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ANNIHILATING POLYNOMIALS AND POSITIVE FORMS

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ABSTRACT. The annihilating polynomials for trace forms, as discovered recently by Conner, are shown to also annihilate many other classes of positive quadratic forms over a field F provided that F satisfies suitable conditions.

Let $W(F)$ denote the Witt ring of non-singular quadratic forms over a field F of characteristic unequal to two. See [4] or [7] for the full definition. $W(F)$ is a commutative ring with an identity element and so, given any polynomial $p(x)$ with integer coefficients and any non-singular quadratic form ϕ over F, we can regard $p(\phi)$ as an element of $W(F)$. We say that the polynomial p *annihilates* the form ϕ if $p(\phi) = 0$ in $W(F)$. In [5] we discovered polynomials $p_n(x)$ with the property that p_n annihilates all non-singular n-dimensional quadratic forms over all fields F of characteristic unequal to two. These polynomials are defined as follows:

$$p_n(x) = \begin{cases} x(x^2-2^2)(x^2-4^2)\cdots(x^2-n^2) & \text{if } n \text{ is even,} \\ (x^2-1^2)(x^2-3^2)\cdots(x^2-n^2) & \text{if } n \text{ is odd.} \end{cases}$$

At about the same time Conner [1] found polynomials $q_n(x)$ which annihilated all n-dimensional trace forms of separable extension fields of F. The polynomials $q_n(x)$ are defined as follows:

$$q_n(x) = \begin{cases} (x-n)(x-n-2)\cdots(x-3)(x-1) & \text{if } n \text{ is odd,} \\ (x-n)(x-n-2)\cdots(x-2)x & \text{if } n \text{ is even.} \end{cases}$$

(We may think of q_n as the "positive half" of p_n).

A *positive form* is a non-singular quadratic form over F whose signature at each ordering of F is non-negative. Trace forms are well-known to be positive forms. In view of Conner's result one may wonder whether every positive n-dimensional quadratic form is annihilated by q_n. However this is easily seen to be false in the case when $n = 1$. In this paper we look at the polynomials $q_n(x)$ in a new way and we are able to show that if F is a field for which I^3 is torsion-free, (I denoting the fundamental ideal in $W(F)$), then, for all $n > 1$, each n-dimensional positive form is annihilated by q_n. For fields F in general we show that any n-dimensional positive form is annihilated by q_m for some $m \geq n$. We also show that if $I^3 = 0$ then every non-singular quadratic form over F is annihilated by some monic integer polynomial of degree two.

We will use the standard notation and terminology for quadratic forms and Witt rings as in [4] or [7].

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Let $\phi = \langle a_1, a_2, \ldots, a_n \rangle$ be a non-singular n-dimensional quadratic form over F, each a_i being a non-zero element of F. We will write $\psi_i = \langle 1, -a_i \rangle$ for each $i = 1, 2, \ldots, n$. Note that ψ_i is a 1-fold Pfister form and thus satisfies $\psi_i^2 = 2\psi_i$ for each i.

We will consider the following polynomials $q_{n,r}(x)$ where r and n are positive integers.

$$q_{n,r}(x) = (n-x)(n-2-x)\cdots(n-2r-x)$$

Note that, for $r = [n/2]$, (the greatest integer less than or equal to $n/2$), $q_{n,r}(x) = (-1)^{r+1}q_n(x)$. Thus to show that $q_n(\phi) = 0$ it suffices to show that $q_{n,r}(\phi) = 0$ for $r = [n/2]$. Observe also that $q_{n,n}(x) = p_n(x)$ for all n.

LEMMA. *For all positive integers n and for all integers r in the range of $1 \leq r \leq n-1$, $q_{n,r}(\phi) = (r+1)!\,(\sum \psi_{i1}\psi_{i2}\ldots\ldots\psi_{ir+1})$ where each $i_j \in \{1, 2, \ldots, n\}$ and the sum is taken over all possible choices of the i_j with $i_1 < i_2 < \cdots < i_r < i_{r+1}$. (In particular $q_{n,r}(\phi)$ is a sum of $(r+1)$-fold Pfister forms).*

PROOF. The proof will be an induction on r. First let $r = 1$.

$$\begin{aligned}\text{Then } q_{n,1}(\phi) = (n-\phi)(n-2-\phi) &= \Big(\sum_{i=1}^{n}\psi_i\Big)\Big(\Big(\sum_{i=1}^{n}\psi_i\Big)-2\Big)\\ &= \Big(\sum_{i=1}^{n}\psi_i^2\Big) + 2\Big(\sum_{i<j}\psi_i\psi_j\Big) - 2\Big(\sum_{i=1}^{n}\psi_i\Big)\\ &= 2\Big(\sum_{i<j}\psi_i\psi_j\Big) \text{ since } \psi_i^2 = 2\psi_i.\end{aligned}$$

Thus the lemma is true for $r = 1$. Now assume the lemma is true for $r = k-1$ where $2 \leq k < n$.

Then

$$\begin{aligned}q_{n,k}(\phi) &= q_{n,k-1}(\phi)(n-2k-\phi)\\ &= (k!)\Big(\sum \psi_{i_1}\psi_{i_2}\ldots\psi_{i_k}\Big)\Big(\Big(\sum_{i=1}^{n}\psi_i\Big)-2k\Big)\end{aligned}$$

using the inductive assumption. (The summation here is over all choices of i_j such that $i_1 < i_2 < \cdots < i_k$).

Thus $q_{n,k} = (k+1)(k!)(\sum \psi_{i_1}\psi_{i_2}\cdots\psi_{i_{k+1}}) + (k!)\{(\sum \psi_{i_1}^2\psi_{i_2}\cdots\psi_{i_k}) + (\sum \psi_{i_1}\psi_{i_2}^2\cdots\psi_{i_k}) + \cdots + (\sum \psi_{i_1}\psi_{i_2}\cdots\psi_{i_k}^2)\} - (2k)(k!)(\sum \psi_{i_1}\psi_{i_2}\cdots\psi_{i_k})$.

(The first summation is over all choices of i_j such that $i_1 < i_2 < \cdots < i_{k+1}$, and the other summations over all i_j such that $i_1 < i_2 < \cdots < i_k$). The fact that $\psi_i^2 = 2\psi_i$ yields the desired result that $q_{n,k}(\phi) = (k+1)!\,(\sum \psi_{i_1}\psi_{i_2}\cdots\psi_{i_{k+1}})$ and completes the proof by induction.

REMARK 1. In the argument of the above proof we have assumed that it is possible to choose $r+1$ distinct integers from $\{1, 2, \ldots, n\}$. This assumption is not valid if $r > n-1$ and is the reason for the upper bound on r in the statement of the lemma.

REMARK 2. Note that when $r = n-1$ there is only one term in the summation so that we have $q_{n,n-1}(\phi) = (n!)\psi_1\psi_2\cdots\psi_n$. This observation provides another way of seeing why the polynomials p_n of [4] annihilate all non-singular n-dimensional quadratic forms. Since $p_n(x) = q_{n,n}(x) = q_{n,n-1}(x)(n-2n-x)$ we have that

$$\begin{aligned} p_n(\phi) &= (n!)\psi_1\psi_2\cdots\psi_n\Big(-2n + (\sum_{i=1}^{n}\psi_i)\Big) \\ &= (-2n)(n!)\psi_1\psi_2\cdots\psi_n + 2n(n!)\psi_1\psi_2\cdots\psi_n \\ &= 0 \end{aligned}$$

THEOREM 1. *Let I denote the fundamental ideal in the Witt ring ($W(F)$ of the field F. If I^3 is torsion-free then, for all $n > 1$, each n-dimensional positive quadratic form over F is annihilated by the polynomial q_n.*

PROOF. It suffices to show that $q_{n,r}(\phi) = 0$ for any n-dimensional positive form ϕ when $r = [n/2]$. We write $\phi = \langle a_1, a_2, \ldots, a_n\rangle$ and $\psi_i = \langle 1, -a_i\rangle$ as earlier.

By the lemma $q_{n,r}(\phi) = (r+1)!\,(\sum \psi_{i_1}\psi_{i_2}\ldots\psi_{i_{r+1}})$. It is important to note that $(r+1) > n/2$ so that in choosing $r+1$ distinct integers i_j from $\{1, 2, \ldots, n\}$ we necessarily must have chosen more than half of the elements of this set.

Since $\phi = \langle a_1, a_2, \ldots, a_n\rangle$ is positive it follows that at least half of the a_i are positive in any given ordering of F. Hence each of the $(r+1)$-fold Pfister forms in the sum for $q_{n,r}(\phi)$ must be a torsion form. (At least one of the ψ_i will have zero signature at any given ordering!).

Now $q_{n,r}(\phi)$ for $r = [n/2]$ will be a sum of torsion elements which are in I^3. (Even when $n = 2$, $r = 1$, the factor 2 outside the sum can be absorbed inside to ensure that $q_{2,1}(\phi)$ is a sum of elements of I^3). Thus, for all $n > 1$, $q_n(\phi) = 0$ for any positive n-dimensional form ϕ. This completes the proof.

COMMENT 1. The fields F for which I^3 is torsion-free are precisely the fields over which quadratic forms are classified up to isometry by the classical invariants (i.e. dimension, discriminant, Witt invariant, and signatures). See [7, 2. 14. 6].

COMMENT 2. Theorem 1 can be generalized as follows: Let k be a positive integer, $k \geq 3$. Then there is some positive integer $m(k)$, depending on k, with the following property: If I^k is torsion-free then, for all $n \geq m(k)$, each positive n-dimensional quadratic form over F is annihilated by q_n. The proof is exactly the same. The integer $m(k)$ will be the least positive integer m such that if $r = [m/2]$ then 2^{k-r-1} divides $(r+1)!$ For example $m(3) = 2$, $m(4) = 4$, $m(5) = m(6) = m(7) = 6$, $m(8) = 8$, etc. The condition on $m(k)$ is to ensure that $q_n(\phi)$ is a sum of elements of I^k.

COMMENT 3. Theorem 1 is false for fields F in general as the following example shows:

Let $F = \mathbb{Q}(x_1, x_2, \ldots, x_8)$, the field of rational functions in $x_1, x_2, \ldots, x_8$, and let $\alpha = x_1^2 + x_2^2 + \cdots + x_8^2$. The two-dimensional form $\phi = \langle \alpha, \alpha\rangle$ is a positive form over F since α is a totally positive element of F. Now $q_2(\phi) = 4\langle 1, \alpha\rangle$ and if $q_2(\phi) = 0$ in $W(F)$ then α is a sum of four squares in F. This contradicts the theorem of Cassels [7, 4. 3. 5]. Hence $q_2(\phi) \neq 0$.

In similar fashion one can construct examples of n-dimensional positive forms which are not annihilated by q_n.

Two special classes of positive forms are:

(1) Sums of Pfister forms. These were studied in [3] where they are called p-forms. See also [2].

(2) Sums of squares in $W(F)$. These were studied in [6].

THEOREM 2. *Let F be any field of characteristic unequal to two.*

(a) Any n-dimensional sum of Pfister forms over F is annihilated by q_n.

(b) Any n-dimensional sum of squares of forms over F is annihilated by q_n.

PROOF. (a) Let $\phi = \sum \nu_i$ be an n-dimensional sum of Pfister forms ν_i. The earlier lemma gives an expression for $q_n(\phi)$ as a sum of $(r+1)$-fold Pfister forms where $r = [n/2]$. Since $r+1 > n/2$ each of these $(r+1)$-fold Pfister forms must necessarily involve more than half of the elements of at least one of the ν_i. Examining the diagonalization of a Pfister form it is easy to see that if we choose more than half of the elements of this diagonalization then we must have included the element 1 or else have included three elements of the kind a, b, ab. Since $\langle 1, -1 \rangle$ is hyperbolic and $\langle 1, -a \rangle \langle 1, -b \rangle \langle 1, -ab \rangle$ is hyperbolic it follows that each of the $(r+1)$-fold Pfister forms is hyperbolic and so $q_n(\phi) = 0$.

(b) The proof is similar to that of (a), using the fact that $\langle a_1, a_2, \ldots, a_m \rangle^2 = (m \times \langle 1 \rangle) \perp (2 \times \langle a_1 a_2, a_1 a_3, \ldots, a_{(m-1)} a_m \rangle)$.

REMARK. Part (a) of Theorem 2 was also proved by Hurrelbrink [2] using a different method.

The result of Conner [1] that q_n annihilates n-dimensional trace forms does not seem to be deducible by our methods. Conner used Burnside rings in proving his result so that perhaps we should not expect our viewpoint to yield the result.

THEOREM 3. *Let F be any field of characteristic unequal to two and let ϕ be a positive n-dimensional quadratic form over F. Then there exists some positive integer $m \geq n$ such that $q_m(\phi) = 0$.*

PROOF. We have seen that $q_n(\phi)$ is a sum of $(r+1)$-fold Pfister forms where $r = [n/2]$ and that these Pfister forms are all torsion forms because ϕ is positive. Since all torsion in $W(F)$ is 2-torsion it follows that a sufficiently high power of two will kill off all of these Pfister forms.

Now $q_{n+2}(\phi) = (n+2-\phi)q_n(\phi) = (2 + \sum \psi_i)q_n(\phi)$ using our earlier notation. Similarly $q_{n+4}(\phi) = (4 + \sum \psi_i)(2 + \sum \psi_i)q_n(\phi)$ etc. Since there are only n distinct ψ_i and $\psi_i^2 = 2\psi_i$ for all i it follows that, for t large enough, $q_{n+2t}(\phi)$ will be a sum in which each term contains a Pfister form multiplied by a sufficiently high power of two to kill it off. Thus $q_{n+2t}(\phi) = 0$ and the theorem is proved.

COLLARY. *Let F be any field of characteristic unequal to two and let ϕ be any non-singular quadratic form over F. Then ϕ is a positive form if and only if ϕ is annihilated in*

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$W(F)$ by some monic polynomial which has integer coefficients and non-negative integer roots.

PROOF. The implication one way is immediate from the theorem. To prove the converse implication suppose that $t(\phi) = 0$ in $W(F)$ where t is a monic polynomial with integer coefficients and non-negative integer roots. Then $t(\mathrm{sig}_P\,\phi) = 0$ for all orderings P of F where $\mathrm{sig}_P\,\phi$ denotes the signature of ϕ at p. (This is because the signature is additive and multiplicative). Hence $\mathrm{sig}_P\,\phi \geq 0$ for all P and ϕ is a positive form.

We finish this paper by showing how our earlier lemma can also be exploited to give results on non-singular quadratic forms which are not necessarily positive.

THEOREM 4. *Let F be a field such that $I^3 = 0$. Then any non-singular quadratic form over F is annihilated in $W(F)$ by some monic integer polynomial of degree* 2.

PROOF. Let ϕ be a non-singular quadratic form of dimension n over F. By the lemma $q_{n,2}(\phi) = 2(\sum_{i<j} \psi_i\psi_j)$. Absorbing the factor 2 into the sum we have that $q_{n,2}(\phi)$ is a sum of 3-fold Pfister forms.

The assumption that $I^3 = 0$ implies that $q_{n,2}(\phi) = 0$, i.e. any form of dimension n is annihilated by the polynomial $(n - x)(n - 2 - x)$.

COMMENT. Similarly to Comment 2 after Theorem 1 we can generalize Theorem 4 as follows: Let F be a field such that $I^k = 0$. Then any non-singular quadratic form over F is annihilated in $W(F)$ by some monic polynomial of degree $m(k)$.

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