \{\lambda\}[\{\tilde{\mu}\}] = \textstyle\sum_\nu c_\nu\{\tilde{\nu}\}$$

provided the integer of which μ is a partition is even. Hence, since $a_2 = \tilde{h}_2$, the S-functions in $h_m[a_2]$ are those whose partitions are conjugate to those occurring in $h_m[h_2]$. The distinguishing mark of these partitions is that each part occurs an even number of times.

From (**18**, p. 238), it also follows that the S-functions in the expansion of $a_m[h_2]$ are those corresponding to partitions which, in the Frobenius notation (see §1), assume one of the following forms:

$$(7.2.3) \qquad \begin{pmatrix} a+1 \\ a \end{pmatrix}, \quad \begin{pmatrix} a+1 & b+1 \\ a & b \end{pmatrix}, \quad \begin{pmatrix} a+1 & b+1 & c+1 \\ a & b & c \end{pmatrix}, \text{ etc.}$$

By (7.1.2) it follows that the S-functions in the expansion of $a_m[a_2]$ are those whose partitions assume one of the following forms:

$$(7.2.4) \qquad \begin{pmatrix} a \\ a+1 \end{pmatrix}, \quad \begin{pmatrix} a & b \\ a+1 & b+1 \end{pmatrix}, \quad \begin{pmatrix} a & b & c \\ a+1 & b+1 & c+1 \end{pmatrix}, \text{ etc.}$$

One further result worth citing is that

$$(7.2.5) \qquad h_2[h_n] = \sum_{v=0}^{[\frac{1}{2}n]} \{2n-2v, 2v\} \qquad \text{(see } \mathbf{17}\text{, p. 336).}$$

Hence by way of examples we have

$$(7.2.6) \qquad h_4[a_2] = \{1^8\} + \{2^2 1^4\} + \{2^4\} + \{3^2 1^2\} + \{4^2\},$$

$$(7.2.7) \qquad a_4[h_2] = \{51^3\} + \{431\},$$

$$(7.2.8) \qquad a_4[a_2] = \{41^4\} + \{32^2 1\},$$

$$(7.2.9) \qquad h_2[h_4] = \{8\} + \{62\} + \{4^2\}.$$

7.3. The expression of the symmetric function $h_m[h_n]$ in terms of S-functions when $m, n > 2$ is a matter of some difficulty. No really satisfactory method for this expansion has been devised, though several methods, each with its peculiar advantages and disadvantages, have been published (**7**; **8**; **9**; **19**; **22**). Thus, for example, it is known that

$$(7.3.1) \qquad h_4[h_3] = \{12\} + \{10, 2\} + \{93\} + \{84\} + \{82^2\} + \{741\} \\ + \{732\} + \{6^2\} + \{642\} + \{62^3\} + \{5421\} + \{4^3\},$$

and many other plethysms for low values of m and n have been calculated.

One method which is immediately suggested by the equivalence of wreath products and plethysms is to calculate $A[B]$ in the form of a polynomial in $s_1, s_2, \ldots$ by direct substitution, and then express this polynomial in terms of S-functions by the use of equation (2.4.4). The first stage rapidly becomes

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unwieldy as m and n increase, while the second stage requires the values of group characters for $\mathscr{S}_{mn}$ to be known. The method has the advantage of being simple and systematic, and therefore more suited than some of the other methods to the use of a computer, in which the group characters can be calculated as required, but it has grave faults of its own. We return to this problem in §10.

8. Applications.

8.1. It will scarcely have escaped the attention of anyone who has perused the literature of enumerative graph theory that the subject is bedevilled by a plethora of "half-results", formulae which are purported to yield the number of graphs of such and such a kind, and in theory do so, but which are in fact quite useless for practical computation. The use of S-functions makes feasible the evaluation of at least some of these formulae.

Thus, for example, it was shown in (**27**) that the number of general graphs (loops and multiple edges allowed) with $2n$ nodes, each of valency 3 (general cubical graphs), is given by

$$N(h_{3n}[h_2] * h_{2n}[h_3]). \tag{8.1.1}$$

If we evaluate this by the original method for finding inner products direct from the cycle-indices, we find that even for the simplest case, $n = 1$, it still takes a good page of algebraic manipulation to calculate the cycle-indices and arrive at the answer, 2, after much effort! For $n = 2$ the computation is considerably more tiresome, and one would be well advised not to attempt $n = 3$. Using expansions of $h_{2n}[h_3]$ in terms of S-functions that have been obtained by others in connection with algebraic applications, we can evaluate (8.1.1) more easily. Thus when $n = 2$ we apply (5.1.4) to $h_6[h_2]$ and $h_4[h_3]$, and since $h_6[h_2]$ is the sum of S-functions whose partitions have only even parts, we have merely to add the coefficients of similar S-functions in (7.3.1). The total thus obtained is 8, which is the required number of general cubical graphs on four nodes.

One can, with justice, object that the use of S-functions in this way does not really make the computation any simpler, but merely affords us the opportunity to stand on the shoulders of others who have already done most of the hard work. This cannot be said of the next application, which is to the enumeration of labelled graphs with nodes of given valency.

It was shown in (**28**) that the number of graphs on n labelled nodes with valencies $v_1, v_2, \ldots, v_n$ is

(i) $N(h_{v_1} h_{v_2} \ldots h_{v_n} * h_m[h_2])$ if loops and multiple edges are allowed;

(ii) $N(h_{v_1} h_{v_2} \ldots h_{v_n} * h_m[a_2])$ if multiple edges are allowed but loops are not;

(iii) $N(h_{v_1} h_{v_2} \ldots h_{n_n} * a_m[h_2])$ if loops are allowed but multiple edges are not;

(iv) $N(h_{v_1} h_{v_2} \ldots h_{v_n} * a_m[a_2])$ if neither loops nor multiple edges are allowed.

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Here, $m = \frac{1}{2}\sum v_i$ denotes the number of edges.

The evaluation of these four expressions by the methods then available (direct operation on the polynomials) represented a considerable amount of tedious computation. Using the S-function technique they present few problems. The product $h_{v_1} h_{v_2} \ldots h_{v_n}$ is expanded in terms of S-functions by the method described in §5. We then have merely to pick out the S-functions whose partitions fall into the four categories mentioned in §7.2. Thus in the expression

$$(8.1.2) \quad h_4 h_2 h_1^2 = \{8\} + 3\{71\} + \{421^2\} + \{42^2\} + 2\{431\} + 4\{521\} + \{4^2\} + 3\{53\} + 4\{62\} + \{51^3\} + 3\{61^2\},$$

which is easily verified, there are $1 + 1 + 1 + 4 = 7$ partitions containing only even parts, i.e., occurring in (7.2.1); one which occurs in (7.2.6); $1 + 2$ occurring in (7.2.7); and none occurring in (7.2.8). Hence the numbers of graphs of the four kinds mentioned above are 7, 1, 3, and 0, respectively. This can be verified from Figure 6. (Note that a double loop must be counted as a multiple edge.)

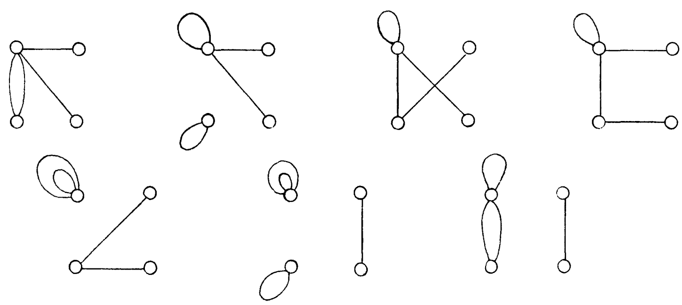

FIGURE 6

8.2. It appears then that algebraic results pertaining to S-functions, plethysms, and so on, can be of use in enumerative graph theory. One may well wonder whether graph theory can be of use to the algebraist. There is reason to believe that this is so, as the following example indicates.

The orthogonal property of S-functions enables one to pick out individual coefficients in an expansion, for $N(A*\{\lambda\})$ will be the coefficient of $\{\lambda\}$ in the expansion of A in terms of S-functions. Let us look for the coefficient of $\{pq\}$ in the expansion of the plethysm $h_m[h_n]$. Naturally $p + q = mn$, and we shall assume that $p \geqslant q$. The required coefficient is

$$(8.2.1) \qquad N(h_m[h_n] * \{pq\}).$$

Since $\{pq\} = h_p\, h_q - h_{p+1}\, h_{q-1}$, by (2.3.13), it suffices to evaluate expressions of the form

$$N(h_m[h_n] * h_p\, h_q). \tag{8.2.2}$$

Now (8.2.2) has a graphical interpretation. It is the number of bigraphs for which one node set, A, consists of m unlabelled nodes, each of valency n; while the other, B, consists of two labelled nodes, one (P) of valency p and the other (Q) of valency q.

Let r be the number of nodes of A which are not joined to P. Then all nr edges from these nodes go to Q, and the graph might appear as in Figure 7 (in which $m = 6$, $n = 3$, $p = 10$, $q = 8$, and $r = 2$).

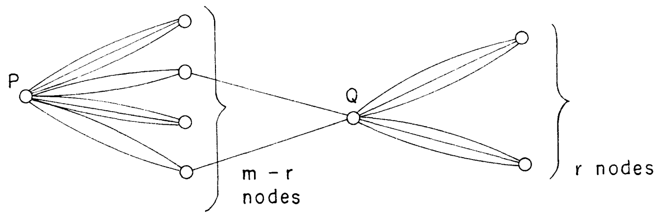

FIGURE 7

Now delete the r nodes not joined to P, and the edges joining them to Q. In the resulting bigraph every node of A is joined by one or more edges to P, and we can remove one of these edges for each node. The nodes and edges thus removed can be restored in a unique way, and hence the bigraphs thus obtained are equinumerous with the original bigraphs for the given value of r. But the resultant bigraph has $m - r$ nodes in A, each of valency $n - 1$; P now has valency $p - m + r$, and Q has valency $q - rn$. The number of such bigraphs is

$$Z(h_{m-r}[h_{n-1}] * h_{p-m+r}\, h_{q-rn}).$$

Hence we have

$$N(h_m[h_n] * h_p\, h_q) = \sum_{r=0}^{m} N(h_{m-r}[h_{n-1}] * h_{p-m+r}\, h_{q-rn}). \tag{8.2.3}$$

Using the similar result for $h_{p+1}\, h_{q-1}$ and subtracting, we find that

$$N(h_m[h_n] * \{pq\}) = \sum_{r=0}^{m} N(h_{m-r}[h_{n-1}] * \{p - m + r, q - rn\}). \tag{8.2.4}$$

Hence the required coefficient can be found if the expansions of the simpler plethysms $h_{m-r}[h_{n-1}]$ are known.

Equation (8.2.4) is possibly not a new result, and perhaps not very interesting, but it is certainly derived without much difficulty. Many other apparently new results can be obtained by similar graphical arguments.

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8.3. Nothing has so far been said about the use of these methods in conjunction with De Bruijn's theorem. The reason is that the added power of that theorem resides in its ability to produce the formulae giving the solutions to combinatorial problems. Once a formula has been obtained, the task of evaluating it is much the same as if it had been obtained by means of the superposition theorem or Pólya's theorem. The final result of De Bruijn's theorem, as quoted in §3, is an expression of the form

$$A(\partial/\partial s_1, \partial/\partial s_2, \ldots, \partial/\partial s_n)_0 B(s_1, s_2, \ldots, s_n), \tag{8.3.1}$$

where A is a polynomial with rational coefficients and B is a polynomial with coefficients which may depend on one or more variables, so that the final result is a generating function. Let us consider the effect of a term

$$A_{(j)}\left(\frac{\partial}{\partial s_1}\right)^{j_1}\left(\frac{\partial}{\partial s_2}\right)^{j_2}\ldots\left(\frac{\partial}{\partial s_n}\right)^{j_n}_0$$

of A acting on a term $B_{(k)}\, s_1^{k_1}s_2^{k_2}\ldots s_n^{k_n}$ of B.

If there is an i such that $j_i \neq k_i$, then the result is zero. For if $j_i > k_i$, the j_i differentiations reduce $s_i^{k_i}$ to zero, while if $j_i < k_i$, the result becomes zero when we put $s_i = 0$. Hence for non-zero terms, the partitions $(1^{j_1}2^{j_2}\ldots)$ and $(1^{k_1}2^{k_2}\ldots)$ must be the same. The effect of

$$A_{(j)}\left(\frac{\partial}{\partial s_1}\right)^{j_1}\left(\frac{\partial}{\partial s_2}\right)^{j_2}\ldots$$

acting on $B_{(j)}\, s_1^{j_1}s_2^{j_2}\ldots$ is clearly to give

$$j_1!\,j_2!\ldots j_n!\,A_{(j)}\,B_{(j)}. \tag{8.3.2}$$

If we let $B_{(j)} = C_{(j)}\,1^{j_1}2^{j_2}\ldots n^{j_n}$, then (8.3.2) can be written as

$$A_{(j)}\,C_{(j)}\,j_1!\,j_2!\ldots j_n!\,1^{j_1}\,2^{j_2}\ldots n^{j_n}, \tag{8.3.3}$$

which is a typical term in the inner product of two polynomials. Now

$$\sum_{(j)} C_{(j)}\,s_1^{j_1}\,s_2^{j_2}\ldots s_n^{j_n} = \sum_{(j)} B_{(j)}\left(\frac{s_1}{1}\right)^{j_1}\left(\frac{s_2}{2}\right)^{j_2}\ldots\left(\frac{s_n}{n}\right)^{j_n}$$
$$= B(s_1, \tfrac{1}{2}s_2, \tfrac{1}{3}s_3, \ldots).$$

Hence formula (8.3.1) can be written as

$$N(A * C), \tag{8.3.4}$$

where $C = B(s_1, \frac{1}{2}s_2, \frac{1}{3}s_3, \ldots)$. But in the statement of De Bruijn's theorem in §3.2, the function B was

$$\prod_{i=1}^{k} Z(H_i; \eta_{i1}, \eta_{i2}, \ldots),$$

where

$$\eta_{it} = \sum_{n=0}^{\infty} Z(S_n; ts_t, 2ts_{2t}, \ldots)[\psi_i(n)]^t.$$

If we replace each s_r by $(1/r)s_r$, then

$$\eta_{it} = \sum_{n=0}^{\infty} Z(S_n; s_t, s_{2t}, \ldots)[\psi_i(n)]^t. \tag{8.3.5}$$

Further, $Z(S_n; s_t, s_{2t}, \ldots) = h_n[s_t]$ in our notation, and we can therefore replace (8.3.5) by

$$\eta_{it} = \sum_{n=0}^{\infty} h_n[s_t][\psi_i(n)]^t. \tag{8.3.6}$$

Hence we can write the right-hand side of De Bruijn's theorem (3.2.2) as

$$N\left(Z(G) * \prod_{i=1}^{k} Z(H_i; \eta_{i1}, \eta_{i2}, \ldots)\right) \tag{8.3.7}$$

with η_{it} given by (8.3.6). This indicates that the formulae obtained by applying De Bruijn's theorem may also be amenable to simplification and evaluation by the methods given above.

8.4. The use by De Bruijn of differential operators in the statement of his theorem resembles closely the use by Foulkes (**7**) of differential operators associated with S-functions. Corresponding to the S-function

$$\{\lambda\} = \frac{1}{n!}\sum g_\rho\, \chi_\rho^{(\lambda)}\, s_1^{j_1}\, s_2^{j_2} \ldots s_n^{j_n},$$

where $\rho = (1^{j_1}2^{j_2} \ldots n^{j_n})$, Foulkes defines the operator

$$D_\lambda = \frac{1}{n!}\sum g_\rho\, \chi_\rho^{(\lambda)}\, 1^{j_1}2^{j_2} \ldots n^{j_n}\left(\frac{\partial}{\partial s_1}\right)^{j_1}\left(\frac{\partial}{\partial s_2}\right)^{j_2} \ldots \left(\frac{\partial}{\partial s_n}\right)^{j_n}, \tag{8.4.1}$$

which corresponds, in the Pólya-De Bruijn notation, to an operator of the form

$$Z(A; \partial/\partial s_1, 2\partial/\partial s_2, 3\partial/\partial s_3, \ldots, n\partial/\partial s_n).$$

Let (λ) be a partition of r, and (μ) a partition of s. It is readily verified, as in 8.3, that if $r = s$, then

$$D_\lambda\{\mu\} = N(\{\lambda\} * \{\mu\}), \tag{8.4.2}$$

and hence that

$$D_\lambda\{\mu\} = \begin{cases} 0 & \text{if } (\lambda) \neq (\mu), \\ 1 & \text{if } (\lambda) = (\mu), \end{cases}$$

as in (5.1.4). This result was obtained by Foulkes.

An interesting point arises here, in that the right-hand side of (8.4.2) is symmetrical in λ and μ, whereas the left-hand side is not. We have defined the right-hand side to be zero if $r \neq s$; but if $r < s$ the left-hand side of (8.4.2) is a symmetric function of weight $s - r$. If $r > s$, the left-hand side of (8.4.2) is zero if it is interpreted as a number, but can be interpreted as an operator. The symmetric function defined by the left-hand side of (8.4.2), where $r < s$, is written $\{\lambda/\mu\}$, and it can be shown that

$$\{\lambda/\mu\} = |h_{\lambda_s - \mu_t - s + t}|. \tag{8.4.3}$$

The greater versatility of these differential operators compared with those in the statement of De Bruijn's theorem lies in the fact that they do not carry with them the injunction to put all $s_r = 0$ after the differentiations have been performed. It is therefore pertinent to wonder what would happen if we were to omit from De Bruijn's theorem the equating to zero of the s_r's. The result of (3.2.2) is then, in general, a polynomial in the s_r's. What, if anything, is the significance of this polynomial?

9. Calculation of inner products.

9.1. We have seen how the superposition theorem yields the number of configurations of a particular type by means of an expression of the form

$$N(A * B * \ldots * K), \tag{9.1.1}$$

and that the results of applications of the theorems of Pólya and De Bruijn can be expressed in this form. Yet (9.1.1) does not give all the information that can be wrung from the cycle-indices $A, B, \ldots, K$. Redfield's master theorem states that the inner product

$$A * B * \ldots * K \tag{9.1.2}$$

is the sum of the cycle-indices of the automorphism groups of the configurations, and this tells us much more. For example, we could use each configuration as a set of "boxes" for a further application of Pólya's theorem, substituting a new figure-counting series into the cycle-indices. The resulting sum can then be found by direct substitution into (9.1.2), that is by substitution in all the cycle-indices at once. De Bruijn, in (**6**), has treated in detail this method of imposing further structures on configurations, though in a slightly different setting.

It is therefore important that there should be ways and means of calculating inner products like (9.1.2). Since $*$ is associative, it is sufficient to confine ourselves to a single inner product $A * B$, and, as a further simplification, we can take A and B to be S-functions. Thus we look for a method of expressing $\{\lambda\} * \{\mu\}$ as a linear combination of S-functions. Clearly (λ), (μ) are partitions of the same integer, say n.

9.2. The original definition of inner product was given by Littlewood (**21**) and arose from the following considerations. Let λ, μ be two partitions of n

and consider the characters $\chi_\rho^{(\lambda)}$ and $\chi_\rho^{(\mu)}$, where ρ runs through all partitions of n. It can be shown that the product $\chi_\rho^{(\lambda)}\chi_\rho^{(\mu)}$ is a linear combination of characters, and we can write

$$\chi_\rho^{(\lambda)}\ \chi_\rho^{(\mu)} = \textstyle\sum_\nu g_{\lambda\mu\nu}\ \chi_\rho^{(\nu)}, \tag{9.2.1}$$

where the $g_{\lambda\mu\nu}$ are, in fact, integers. Littlewood defined the inner product of S-functions such that its expansion would be analogous to (9.2.1), i.e., such that

$$\{\lambda\} \circ \{\mu\} = \sum g_{\lambda\mu\nu}\{\nu\}, \tag{9.2.2}$$

where $\{\lambda\} \circ \{\mu\}$ is the notation he originally used (in **21**); later (in **23**) he used $\{\lambda\} \cdot \{\mu\}$.

We shall continue to use $\{\lambda\} * \{\mu\}$, mainly because it is more distinctive, and does not conflict with the natural tendency to use a dot, as well as simple juxtaposition, to denote ordinary algebraic multiplication.

From (5.1.3) we have

$$\begin{aligned}\{\lambda\} * \{\mu\} &= \frac{1}{n!}\sum_\rho g_\rho\ \chi_\rho^{(\lambda)}\chi_\rho^{(\mu)} s_\rho = \frac{1}{n!}\sum_\rho g_\rho\Big(\sum_\nu g_{\lambda\mu\nu}\ \chi_\rho^{(\nu)}\Big)s_\rho \\ &= \sum_\nu g_{\lambda\mu\nu}\Big(\frac{1}{n!}\sum_\rho g_\rho\ \chi_\rho^{(\nu)} s_\rho\Big) = \sum_\nu g_{\lambda\mu\nu}\{\nu\},\end{aligned} \tag{9.2.3}$$

verifying that our $*$ and Littlewood's $\circ$ represent the same operation. If we write σ in place of ν in (9.2.3) and form the inner product with another S-function $\{\nu\}$, we get

$$\{\lambda\} * \{\mu\} * \{\nu\} = \textstyle\sum_\sigma g_{\lambda\mu\sigma}\, N(\{\sigma\} * \{\nu\}),$$

whence

$$N(\{\lambda\} * \{\mu\} * \{\nu\}) = \textstyle\sum_\sigma g_{\lambda\mu\sigma}\, N(\{\sigma\} * \{\nu\}) = g_{\lambda\mu\nu} \tag{9.2.4}$$

since $N(\{\sigma\} * \{\nu\}) = 0$ unless $\sigma = \nu$. On the other hand, from (3.2.2),

$$N(\{\lambda\} * \{\mu\} * \{\nu\}) = \frac{1}{n!}\sum_\rho g_\rho\ \chi_\rho^{(\lambda)}\chi_\rho^{(\mu)}\chi_\rho^{(\nu)}. \tag{9.2.5}$$

Hence

$$g_{\lambda\mu\nu} = \frac{1}{n!}\sum_\rho g_\rho\ \chi_\rho^{(\lambda)}\chi_\rho^{(\mu)}\chi_\rho^{(\nu)}. \tag{9.2.6}$$

From the symmetry of this expression it follows that the values of $g_{\lambda\mu\nu}$ are independent of the order of the subscripts.

Armed with the values of the relevant $g_{\lambda\mu\nu}$ we can readily evaluate inner products, since $g_{\lambda\mu\nu}$ is the coefficient of $\{\nu\}$ in the expansion of $\{\lambda\} * \{\mu\}$. Thus the whole problem turns on the feasibility of computing these numbers. This is not altogether easy. Equation (9.2.6) gives a simple, direct method, but it has the disadvantage, as far as hand calculation goes, of needing the group characters. The use of (9.2.6) for evaluation by computer is feasible, and will be considered in the next section.

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9.3. A different method of evaluating inner products was given by Littlewood (**21**) as a generalization of a method of Robinson and Taulbee (**33**). Mention has already been made of an algorithm for expressing the product of two S-functions in terms of S-functions (§6), and we can therefore define a set of numbers $\Gamma_{\rho\sigma\nu}$ by

$$\{\rho\}\{\sigma\} = \textstyle\sum_\nu \Gamma_{\rho\sigma\nu}\{\nu\}. \tag{9.3.1}$$

Littlewood's result (Theorem III of (**21**)) then reads

$$\{\lambda\}\{\mu\} * \{\nu\} = \sum \Gamma_{\rho\sigma\nu}(\{\lambda\} * \{\rho\})(\{\mu\} * \{\sigma\}) \tag{9.3.2}$$

summed for all partitions ρ, σ of the same weight as λ and μ, respectively.

If we take the special case for which $\{\lambda\} = h_r$, $\{\mu\} = h_s$, and ν is a partition of $r + s$, we get the result of Robinson and Taulbee. Equation (9.3.2) becomes

$$h_r\, h_s * \{\nu\} = \sum \Gamma_{\rho\sigma\nu}(h_r * \{\rho\})(h_s * \{\sigma\}) = \sum \Gamma_{\rho\sigma\nu}\{\rho\}\{\sigma\} \tag{9.3.3}$$

since $h_r * \{\rho\} = \{\rho\}$, etc. Here ρ and σ are partitions of r and s, respectively. In the same way we have

$$h_{r_1}\, h_{r_2} \ldots h_{r_k} * \{\nu\} = \sum \Gamma_{\rho_1\rho_2\ldots\rho_k\nu}\{\rho_1\}\{\rho_2\} \ldots \{\rho_k\}, \tag{9.3.4}$$

where ρ_i is a partition of r_i, ν is a partition of $r_1 + r_2 + \ldots + r_k$, and $\Gamma_{\rho_1\rho_2\ldots\rho_k\nu}$ is the coefficient of $\{\nu\}$ in $\{\rho_1\}\{\rho_2\} \ldots \{\rho_k\}$.

From (9.3.4) we can calculate inner products for which one polynomial is a product of h's. But since

$$\{\nu\} = |h_{\nu_s - s + t}|,$$

any S-function can be expressed linearly in terms of such products, and hence the inner product of any two S-functions can be found.

The need to split up one of the S-functions into products of h's in order to apply this method makes it less simple than the one described in §9.2; furthermore it requires the computation of a great number of coefficients $\Gamma_{\rho_1\rho_2\ldots\rho_k\nu}$. However, it has the advantage of not requiring the values of the group characters.

10. Use of computers.

10.1. Many of the formulae that have been quoted in the previous sections appear concise and simple, but frequently this appearance is misleading, the formulae being extremely difficult to evaluate. Often the difficulty is not in the operations that need to be performed but rather in the sheer amount of work that has to be done. At other times the difficulty is the need for data that are not readily available, such as in the many problems which call for a knowledge of group characters of $\mathscr{S}_n$ for $n > 14$.

This is a situation which is ideally suited to the employment of a digital computer. The basic unit of information required in the handling of problems of this kind is that consisting of a number (usually an integer) and a partition.

This partition may represent an actual partition, or the S-function associated with that partition, or perhaps the monomial $s_1^{j_1}s_2^{j_2} \dots s_n^{j_n}$ associated with it. The number is the coefficient of whatever the partition represents. The handling of these partitions calls for some interesting programming approaches, but once a few routines have been compiled for performing certain basic operations, programs for many kinds of problems can be built up comparatively easily. We discuss a few of these below.

10.2. Characters of the symmetric group. Although tables of characters of the symmetric groups $\mathscr{S}_n$ have been published only for $n \leqslant 14$, there are methods for calculating individual group characters when they are required. Methods of doing this have been discussed by Comét (**2**; **3**), who relied on a result of Nakayama (**25**). A good part of Comét's discussion was concerned with the programming problem of storing partitions efficiently in a fixed word-length binary computer (see also **4**). This problem is simplified if the computer is a variable word-length decimal machine, and a program for calculating group characteristics has been written for, and successfully run on such a computer (an IBM 1620) at the University of the West Indies.

10.3. Partitions. Many of the formulae given in this paper require summation over all partitions of a given integer. Therefore a computer program is required for producing these partitions. The brute force, recursive method of doing this is probably as good as any. The first partition is n; the next is $n - 1$ followed by all partitions of 1, etc., $n - r$ followed by all partitions of r, etc., 1 followed by all partitions of $n - 1$.

10.4. Products of S-functions. Given a partition (λ) and an integer r, we wish to find the S-functions in the expansion of $\{\lambda\}h_r$. This requires a program to carry out the manipulations of the Young diagram described in §5. By using the output of such a program as the input to the same program (supplying the value of r again) the expansions of expressions of the form

$$h_{n_1}\, h_{n_2} \dots h_{n_k}$$

are obtained.

10.5. Inner products of S-functions. As was shown in §9, inner products of S-functions can be found if the values of the coefficients

$$g_{\lambda\mu\nu} = N(\{\lambda\} * \{\mu\} * \{\nu\})$$

are known. Since

$$g_{\lambda\mu\nu} = \frac{1}{n!}\sum_{\rho} g_\rho\, \chi_\rho^{(\lambda)}\chi_\rho^{(\mu)}\chi_\rho^{(\nu)},$$

this coefficient can be computed if programs to generate the partitions ρ, and the characteristics $\chi_\rho^{(\lambda)}$ are available. Also needed is a program to calculate g_ρ, a relatively simple matter. A program has been written in this way for the IBM 1620 for the calculation of $g_{\lambda\mu\nu}$, and it is hoped to produce tables of these coefficients in the near future. Note that since $g_{\lambda\mu\nu}$ is symmetrical in its arguments, it is only necessary to tabulate these coefficients for $\lambda \geqslant \mu \geqslant \nu$, where $\geqslant$ is some ordering relation among the partitions such as, for example, the order in which they are generated by the method of §10.3.

The above programs could also be combined to calculate inner products by Littlewood's method, described in §9. This has the advantage of not requiring the calculation of group characters, but requires multiplications of S-functions over many combinations of partitions, followed by manipulation of determinants of differing size.

10.6. Plethysms. The expansion of plethysms gives rise to many more difficulties than the other problems considered. One possible method has already been mentioned in §8, viz., to substitute one polynomial directly into the other, and then express the polynomial thus obtained in terms of S-functions by use of (2.4.4). This would require a computer program to perform algebraic substitutions, and although there is nothing impossible in this, it is an even further cry from the normal arithmetical operations of computers than is the handling of partitions, and would be a task not to be undertaken lightly.

One method which seems to have possibilities is the following. By differentiating the generating function (4.2.3) with respect to t we obtain

$$(10.6.1)\qquad \sum_{m=0}^{\infty} mh_m t^{m-1} = \left(\sum_{n=0}^{\infty} h_n t^n\right)(s_1 + s_2 t + s_3 t^2 + \ldots + s_r t^{r-1} + \ldots).$$

Equating coefficients of t^m, we have

$$(10.6.2)\qquad (m+1)h_{m+1} = s_1 h_m + s_2 h_{m-1} + \ldots + s_r h_{m-r} + \ldots + s_m h_0,$$

where $h_0 = 1$ by convention. Hence

$$(10.6.3)\qquad (m+1)h_{m+1}[h_n] = \sum_{r=1}^{m} s_r[h_n]h_{m-r}[h_n],$$

which gives $h_{m+1}[h_n]$ in terms of simpler plethysms $h_{m-r}[h_n]$ provided that the expressions $s_r[h_n]$ can be evaluated.

Now $s_r[h_n]$ is the result of writing h_n, with its subscripts multiplied by r, in place of s_r; it is therefore just h_n with every subscript multiplied by r. Consider now $h_n[s_r]$. This is the result of replacing every s_p in h_n by s_r with its subscript multiplied by p; that is, we replace each s_p in h_n by s_{pr}. Hence these two results are the same. We have

$$(10.6.4)\qquad s_r[h_n] = h_n[s_r].$$

Note that this still holds if, in place of h_n, we put any symmetric function. The expression $\{\mu\}[s_r]$ was written by Foulkes (**7**) as $\{\mu\}^{(r)}$, and plays an important part in several suggested methods for evaluating plethysms (see **7**; **35**).

To express $h_n[s_r]$ in terms of S-functions, using a computer, is comparatively easy. Since

$$h_n = \sum_{\rho} \frac{g_\rho}{n!} s_\rho,$$

we have

$$h_n[s_r] = \sum_{\rho} \frac{g_\rho}{n!} s_\sigma,$$

where σ denotes the partition of nr obtained by multiplying every part of ρ by r. Then, since $s_\sigma = \sum_\lambda \chi_\sigma^{(\lambda)}\{\lambda\}$, we obtain

$$(10.6.5) \qquad h_n[s_r] = \sum_{\rho} \sum_{\lambda} \frac{g_\rho}{n!} \chi_\sigma^{(\lambda)}\{\lambda\} = \sum_{\lambda} \frac{1}{n!} \left(\sum_{\rho} g_\rho \, \chi_\sigma^{(\lambda)} \right) \{\lambda\}.$$

By means of the program for calculating group characters, (10.6.5) can be used to find the coefficients in the expansion of $h_n[s_r]$. To evaluate each term in (10.6.3) then requires a program to multiply any two S-functions, which would be only a little more complicated than the program already described for evaluating $\{\lambda\}h_n$. The fact that all the coefficients on the right-hand side of (10.6.3) must be divisible by $m + 1$ provides a useful check on the accuracy if the method is being applied by hand, or in the debugging stage of a computer application. The main advantage is that no algebraic substitution is required. This method was suggested by Littlewood (**20**), and a method of calculating $h_n[s_r]$ without finding the group characteristics explicitly was given by Todd (**35**).

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